Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 217 (2008) 2377–2400
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Subword complexity and projection bodies

Christian Steineder

Vienna University of Technology, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstraße 8-10, 1040 Vienna, Austria

Received 2 May 2007; accepted 12 November 2007

Available online 29 January 2008

Communicated by Michael J. Hopkins

Abstract

A polytope $P \subseteq [0,1)^d$ and an $\vec{\alpha} \in [0,1)^d$ induce a so-called Hartman sequence $\mathbf{h}(P,\vec{\alpha}) \in \{0,1\}^{\mathbb{Z}}$ which is by definition 1 at the kth position if $k\vec{\alpha} \bmod 1 \in P$ and 0 otherwise, $k \in \mathbb{Z}$. We prove an asymptotic formula for the subword complexity of such a Hartman sequence. This result establishes a connection between symbolic dynamics and convex geometry: If the polytope P is convex then the subword complexity of $\mathbf{h}(P,\vec{\alpha})$ asymptotically equals the volume of the projection body ΠP of P for almost all $\vec{\alpha} \in [0,1)^d$.

MSC: 11K31; 37B10; 52B11

Keywords: Distribution mod 1; Ergodic group translation; Hartman sequence; Projection body; Sturmian sequence; Subword complexity

1. Introduction

Let $\mathcal{K}^d$ be the set of convex bodies, i.e. compact convex sets, in $\mathbb{E}^d$, the standard Euclidean space equipped with the usual inner product $x \cdot y$, $x, y \in \mathbb{R}^d$. Each $K \in \mathcal{K}^d$ is uniquely determined by its support function $h_K : \mathbb{S}^{d-1} \to \mathbb{R}$ defined by $h_K(u) = \sup\{x \cdot u \colon x \in K\}$. Here $\mathbb{S}^{d-1}$ denotes the $(d-1)$-dimensional unit sphere. For $u \in \mathbb{S}^{d-1}$, let $K|_u$ be the projection of K onto the hyperplane $x \cdot u = 0$ and λ^d the d-dimensional Lebesgue measure. Then, each $K \in \mathcal{K}^d$ induces the convex body $\Pi K \in \mathcal{K}^d$ whose support function is

$$h_{\Pi K}(u) = \lambda^{d-1}(K|_u).$$

E-mail address: christian.steineder@tuwien.ac.at.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2007.11.006

ΠK is called the projection body of K. This concept was introduced by Minkowski at the turn of the last century and has proved to be an important tool for the study of projections (cf. [7]).

Questions regarding the volume of the projection body (or its polar) have been central to convex geometry for some time now (see e.g. [5,6,8–10,13–18,20–22,25,28,31,32,37,38]).

In spite of every effort the projection body holds some unsolved questions of central importance. One major open problem is: which convex bodies of given volume have a projection body of maximal and minimal volume (see [19]).

In this paper it is shown that the volume of the projection body arises in a totally unexpected setting, namely in the context of the so-called Hartman sequences which play a role in the theory of symbolic dynamics and can be seen as a generalization of the classical Sturmian sequences. Here a brief introduction to this concept:

Let G be a compact topological group which is monothetic, i.e. there exists a generating $g \in G$ such that $(kg)_{k\in\mathbb{Z}}$ is dense in G. Hence G is abelian. We mainly focus on the special case $G = \mathbb{T}^d$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the torus group, with generator $\vec{\alpha} = (\alpha_1, \dots, \alpha_d)$. Such an $\vec{\alpha}$ is a generating element of $\mathbb{T}^d$ if and only if it is strongly irrational, i.e. its components together with 1 are linearly independent over $\mathbb{Z}$. To disburden the notation we interpret the translation by $\vec{\alpha}$ as a group action on $\mathbb{T}^d$ and hence omit the term mod(ulo) 1 in the sequel.

On such a group G there exists a unique Haar probability measure μ_G. A subset $M \subseteq G$ is called a (μ_G-)continuity set, if the Haar measure of its topological boundary, ∂M, is 0. Hence continuity sets are generalized Jordan measurable sets. The continuity sets we concentrate on are the polytopes in $\mathbb{T}^d$ (for the definition of polytopes in $\mathbb{T}^d$ see Section 2.1). A continuity set $M \subseteq G$ defines the binary biinfinite sequence, called Hartman sequence, $\mathbf{h} = (h(M,g)_k)_{k=-\infty}^{\infty} \in \{0,1\}^{\mathbb{Z}}$ which is 1 at the kth position if $kg \in M$ and 0 otherwise. The unique ergodicity of minimal group translations (cf. [34]) yields one fundamental property of Hartman sequences, namely their uniform density. More precisely:

$$\lim_{N\to\infty} \frac{\sum_{k=l}^{l+N-1} h(M,g)_k}{N} = \mu_G(M)$$

uniformly in $l \in \mathbb{Z}$. In our setting, i.e. $G = \mathbb{T}^d$ with generator $\vec{\alpha} \in \mathbb{T}^d$, the unique ergodicity translates to the classical well-distribution of the sequence $(k\vec{\alpha})_{k=0}^{\infty}$ (cf. [11] or Weyl's famous work [35]).

In [36] the author shows that a Hartman sequence $\mathbf{h} = (h(M,g)_k)_{k=-\infty}^{\infty}$ contains essentially all information about the underlying continuity set M and the generator g. But how much information is contained in the language generated by $\mathbf{h}$?

One possibility to quantify the richness of the language of a sequence is given by the complexity function: Let $\mathbf{h} = (h_k)_{k=-\infty}^{\infty} \in \{0,1\}^{\mathbb{Z}}$. We call, for $k \in \mathbb{Z}$ and $N \in \mathbb{N}$, a segment $h_k h_{k+1} \dots h_{k+N-1} \in \{0,1\}^N$ of $\mathbf{h}$ a (sub)word of $\mathbf{h}$ of length N. The (subword) complexity $\mathcal{P}_{\mathbf{h}}(N)$ is per definition the number of distinct words of $\mathbf{h}$ of length $N \in \mathbb{N}$. Hence $1 \leqslant \mathcal{P}_{\mathbf{h}}(N) \leqslant 2^N$. The complexity function, introduced in [23] and [24], is a well-studied function in combinatorics and symbolic dynamics and closely related to the concept of entropy in ergodic theory (cf. [12,26]). To emphasize the large range of applications of the complexity function $\mathcal{P}(N)$ we instance two recent number theoretical articles, [1] and [2], containing a combinatorial transcendence criterion based on $\mathcal{P}(N)$. Questions concerning the subword complexity of sequences arising from ergodic group translations have a long tradition—the most prominent representatives of such sequences are the Sturmian ones which are the non-(eventually)-periodic sequences of minimal complexity, namely $\mathcal{P}(N) = N + 1$ (see, i.e., [4,26, Chapter 6] and [12, Chapter 2]). For first results concerning the subword complexity of Hartman sequences we refer to [33].

Because of several reasons it is natural to interpret Hartman sequences as generalized Sturmian sequences. For instance, every Hartman sequence can be approximated arbitrarily well (w.r.t. the density) by finite intersections and unions of Sturmian sequences (see [27]). If $\mathbf{h} = (h(M,g)_k)_{k=-\infty}^{\infty} \in \{0,1\}^{\mathbb{Z}}$ is a Hartman sequence induced by the continuity set M and the generator $g \in G$ we write $\mathcal{P}_{(M,g)}(N)$ instead of $\mathcal{P}_{\mathbf{h}}(N)$ to emphasize the relation between the complexity and M and g.

The main result of the present work, Theorem 5, connecting convex geometry and symbolic dynamics, says that the identity

$$\lim_{N\to\infty} \frac{\mathcal{P}_{(P,\vec{\alpha})}(N)}{N^d} = \lambda^d(\Pi P)$$

holds whenever $P \subseteq \mathbb{T}^d$ and $\vec{\alpha} \in \mathbb{T}^d$ fulfil some independence condition which is, by Proposition 23, satisfied by almost all $\vec{\alpha} \in \mathbb{T}^d$ in the measure theoretical as well as in the topological sense if P is a fixed d-dimensional convex polytope in $\mathbb{T}^d$.

The essential tool to prove Theorem 5 is the so-called local complexity which is defined in Section 2.2.

The paper is organized in the following way: After the introduction we present in Section 2 further notation and facts needed for a concise formulation of the main result. In Section 3 the main result is presented and verified by applying the formula for the local complexity given by Proposition 6. Section 4 is devoted to the geometrical proof of Proposition 6. In Section 5 we conclude by showing Proposition 23 mentioned above.

2. Further notation and preliminaries

2.1. *Polytopes in* $\mathbb{T}^d$

For $u \in \mathbb{S}^{d-1}$ and $\lambda \in \mathbb{R}$, the set $H_{u,\lambda} = \{x \in \mathbb{E}^d\colon\ x \cdot u = \lambda\}$ is a hyperplane in $\mathbb{E}^d$. $H^+_{u,\lambda} = \{x \in \mathbb{E}^d\colon\ x \cdot u \gtrless \lambda\}$ and $H^-_{u,\lambda} = \{x \in \mathbb{E}^d\colon\ x \cdot u \lesseqgtr \lambda\}$ are the induced $\substack{\text{open}\\ \text{closed}}$ halfspaces. (For the sake of simplicity we do not introduce a different notation for closed and open halfspaces.) As usual, a polytope is a bounded region of $\mathbb{E}^d$ enclosed by a finite set of hyperplanes. In particular, a convex polytope $P \subseteq \mathbb{E}^d$ is defined by $P = \bigcap_{r=1}^{L} H^-_{u_r,\lambda_r}$, $u_r \in \mathbb{S}^{d-1}$ and $\lambda_r \in \mathbb{R}$, $r = 1,\dots,L$.

We call a $(d-1)$-dimensional subset of the boundary ∂P of a polytope $P \subseteq \mathbb{E}^d$ appearing as the intersection of ∂P with a tangent hyperplane of P a face of P and, for $i = 0,\dots,d-2$, an i-dimensional subset of ∂P defined by the intersection of some faces of P, an i-dimensional face of P if $i \in \{2,\dots,d-2\}$, an edge of P if $i = 1$ and a vertex of P if $i = 0$.

We say the d hyperplanes H_{u_r,λ_r}, $r = 1,\dots,d$, are in general position if the vectors u_r span $\mathbb{E}^d$. Accordingly, d faces of a polytope are in general position if they are contained in d hyperplanes in general position.

$\mathbb{T}^d$ can be obtained from $[0,1)^d$ by identifying the opposite faces of $[0,1)^d$. Having this gluing process in mind, a set $C \subseteq [0,1)^d \subseteq \mathbb{E}^d$ can be interpreted as a subset of $\mathbb{T}^d$ and vice versa. To keep notation simple we only distinguish between these different points of view if necessary. As a general principle we interpret $[0,1)^d$ as a subset of $\mathbb{E}^d$ if we need to define sets and, on the other hand, as $\mathbb{T}^d$ if the translation by $\vec{\alpha}$ comes to the fore.

For the rest of the article:

- P denotes a polytope in $[0, 1)^d$ with nonempty interior. In fact, because of technical reasons we assume $P \subseteq (0, 1)^d$. Since what follows is invariant under translations this restriction is without loss of generality. According to the above-mentioned, each (translate mod 1 of such a) $P \subseteq [0, 1)^d$ can be interpreted as a polytope in $\mathbb{T}^d$. We denote the L faces of P by F_r, $r = 1, \ldots, L$, and write $F \subseteq H_{u,\lambda}$, $u \in \mathbb{S}^{d-1}$, $\lambda \in \mathbb{R}$, for a face F of a polytope $P \subseteq \mathbb{T}^d$ if the face F of P, interpreted as subset of $[0, 1)^d$, lies in the hyperplane $H_{u,\lambda}$ of $\mathbb{E}^d$.
- $B_r(x)$ denotes the open ball with center x and radius r and $B_r = B_r(0)$.
- $C(\sigma, x) = [x - \sigma/2, x + \sigma/2)^d$ denotes a half open cube with center x and side length σ.
- λ^d denotes the d-dimensional Haar measure on $\mathbb{T}^d$ as well as the d-dimensional Lebesgue measure on $[0, 1)^d$.

2.2. *Partition sets and the local complexity*

Let M be a continuity set, C an arbitrary set in $\mathbb{T}^d$, $\vec{\alpha}$ strongly irrational and $\mathbf{h} = (h(M, \vec{\alpha})_k)_{k=-\infty}^{\infty} \in \{0, 1\}^{\mathbb{Z}}$ the induced Hartman sequence. The following concept, introduced in [33], is of central importance in the sequel.

We say that a word $w = h_k h_{k+1} \ldots h_{k+N-1} \in \{0, 1\}^N$ of $\mathbf{h}$ starts in $C \subseteq \mathbb{T}^d$ if $k\vec{\alpha} \in C$, $k \in \mathbb{Z}$. Let

$$\begin{aligned} W(C, N) &= W(C, N, M, \vec{\alpha}) \\ &= \left\{ w = h_k h_{k+1} \ldots h_{k+N-1} \in \{0, 1\}^N \colon\ k\vec{\alpha} \in C, k \in \mathbb{Z} \right\} \end{aligned}$$

be the set of all words of $\mathbf{h} = (h(M, \vec{\alpha})_k)_{k=-\infty}^{\infty}$ of length $N \in \mathbb{N}$ starting in C. This set and in particular its cardinality are of central importance in the sequel.

Definition 1. We call

$$\mathcal{P}(C, N) = \mathcal{P}(C, N, M, \vec{\alpha}) = \left| W(C, N, M, \vec{\alpha}) \right|$$

the local complexity of C induced by M and $\vec{\alpha}$.

For $N \in \mathbb{N}$, it is easy to see (cf. [27]) that a word $w = w_0 w_1 \ldots w_{N-1} \in \{0, 1\}^N$ is an element of $W(C, N)$ if and only if

$$M_w(C) = \left(\bigcap_{j=0}^{N-1} \left(M^{w_j} - j\vec{\alpha} \right) \right) \cap C,$$

$M^1 = M$ and $M^0 = \mathbb{T}^d \setminus M$, is nonempty. The sets $M_w(C)$, $w \in \{0, 1\}^N$, generate a partition of C. Thus, the local complexity of C is the number of partition sets $M_w(C)$ in C induced by N translates of ∂M by $-\vec{\alpha}$. Under certain assumptions, introduced in the next section, we can concentrate on the d-dimensional partition sets $M_w(C)$ (see also Lemma 17).

2.3. Independence of $P \subseteq \mathbb{T}^d$ and $\vec{\alpha} \in \mathbb{T}^d$

Let $P \subseteq \mathbb{T}^d$ be a polytope with L faces $F_r \subseteq H_{u_r,\lambda_r}$, $r = 1, \dots, L$, and $\vec{\alpha} \in \mathbb{T}^d$ strongly irrational. We call a point $x \in \mathbb{T}^d$ a vertex after N translations if there exist, for $j = 1, \dots, d$, numbers $n_j \in \{0, 1, \dots, N-1\}$, and faces F_{r_j}, $r_j \in \{1, \dots, L\}$, in general position such that $x + n_j\vec{\alpha} \in F_{r_j}$. The name vertex is clearly motivated by the fact that such an x can be written as $\{x\} = \bigcap_{j=1}^{d}(F_{r_j} - n_j\vec{\alpha})$. For our estimates we need a condition which guarantees that there are not too many over-determined vertices in $\mathbb{T}^d$ induced by the orbit $(\partial P - n\vec{\alpha})_{n\in\mathbb{N}}$. In fact, for technical reasons, we even assume that the orbit of a slightly enlarged version of P does not generate too many over-determined vertices. More precisely: Let $c > 0$. Set, in $\mathbb{E}^d$,

$$F_r^{+c} = (F_r \tilde{+} B_c) \cap H_{u_r,\lambda_r} \quad \text{and} \quad F_r^{-c} = F_r \Bigg\backslash \left(\bigcup_{\substack{s=1\\ s\neq r}}^{L} (F_s \tilde{+} B_c)\right),$$

where $\tilde{+}$ denotes the usual set theoretical (Minkowski) sum. Fix $\sigma' > 0$ such that $P \tilde{+} B_{\sigma'} \subseteq [0,1)^d$. This is possible since we assume $P \subseteq (0,1)^d$ (cf. Section 2.1). Then, clearly, $F_r^{\sigma'} \subseteq [0,1)^d$, for all $r \in \{1, \dots, L\}$. Set $\partial P^{\sigma'} = \bigcup_{r=1}^{L} F_r^{\sigma'}$.

Fix $F_r^{\sigma'} \subseteq H_{u_r,\lambda_r}$, $r \in \{1, \dots, L\}$. An $x \in F_r^{\sigma'}$, for which there exist d further faces $F_{r_j}^{\sigma'} \subseteq H_{u_{r_j},\lambda_{r_j}}$ of P such that $F_{r_j}^{\sigma'} \neq F_r^{\sigma'}$ and $\operatorname{span}(u_r, u_{r_1}, \dots, u_{r_d}) = \mathbb{E}^d$, as well as d integers $n_j \in \{0, \dots, N-1\}$, $j = 1, \dots, d$, such that $x + n_j\vec{\alpha} \in F_{r_j}^{\sigma'}$ for all j, is called over-determined vertex on $F_r^{\sigma'}$ after N steps. Let $Q_r(N, \sigma')$ be the set of all over-determined vertices on $F_r^{\sigma'}$ after N steps.

Definition 2. For $\sigma' > 0$ a polytope $P \subseteq \mathbb{T}^d$ and a strongly irrational $\vec{\alpha} \in \mathbb{T}^d$ are called σ'-asymptotically independent (abbreviated σ'-a.i.) if

(1) $P \tilde{+} B_{\sigma'} \subseteq [0,1)^d$,
(2) $F_r^{-\sigma'}$ is a $(d-1)$-dimensional set for every $r = 1, \dots, L$,
(3) $\max_{r\in\{1,\dots,L\}} |Q_r(N, \sigma')| = o(N^{d-1})$, and
(4) there exists no $n \in \mathbb{N} \setminus \{0\}$ such that $(\partial P^{\sigma'} - n\vec{\alpha}) \cap \partial P^{\sigma'}$ contains a $(d-1)$-dimensional set.

As shown in Section 5, a polytope $P \subseteq \mathbb{T}^d$ and an $\vec{\alpha} \in \mathbb{T}^d$ are typically σ'-a.i.

2.4. Definition of a measure preserving mapping

Let $C_0 = [0, \sigma)^d$, $\sigma > 0$, be a half open cupe rooted at 0. Let $W = \{w_1, \dots, w_d\}$ be a set of d linearly independent vectors in $\mathbb{S}^{d-1}$. Let $M_W = (w_1, \dots, w_d)$ be the $d \times d$-matrix whose columns are the vectors $w_1, \dots, w_d$.

We define the matrix

$$\tau_W = \left|\operatorname{Det}(M_W)\right|^{1/d} \left(M_W^{-1}\right)^t.$$

Observe that

(1) τ_W is measure preserving since $|\operatorname{Det}(\tau_W)| = 1$,
(2) for $\tilde{w}_i = \tau_W(\sigma e_i)$, $i, j \in \{1, \dots, d\}$, $\sigma > 0$,

$$\tilde{w}_i \cdot w_j = \begin{cases} 0 & \text{if } i \neq j, \\ \sigma |\operatorname{Det}(M_W)|^{1/d} & \text{if } i = j. \end{cases}$$

Here e_i, $i = 1, \dots, d$, denotes the ith Euclidean unit vector.

In particular τ_W maps C_0, spanned by the vectors σe_i, $i = 1, \dots, d$, measure preservingly to the parallelepiped $T_W(\sigma)$, spanned by the vectors $\tilde{w}_i$. The faces of $T_W(\sigma)$ lie, by (2), in the hyperplanes $H_{w_1,0}, \dots, H_{w_d,0}, H_{w_1,\rho}, \dots, H_{w_d,\rho}$, $\rho = \sigma |\operatorname{Det}(M_W)|^{1/d}$, i.e. its unit normal vectors are the elements of W.

Fix, moreover, $T_W(\sigma) = \emptyset$ if the elements of W are linearly dependent.

For a polytope in $[0, 1)^d$ with faces $F_r \subseteq H_{u_r,\lambda_r}$, i.e., with normal vectors u_r, $r = 1, \dots, L$, and $\sigma > 0$, we define

$$\xi(P, \sigma) = \max_{\substack{W=\{u_{i_1},u_{i_2},\dots,u_{i_d}\}\subseteq \\ \{u_1,u_2,\dots,u_L\}}} \bigl(\operatorname{diam}\bigl(T_W(\sigma)\bigr)\bigr).$$

2.5. The separation number

Back in the abstract setting let G be a compact monothetic metrizable group with generator g and Haar measure μ_G. Let $d(\cdot,\cdot)$ denote a metric on G compatible with its topology. We call, following [36], a (μ_G-)continuity set $M \subseteq G$ aperiodic if $\mu_G(M \triangle (M + x)) = 0$ implies $x = 0$. Recall from Section 2.2 that, for a word $w = w_0 w_1 \dots w_{N-1} \in \{0, 1\}^N$ of length N, the set M_w of starting points of w is given by $M_w = \bigcap_{j=0}^{N-1} (M^{w_j} - jg)$, $M^1 = M$ and $M^0 = G \setminus M$. Although the next result seems rather natural in the context of ergodic theory we like to present the following nice (and so far unpublished) proof of this statement due to the author of [36].

Lemma 3. *Let $M \subseteq G$ be an aperiodic continuity set and g a generator of G. Let $\varepsilon > 0$. Then there exists an $N_s = N_s(M, g, \varepsilon) \in \mathbb{N}$, the so-called ε-separation number of M and g, such that for every set M_w, $w \in \{0, 1\}^N$,* $\operatorname{diam}(M_w) < \varepsilon$ *whenever N, the length of the word w, is greater than N_s.*

Proof. Fix, for $z \in G$, the notation $w(z) = (w(z)_k)_{k=0}^{\infty} = (1_M(z + kg))_{k=0}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ and $w(z)|_n = (w(z)_k)_{k=0}^{n-1}$ where $1_M : G \to \{0, 1\}$ is the characteristic function of M.

In order to prove that for every $\varepsilon > 0$ there exists an $N_s \in \mathbb{N}$ such that for all $n \geqslant N_s$ and all $x, y \in G$ $w(x)|_n = w(y)|_n$ implies $d(x, y) \leqslant \varepsilon$ assume that there exists an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there exist $x_n, y_n \in G$ such that $w(x_n)|_n = w(y_n)|_n$ but $d(x_n, y_n) \geqslant \varepsilon$.

$G \times G$ is compact since G is. Thus there exists a subsequence $(x_{k_n}, y_{k_n})_{n=0}^{\infty}$ of $(x_n, y_n)_{n=0}^{\infty}$ converging to, say, (x, y). Observe that $x \neq y$. Choose an arbitrary $z \in M^o$ and a sequence $(l_n)_{n=0}^{\infty}$ in $\mathbb{Z}$ such that $x_{k_n} + l_n g \to z$. Such a sequence $(l_n)_{n=0}^{\infty}$ exists because g is a generator of G. Hence $l_n g \to z - x$. We may assume, w.l.o.g., that $l_n \leqslant k_n$ for all $n \in \mathbb{N}$ (if necessary, we can pick a sufficiently sparse subsequence of $(k_n)_{n=0}^{\infty}$). Since $x_{k_n} + l_n g \in M$ iff $w(x_{k_n})_{l_n} = 1$ iff $w(y_{k_n})_{l_n} = 1$ (because of $w(x_n)|_n = w(y_n)|_n$ and $l_n \leqslant k_n$) iff $y_{k_n} + l_n g \in M$ we conclude that $z \in M^o$ iff $y + z - x \in \bar{M}$. z is chosen arbitrarily and M is a continuity set. So we obtain

$\mu_G(M \setminus (M + y - x)) = 0$. The same argument yields $\mu_G((M + y - x) \setminus M) = 0$. But $y - x \neq 0$ contradicts the aperiodicity of M. $\square$

Remark 4. A d-dimensional polytope $M = P$ in $G = \mathbb{T}^d$ (with $g = \vec{\alpha} \in \mathbb{T}^d$ strongly irrational) is clearly aperiodic.

3. The main result

Theorem 5. *Let P be a polytope in $\mathbb{T}^d$ with L faces F_r and normal vectors u_r, $r = 1, \ldots, L$. Let $\vec{\alpha} \in \mathbb{T}^d$ be strongly irrational and $\sigma' > 0$ sufficiently small such that P and $\vec{\alpha}$ are σ'-a.i. Then*

$$\lim_{N\to\infty} \frac{\mathcal{P}_{(P,\vec{\alpha})}(N)}{N^d} = \frac{1}{d!} \sum_{r_1=1}^{L} \cdots \sum_{r_d=1}^{L} \left(\left|\mathrm{Det}(u_{r_1}, \ldots, u_{r_d})\right| \prod_{j=1}^{d} \lambda^{d-1}(F_{r_j}) \right).$$

Hence, if P is a convex polytope in $\mathbb{T}^d$ and ΠP its projection body then

$$\lim_{N\to\infty} \frac{\mathcal{P}_{(P,\vec{\alpha})}(N)}{N^d} = \lambda^d(\Pi P).$$

The main tool for the proof of Theorem 5 is the following estimate of the local complexity $\mathcal{P}(C, N, P, \vec{\alpha})$.

Proposition 6. *Let P be a polytope in $\mathbb{T}^d$ with L faces F_r and normal vectors u_r, $r = 1, \ldots, L$. Let $\vec{\alpha} \in \mathbb{T}^d$ be strongly irrational and $\sigma' > 0$ sufficiently small such that P and $\vec{\alpha}$ are σ'-a.i. Let $C = C(\sigma, x)$ be a half open cube in $\mathbb{T}^d$ with side length $\sigma > 0$ and center x, σ small enough such that $\xi(P, \sigma) < \sigma'/2$. Then*

$$\begin{aligned} &\frac{\sigma^d}{d!} \left(\sum_{r_1=1}^{L} \cdots \sum_{r_d=1}^{L} \left(\left|\mathrm{Det}(u_{r_1}, \ldots, u_{r_d})\right| \prod_{j=1}^{d} \lambda^{d-1}\left(F_{r_j}^{-2\xi(P,\sigma)}\right) \right) \right) \\ &\quad \leqslant \liminf_{N\to\infty} \frac{\mathcal{P}(C, N, P, \vec{\alpha})}{N^d} \leqslant \limsup_{N\to\infty} \frac{\mathcal{P}(C, N, P, \vec{\alpha})}{N^d} \\ &\quad \leqslant \frac{\sigma^d}{d!} \left(\sum_{r_1=1}^{L} \cdots \sum_{r_d=1}^{L} \left(\left|\mathrm{Det}(u_{r_1}, \ldots, u_{r_d})\right| \prod_{j=1}^{d} \lambda^{d-1}\left(F_{r_j}^{+2\xi(P,\sigma)}\right) \right) \right). \end{aligned}$$

Proof of Theorem 5. Pick $k \in \mathbb{N}$ sufficiently large such that $\xi(P, 1/k) < \sigma'/2$. Set $\sigma = 1/k$ and cover $\mathbb{T}^d$ by k^d disjoint cubes $C_i(1/k) = [x_i - \frac{1}{2k}, x_i + \frac{1}{2k})^d$. For $\varepsilon \in (0, 1/k)$ let N_s be the ε-separation number of P and $\vec{\alpha}$ given by Lemma 3. Then, for $C_i^{-\varepsilon} = [x_i - \frac{1}{2k} + \varepsilon, x_i + \frac{1}{2k} - \varepsilon)^d$, $i \neq j$ and $N \geqslant N_s$ imply $W(C_i^{-\varepsilon}, N, P, \vec{\alpha}) \cap W(C_j^{-\varepsilon}, N, P, \vec{\alpha}) = \emptyset$.

Thus, considering the local complexities of all cubes $C_i^{-\varepsilon}$, $i = 1, \ldots, k^d$, simultaneously, Proposition 6 gives the lower bound

$$\liminf_{N\to\infty}\frac{\mathcal{P}_{(P,\vec{\alpha})}(N)}{N^d}$$
$$\geqslant \frac{k^d(\frac{1}{k}-\varepsilon)^d}{d!}\left(\sum_{r_1=1}^{L}\cdots\sum_{r_d=1}^{L}\left(\left|\mathrm{Det}(u_{r_1},\dots,u_{r_d})\right|\prod_{j=1}^{d}\lambda^{d-1}\left(F_{r_j}^{-2\xi(P,\frac{1}{k}-\varepsilon)}\right)\right)\right).$$

This holds for all $\varepsilon > 0$. Therefore we have

$$\liminf_{N\to\infty}\frac{P_{(P,\vec{\alpha})}(N)}{N^d}$$
$$\geqslant \frac{k^d(\frac{1}{k})^d}{d!}\left(\sum_{r_1=1}^{L}\cdots\sum_{r_d=1}^{L}\left(\left|\mathrm{Det}(u_{r_1},\dots,u_{r_d})\right|\prod_{j=1}^{d}\lambda^{d-1}\left(F_{r_j}^{-2\xi(P,\frac{1}{k})}\right)\right)\right)$$
$$=\frac{1}{d!}\sum_{r_1=1}^{L}\cdots\sum_{r_d=1}^{L}\left(\left|\mathrm{Det}(u_{r_1},\dots,u_{r_d})\right|\prod_{j=1}^{d}\lambda^{d-1}\left(F_{r_j}^{-2\xi(P,\frac{1}{k})}\right)\right).$$

Analogously Proposition 6 yields the upper bound

$$\limsup_{N\to\infty}\frac{\mathcal{P}_{(P,\vec{\alpha})}(N)}{N^d}$$
$$\leqslant \frac{k^d(\frac{1}{k})^d}{d!}\left(\sum_{r_1=1}^{L}\cdots\sum_{r_d=1}^{L}\left(\left|\mathrm{Det}(u_{r_1},\dots,u_{r_d})\right|\prod_{j=1}^{d}\lambda^{d-1}\left(F_{r_j}^{+2\xi(P,\frac{1}{k})}\right)\right)\right)$$
$$=\frac{1}{d!}\sum_{r_1=1}^{L}\cdots\sum_{r_d=1}^{L}\left(\left|\mathrm{Det}(u_{r_1},\dots,u_{r_d})\right|\prod_{j=1}^{d}\lambda^{d-1}\left(F_{r_j}^{+2\xi(P,\frac{1}{k})}\right)\right).$$

$\lim_{k\to\infty}\lambda^{d-1}(F_r^{\pm 2\xi(P,\frac{1}{k})}) = \lambda^{d-1}(F_r)$ for all $r = 1,\dots,L$ proves the formula. If P is a convex polytope then our result for the asymptotic complexity just coincides with the formula for the volume of the projection body of P (cf. [29, p. 415]). □

Remark 7. [30] contains results related to Proposition 6 in the context of stochastic geometry.

4. Proof of Proposition 6

4.1. Overview

As discussed in Section 2.1, we interpret according to the context our fixed polytope $P \subseteq (0,1)^d$ with L faces $F_r \subseteq H_{u_r,\lambda_r}$, $r = 1,\dots,L$, as a subset of either $\mathbb{E}^d$ or $\mathbb{T}^d$. Let $\vec{\alpha}\in\mathbb{T}^d$ be a strongly irrational translation vector and σ' sufficiently small such that P and $\vec{\alpha}$ are σ'-asymptotically independent. $\sigma > 0$ is a real number such that $\xi(P,\sigma) < \sigma'/2$.

$C(\sigma,x) \subseteq \mathbb{T}^d$ is again the half open cube with side length σ and center x. Let $w = w_0w_1\dots w_{N-1} \in \{0,1\}^N$ be a word of length $N \in \mathbb{N}$. Following Section 2.2, let $P_w(C(\sigma,x)) = C(\sigma,x)\cap\bigcap_{j=0}^{N-1}(P^{w_j} - j\vec{\alpha})$, where $P^1 = P$ and $P^0 = \mathbb{T}^d \setminus P$. These sets directly yield the local complexity, namely $\mathcal{P}(C(\sigma,x),N) = |\{P_w(C(\sigma,x))\colon\ w\in\{0,1\}^N\}|$.

In order to estimate the number of partition cells of $C(\sigma, x)$ of the form $P_w(C(\sigma, x))$, $w \in \{0,1\}^N$, we shall construct two other partitions of $C(\sigma, x)$, $\Pi^1(C(\sigma, x), N)$ and $\Pi^2(C(\sigma, x), N)$, induced by certain sequences of hyperplanes such that $|\Pi^1(C(\sigma, x), N| \leqslant \mathcal{P}(C(\sigma, x), N) \leqslant |\Pi^2(C(\sigma, x), N)|$. Partitions induced by hyperplanes are more favorable since the number of vertices, i.e. intersection points of d hyperplanes in general position, essentially equals the number of d-dimensional partition cells.

A general result in this direction follows in Section 4.2. In Section 4.3, we define the sequences of hyperplanes yielding the partitions $\Pi^i(C(\sigma, x), N)$, $i = 1, 2$. In Section 4.4 we show that the number of partition cells in $\Pi^i(C(\sigma, x), N)$ asymptotically coincides with the number of vertices in C. This and the well-distribution of the sequence $(n\vec{\alpha})_{n \geqslant 0}$ in $\mathbb{T}^d$ allow to deduce an explicit formula in Section 4.5. In Section 4.6 we finish the proof of Proposition 6.

4.2. A basic lemma

Let $(H_i)_{i=1}^{\infty}$ be a sequence of distinct hyperplanes in $\mathbb{E}^d$ and C a d'-dimensional (not necessarily closed) bounded convex set, $d' \in \{1, \dots, d\}$. The hyperplanes $(H_i)_{i=1}^N$ induce a partition in C—the partition cells are of the form $C \cap \bigcap_{i=1}^N H_i^{\pm}$. $H_i^{\pm}$ denotes the positive respectively the negative halfspace induced by H_i. It is natural to assume some relation between the d'-dimensional partition cells and the vertices, i.e. those $x \in C$ with $\{x\} = \bigcap_{j=1}^{d'} H_{i_j}$, for suitable $i_j \in \{1, \dots, N\}$. To establish such a relation we introduce the concept of the weight.

For every point $x \in C$ we define its weight $w(x, N, C) \in \mathbb{N}$ recursively on N, the number of hyperplanes $(H_i)_{i=1}^N$, and d', $1 \leqslant d' \leqslant d$, the dimension of C, in the following way:

(1) $w(x, 0, C) = 0$ for all d'-dimensional C, $d' \in \{1, \dots, d\}$, and for all $x \in C$.
(2) If $d' = 1$ set

$$w(x, N, C) = \begin{cases} 1 & \text{if } \{x\} = H_i \cap \operatorname{rel\,int} C \text{ for some } i \in \{1, \dots, N\}, \\ 0 & \text{otherwise.} \end{cases}$$

(As usual, $\operatorname{rel\,int} S$ denotes the relative interior of S, i.e. the interior of S considered as a subset of the affine hull of S. Analogously we define the relative boundary of a set.)
(3) Assume $w(x, N', C')$ is defined for every d''-dimensional C', $1 \leqslant d'' \leqslant d'$, and every $1 \leqslant N' < N$. Let C be d'-dimensional. Set $C_N = C \cap H_N$ and

$$w(x, N, C) = \begin{cases} w(x, N-1, C) & \text{if } x \in H_N \cap \operatorname{rel\,int} C, C \nsubseteq H_N \\ \quad + w(x, N-1, C_N) & \text{and } C_N \neq H_i \cap C \text{ for all } 1 \leqslant i < N, \\ w(x, N-1, C) & \text{otherwise.} \end{cases}$$

(2) and (3) guarantee

$$\begin{array}{l} C \subseteq H_N \text{ or } x \notin C \cap H_N \\ \text{or } C \cap H_N = H_i \cap C \\ \text{for some } 1 \leqslant i < N \end{array} \quad \Longrightarrow \quad w(x, N, C) = w(x, N-1, C). \tag{1}$$

Moreover, $w(x, N, C) = 0$ for all x on the relative boundary of C, all $N \in \mathbb{N}$ and all d'-dimensional C.

Before coming to the basic lemma, Lemma 10, we present immediate lower and upper bounds for the weight.

Lemma 8. *Assume $x \in \operatorname{relint} C$ where C is a d'-dimensional bounded convex set. Let $S(x) = \{i \in \{1, \dots, N\}\colon x \in H_i\}$. Then*

$$w(x, N, C) \begin{cases} = 0 & \text{if } \{x\} \neq \bigcap_{i \in S(x)} H_i \cap C, \\ = 1 & \text{if } \{x\} = \bigcap_{i \in S(x)} H_i \cap C \text{ and } |S(x)| = d', \\ \geqslant 1 & \text{if } \{x\} = \bigcap_{i \in S(x)} H_i \cap C \text{ and } |S(x)| > d'. \end{cases}$$

Proof. $w(x, N, C) = 0$ if $\{x\} \neq \bigcap_{i \in S(x)} H_i \cap C$ follows immediately by induction from the definition of the weight. We show that $w(x, N, C) = 1$ if $\{x\} = \bigcap_{i \in S(x)} H_i \cap C$ and $|S(x)| = d'$ by induction on d'. If $d' = 1$ the assertion is clear. Assume it is true for any d''-dimensional C', $d'' \leqslant d' - 1$ and let C be a d'-dimensional convex set. If $S(x) = \{i_1, \dots, i_{d'}\}$ such that $1 \leqslant i_1 < \dots < i_{d'} \leqslant N$ then, by induction hypotheses, $w(x, i_{d'} - 1, C \cap H_{i_{d'}}) = 1$ and $w(x, i_{d'} - 1, C) = 0$ according to the first equality of this lemma. Thus, $w(x, N, C) = w(x, i_{d'}, C) = w(x, i_{d'} - 1, C) + w(x, i_{d'} - 1, C \cap H_{i_{d'}}) = 1$ by the definition of the weight. The last case, $w(x, N, C) \geqslant 1$, follows directly from the monotonicity of the weight w.r.t. the number of hyperplanes x is contained in. $\square$

Lemma 9. *Let $x \in \operatorname{relint} C$ where C is a d'-dimensional bounded convex set. If there exist $L \geqslant d'$ hyperplanes H_{i_j}, $j = 1, \dots, L$, among $(H_i)_{i=1}^N$ such that $C \nsubseteq H_{i_j}$ and $x \in H_{i_j} \cap C$ then*

$$w(x, N, C) \leqslant u(L, d'), \tag{2}$$

where

$$u(L, d') = \begin{cases} 1 & \text{if } d' = 1, \\ \sum_{i_1=1}^{L+1-d'} \sum_{i_2=1}^{i_1} \cdots \sum_{i_{d'-1}=1}^{i_{d'-2}} 1 & \text{if } d' > 1. \end{cases}$$

Proof. By induction on L, the number of hyperplanes x is contained in, and d', the dimension of C:

$d' = 1$, $L \in \mathbb{N}$: Clear.
$d' - 1 \to d'$ and $L \leqslant d'$: $u(d', d') = 1$ is an upper bound due to Lemma 8.
$d' - 1 \to d'$ and $L - 1 \to L$: According to the induction hypothesis, assume $w(x, N - 1, C') \leqslant u(L', d'')$ for any d''-dimensional C' whenever either $L' \in \mathbb{N}$ and $d'' < d'$ or $L' \leqslant L - 1$ and $d'' = d'$.

For the step $L - 1 \to L$, let $d'' = d'$, $L' = L - 1$ and $x \in H_N = H_{i_L}$. Then, for $C_N = C \cap H_N$, the definition of the weight implies

$$\begin{aligned} w(x, N, C) &\leqslant w(x, N - 1, C) + w(x, N - 1, C_N) \\ &\leqslant \left(\sum_{i_1=1}^{L-d'} \sum_{i_2=1}^{i_1} \cdots \sum_{i_{d'-1}=1}^{i_{d'-2}} 1 \right) + \left(\sum_{i_1=1}^{L-(d'-1)} \sum_{i_2=1}^{i_1} \cdots \sum_{i_{d'-2}=1}^{i_{d'-3}} 1 \right) \\ &= u(L, d'). \qquad \square \end{aligned}$$

For $N \in \mathbb{N}$ let $G(C, N) = \{x \in C\colon\ w(x, N, C) \geqslant 1\}$. $G(C, N)$ is, by Lemma 8, the (finite) set of all intersection points in C induced by the hyperplanes H_i, $i = 1, \ldots, N$.

As noticed at the beginning of this section, the H_i, $i = 1, \ldots, N$, intersecting the d'-dimensional convex bounded set C induce a partition of C. Let $\Pi_{d'}(C, N)$ be the set of all d'-dimensional partition cells in C induced by H_i, $i = 1, \ldots, N$. In particular, an element $\pi \in \Pi_{d'}(C, N)$ is called an inner partition cell, if its closure does not intersect the relative boundary of C. Let $\Pi_{d'}(C, N)^o$ be the set of all d'-dimensional inner partition cells.

Lemma 10. *Let $d' \in \{1, \ldots, d\}$. Let C be a d'-dimensional bounded convex set. Let $N \in \mathbb{N}$. Then*

$$\left|\Pi_{d'}(C, N)^o\right| \leqslant \sum_{x \in G(C,N)} w(x, N, C) \leqslant \left|\Pi_{d'}(C, N)\right|.$$

Proof. Assuming $C_N = H_N \cap C \neq \emptyset$ and $C \nsubseteq H_N$, i.e., C_N is $(d' - 1)$-dimensional, as well as $d' \geqslant 2$ and $C_N \neq H_i \cap C$, for all $i = 1, \ldots, N - 1$, we first verify the equations

$$\left|\Pi_{d'}(C, N)\right| = \left|\Pi_{d'}(C, N - 1)\right| + \left|\Pi_{d'-1}(C_N, N - 1)\right|, \tag{3}$$

$$\left|\Pi_{d'}(C, N)^o\right| \leqslant \left|\Pi_{d'}(C, N - 1)^o\right| + \left|\Pi_{d'-1}(C_N, N - 1)^o\right|. \tag{4}$$

Every d'-dimensional partition cell is a convex set in C. Let $\pi \in \Pi_{d'}(C, N - 1)$. Then $\operatorname{rel\,int}\pi \cap H_N \neq \emptyset$ is equivalent to the fact that by the Nth intersection π is split into two d'-dimensional partition cells $\pi_1 = \pi \cap H_N^-$ and $\pi_2 = \pi \cap H_N^+ \in \Pi_{d'}(C, N)$. $\pi \cap H_N$ is the $(d' - 1)$-dimensional splitting set in C_N separating π_1 and π_2. Since $C_N \neq H_i \cap C$ for all $i = 1, \ldots, N - 1$ the number of additional d'-dimensional cells induced by H_N equals the number of $(d' - 1)$-dimensional cells in C_N proving (3). The same argument applies to (4). But an inner splitting set in $\Pi_{d'-1}(C_N, N - 1)^o$ does not generate a new d'-dimensional inner partition cell if it intersects an element of $\Pi_{d'}(C, N - 1)$ which has a nonempty intersection with two disjoint regions of the boundary of C. Therefore we only obtain an inequality.

We prove the assertion by a twofold induction on the dimension $d' \geqslant 1$ and $N \in \mathbb{N}$, the number of hyperplanes.

$d' = 1$, $N \in \mathbb{N}$ and $d' - 1 \to d'$, $N = 1$: Clear by the definition of $w(x, N, C)$.
$d' - 1 \to d'(\geqslant 2)$, $N - 1 \to N$:

Case 1. $C_N = H_N \cap C \neq \emptyset$, $C_N \neq H_i \cap C$ for all $i = 1, \ldots, N - 1$ and $C \nsubseteq H_N$. Using (3), (4) and the induction hypothesis we obtain

$$\begin{aligned}\left|\Pi_{d'}(C, N)^o\right| &\leqslant \left|\Pi_{d'}(C, N - 1)^o\right| + \left|\Pi_{d'-1}(C_N, N - 1)^o\right| \\ &\leqslant \underbrace{\sum_{x \in G(C,N-1)} w(x, N - 1, C) + \sum_{x \in G(C_N,N-1)} w(x, N - 1, C_N)}_{=\sum_{x \in G(C,N)} w(x,N,C) \text{ by definition}} \\ &\leqslant \left|\Pi_{d'}(C, N - 1)\right| + \left|\Pi_{d'-1}(C_N, N - 1)\right| = \left|\Pi_{d'}(C, N)\right|.\end{aligned}$$

Case 2. $C_N = H_N \cap C = \emptyset$ or $C_N = H_i \cap C$ for a number $i \in \{1, \ldots, N - 1\}$ or $C \subseteq H_N$. Then, $|\Pi_{d'}(C, N)| = |\Pi_{d'}(C, N - 1)|$ and $|\Pi_{d'}(C, N)^o| = |\Pi_{d'}(C, N - 1)^o|$. Hence, by the induction hypothesis on N,

$$\begin{aligned}\left|\Pi_{d'}(C,N)^o\right| &= \left|\Pi_{d'}(C,N-1)^o\right| \\ &\leqslant \sum_{x\in G(C,N-1)} w(x,N-1,C) \\ &\leqslant \left|\Pi_{d'}(C,N-1)\right| = \left|\Pi_{d'}(C,N)\right|.\end{aligned}$$

Under the assumptions of Case 2, (1) guarantees

$$\sum_{x\in G(C,N-1)} w(x,N-1,C) = \sum_{x\in G(C,N)} w(x,N,C). \qquad \square$$

4.3. Construction of the partitions $\Pi^i(C(\sigma,x),N)$, $i=1,2$

From now on fix $C=C(\sigma,x)$. As announced in Section 4.1, the aim of this section is to define hyperplanes inducing the partitions $\Pi^i(C,N)$, $i=1,2$, such that $|\Pi^1(C,N)|\leqslant \mathcal{P}(C,N)\leqslant |\Pi^2(C,N)|$. In Section 4.5 we refer to these hyperplanes in a slightly more general setting. Therefore we formulate the following definitions not only for cubes but for parallelepipeds.

We use the notation fixed at the beginning of Section 4.1 and interpret the polytope P as a subset of $\mathbb{E}^d$. Let $\sigma>0$ be small enough such that $\xi(P,\sigma)<\sigma'/2$. Set $B(\sigma)=B_{\frac{3}{2}\xi(P,\sigma)}$ and let $E\subseteq\mathbb{E}^d$ be a translate of a parallelepiped in $[0,1)^d$ with center e and $\operatorname{diam}(E)\leqslant\xi(P,\sigma)$. Let $E'=E-e$ be the translate of E with center 0. We define, for a face $F_r\subseteq H_{u_r,\lambda_r}$ of P, $r\in\{1,\dots,L\}$,

$$\begin{aligned}
\phi_{u_r}(\sigma) &= F_r \Big\backslash \left(\bigcup_{s=1,\, s\neq r}^{L} \left(F_s \tilde{+} B(\sigma)\right)\right) \quad \left(=F_r^{-\frac{3}{2}\xi(P,\sigma)}\right),\\
\Phi_{u_r}(\sigma) &= F_r^{\sigma'} \cap \left(F_r \tilde{+} B(\sigma)\right) \qquad\qquad\quad \left(=F_r^{+\frac{3}{2}\xi(P,\sigma)}\right),\\
J_{u_r}(E) &= \{\lambda\in\mathbb{R}\colon\ H_{u_r,\lambda}\cap E'\neq\emptyset\},\\
\gamma_{u_r}(E) &= \left\{z\in\mathbb{E}^d\colon\ z=y+\lambda u_r,\ y\in\phi_{u_r}(\sigma),\ \lambda\in J_{u_r}(E)\right\},\\
\Gamma_{u_r}(E) &= \left\{z\in\mathbb{E}^d\colon\ z=y+\lambda u_r,\ y\in\Phi_{u_r}(\sigma),\ \lambda\in J_{u_r}(E)\right\},\\
\gamma(E) &= \bigcup_{r=1}^{L}\gamma_{u_r}(E), \qquad \Gamma(E)=\bigcup_{r=1}^{L}\Gamma_{u_r}(E).
\end{aligned}$$

Thus, for $r=1,\dots,L$, $\gamma_{u_r}(E)\subseteq\Gamma_{u_r}(E)$ since $\phi_{u_r}(\sigma)\subseteq\Phi_{u_r}(\sigma)$. Condition (2) of Definition 2 implies that $\phi_{u_r}(\sigma)$ and $\Phi_{u_r}(\sigma)$ are $(d-1)$-dimensional subsets of the enlarged face $F_r^{\sigma'}$. $\genfrac{}{}{0pt}{}{\phi_{u_r}(\sigma)}{\Phi_{u_r}(\sigma)}$ is parallel to two faces of the rectangular parallelepiped $\genfrac{}{}{0pt}{}{\gamma_{u_r}(E)}{\Gamma_{u_r}(E)}$ which (partly) covers the face F_r. The height of both, $\gamma_{u_r}(E)$ and $\Gamma_{u_r}(E)$, equals the length of the interval $J_{u_r}(E)$ and $J_{u_r}(E)\subseteq[-\frac{\xi(P,\sigma)}{2},\frac{\xi(P,\sigma)}{2}]$. Hence, fixing $B(\sigma)=B_{\frac{3}{2}\xi(P,\sigma)}$ guarantees that the sets $\gamma_{u_r}(E)$ are pairwise disjoint. Moreover,

$$\begin{aligned}
\lambda^d\left(\gamma_{u_r}(E)\right) &= \left|J_{u_r}(E)\right|\lambda^{d-1}\left(\phi_{u_r}(\sigma)\right),\\
\lambda^d\left(\Gamma_{u_r}(E)\right) &= \left|J_{u_r}(E)\right|\lambda^{d-1}\left(\Phi_{u_r}(\sigma)\right).
\end{aligned} \tag{5}$$

Note that $\gamma(E)$ and $\Gamma(E)$ are contained in $[0,1)^d$. Also each translate of E whose center lies in $\gamma(E)$ or $\Gamma(E)$ is contained in $[0,1)^d$.

Switching to $\mathbb{T}^d$, i.e. interpreting $\gamma_{u_r}(E)$ and $\Gamma_{u_r}(E)$ as subsets of $\mathbb{T}^d$, set, for $r \in \{1, \dots, L\}$,

$$N_r^i(E) = \left\{ n \in \mathbb{N}\colon\ e + n\vec{\alpha} \in \begin{array}{ll} \gamma_{u_r}(E) & \text{if } i = 1 \\ \Gamma_{u_r}(E) & \text{if } i = 2 \end{array} \right\},$$

$$N^i(E) = \bigcup_{r=1}^{L} N_r^i(E), \quad i = 1, 2.$$

Of course we take here e, the center of E, on $\mathbb{T}^d$, i.e. mod 1. The sets $N_r^1(E)$ are pairwise disjoint since the sets $\gamma_{u_r}(E)$ are. This does not hold for the sets $N_r^2(E)$ and $\Gamma_{u_r}(E)$. Eq. (5) and the well-distribution of the sequence $(n\vec{\alpha})_{n=0}^{\infty}$ imply

$$\begin{aligned} &\left|N_r^i(E) \cap \{0, 1, \dots, N-1\}\right| \\ &\quad = N \begin{array}{ll} \left|J_{u_r}(E)\right| \lambda^{d-1}\big(\phi_{u_r}(\sigma)\big) & (i = 1) \\ \left|J_{u_r}(E)\right| \lambda^{d-1}\big(\Phi_{u_r}(\sigma)\big) & (i = 2) \end{array} + o(N). \end{aligned} \tag{6}$$

Now we can define the hyperplanes yielding the partitions $\Pi^i(E, N)$, $i = 1, 2$, of E:

Every $n \in N^i(E)$ corresponds to some hyperplanes $H_{u,\mu} = H_{u_r,\mu}^{(n)} \subseteq \mathbb{E}^d$ for which there exists a face $F_r \subseteq H_{u_r,\lambda_r}$, $r \in \{1, \dots, L\}$, such that

$$\big((H_{u,\mu} \cap E) + n\vec{\alpha}\big) \subseteq F_r^{\sigma'} \tag{7}$$

holds mod 1, i.e. on $\mathbb{T}^d$. Such a face F_r exists whenever $n \in N_r^i(E)$. Observe that $\begin{array}{c} n\in N^1(E) \\ n\in N^2(E) \end{array}$ implies that (7) holds for $\begin{array}{c} \text{exactly one} \\ \text{one or more} \end{array}$ $r \in \{1, \dots, L\}$. Related to such a hyperplane $H_{u_r,\mu}^{(n)}$ are the induced halfspaces $(H_{u_r,\mu}^{(n)})^{\pm}$. For obvious reasons we define $(H_{u_r,\mu}^{(n)})^{+}$ to be closed if, in $\mathbb{E}^d$, $H_{u_r,\lambda_r}^{+} \cap B_\varepsilon(x) \cap \begin{array}{c} P \\ \mathbb{T}^d \setminus P \end{array}$ is nonempty for every $\varepsilon > 0$ and every $x \in F_r \subseteq H_{u_r,\lambda_r}$ and $F_r \subseteq \begin{array}{c} P \\ [0,1)^d \setminus P \end{array}$ and open otherwise. However, to avoid cumbersome notation we simply write $(H_{u_r,\mu}^{(n)})^{\pm}$ for the induced halfspaces and tacitly assume that the one satisfying the just mentioned condition defining closedness is indeed closed.

Let $\prec$ be the lexicographical order on $\mathbb{N} \times \{1, \dots, L\}$, i.e. $(n, r) \prec (n', r')$ iff $n < n'$ or $n = n'$ and $r < r'$. Then, $\mathcal{H}^i(E) = \{H_1, H_2, \dots\}$, $i = 1, 2$, denotes the set of all hyperplanes $H_{u_r,\mu}^{(n)} \subseteq \mathbb{E}^d$, $n \in N^i(E)$, for which (7) holds and that are enumerated increasingly w.r.t. $\prec$, i.e., $H_{u_r,\mu}^{(n)} \prec H_{u_{r'},\mu'}^{(n')}$ if $(n, r) \prec (n', r')$. This enumeration makes sense since the σ'-a.i. guarantees that all elements of $\mathcal{H}^i(E)$ are distinct.

For $i = 1, 2$, $\mathcal{H}^i(E, N) = \{H_1, H_2, \dots, H_{J^i}\} \subseteq \mathcal{H}^i(E)$ is the set of all hyperplanes $H_{u,\mu}^{(n)}$ in $\mathcal{H}^i(E)$ such that $n \in \{0, \dots, N-1\}$. For $W \subseteq \{u_1, \dots, u_L\} \subseteq \mathbb{S}^{d-1}$, $\mathcal{H}^i(E, N, W) = \{H_{j_1}, \dots, H_{j_s}\} \subseteq \mathcal{H}^i(E, N)$, $j_1, \dots, j_s$ suitable, denotes the set of all $H_{u,\mu}$ in $\mathcal{H}^i(E, N)$ such that $u \in W$.

Eq. (7) and the fact that all the elements of $\mathcal{H}(E)$ are distinct imply

$$\left|\mathcal{H}^i\big(E, N, \{u_r\}\big)\right| = \left|N_r^i(E) \cap \{0, 1, \dots, N-1\}\right|. \tag{8}$$

The elements of $\mathcal{H}^i(E,N)$ are defined in such a way that the induced partitions $\Pi^i(E,N)$ (the partition cells are given by $\bigcap_{j=1}^{J^i} H_j^{\pm}$, $J^i = |\mathcal{H}^i(E,N)|$) can be used to estimate the local complexity $\mathcal{P}(E,N)$. Since sufficient for our needs we formulate the next lemma only for the special case where $E = C(\sigma, x)$ is a cube.

Lemma 11. *Let $C = C(\sigma, x) \subseteq \mathbb{T}^d$ where $\sigma > 0$ is small enough such that $\xi(P,\sigma) < \sigma'/2$. Then*

$$\left|\Pi^1(C,N)\right| \leqslant \mathcal{P}(C,N) \leqslant \left|\Pi^2(C,N)\right|.$$

Proof. Recall that $\mathcal{P}(C,N)$ is the cardinality of the sets $P_w(C)$, $w \in \{0,1\}^N$ (Section 2.2). The sets $P_w(C)$ are polytopes (not necessarily convex) whose boundary is given by intersections of translates of C by $n\vec{\alpha}$ on $\mathbb{T}^d$, $n = 0,\dots,N-1$, with ∂P. The sets in $\Pi^2(C,N)$ ($\Pi^1(C,N)$) are, according to (7), induced by (some) intersections of translates of C by $n\vec{\alpha}$ on $\mathbb{T}^d$, $n = 0,\dots,N-1$, with $\partial P^{\sigma'}$ (∂P). Also note that $N^1(C) \subseteq \{n \in \mathbb{N}\colon (C + n\vec{\alpha}) \cap \partial P \neq \emptyset\} \subseteq N^2(C)$.

So, if $\pi \in \Pi^1(C,N)$, i.e., $\pi = \bigcap_{j=1}^{J} H_j^{\pm}$, $J = |\mathcal{H}^1(C,N)|$, $H_j \in \mathcal{H}^1(C,N)$, then $\pi = \bigcap_{j\in K}\left(\genfrac{}{}{0pt}{}{P}{\mathbb{T}^d\setminus P} - i_j\vec{\alpha}\right)$, $K \subseteq \mathbb{N}$ and $i_j \in \{0,1,\dots,N-1\}$ suitable. Therefore every $\pi \in \Pi^1(C,N)$ coincides with a (union of) set(s) $P_w(C)$, $w \in \{0,1\}^N$ suitable, proving the first inequality. The same argument guarantees that every $P_w(C)$, $w \in \{0,1\}^N$ is the union of $\pi \in \Pi^2(C,N)$. $\square$

In the sequel we interpret, for $i = 1,2$, the hyperplanes in $\mathcal{H}^i(E,N)$ again as subsets of $\mathbb{E}^d$. Motivated by Lemma 11, we focus on the estimate of $|\Pi^i(E,N)|$. Lemma 10 together with the σ'-a.i. of P and $\vec{\alpha}$ allow to establish a connection between $|\Pi^i(E,N)|$ and the number of so-called intersecting d-tuples, defined as follows.

Let $W \subseteq \{u_1,\dots,u_L\}$ and $\mathcal{H}^i(E,N) = \{H_1,\dots,H_{J^i}\}$, increasingly ordered w.r.t. $\prec$. Then we define:

$$V^i(E,N) = \left\{ (j_1,\dots,j_d)\colon \begin{array}{l} 1 \leqslant j_1 < j_2 < \cdots < j_d \leqslant J^i, \\ \exists x \in E \text{ with } \{x\} = \bigcap_{k=1}^d H_{j_k} \\ \text{and } H_{j_k} \in \mathcal{H}^i(E,N) \end{array} \right\},$$

$$V^i(E,N,W) = \left\{ (j_1,\dots,j_d)\colon \begin{array}{l} 1 \leqslant j_1 < j_2 < \cdots < j_d \leqslant J^i, \\ \exists x \in E \text{ with } \{x\} = \bigcap_{k=1}^d H_{j_k} \\ \text{and } H_{j_k} \in \mathcal{H}^i(E,N,W) \end{array} \right\},$$

$$V_0^i(E,N) = \left\{ (j_1,\dots,j_d) \in V^i(E,N)\colon \begin{array}{l} \bigcap_{k=1}^d H_{j_k} \nsubseteq H_{j'} \\ \text{for all } H_{j'} \in \mathcal{H}^i(E,N)\setminus \\ \{H_{j_1},\dots,H_{j_d}\} \end{array} \right\}.$$

$V^i(E,N)$ corresponds to the set of intersection points of d hyperplanes of $\mathcal{H}^i(E,N)$ in general position. Thus, we call its elements intersecting d-tuples. Accordingly, $V_0^i(E,N)$ represents the set of all points in E contained in exactly d hyperplanes of $\mathcal{H}^i(E,N)$ in general position. It is natural to call its elements uniquely intersecting d-tuples. Obviously $V_0^i(E,N) \subseteq V^i(E,N)$. Additionally note that

$$V^i(E,N) = \bigcup_{\substack{W=\{w_1,\dots,w_d\}\\ \subseteq\{u_1,\dots,u_L\}}} V^i(E,N,W), \quad i=1,2, \tag{9}$$

where all the occurring sets $V^i(E,N,W)$ are pairwise disjoint.

4.4. Consequences of Lemma 10 and the σ'-asymptotic independence

We still refer to the notation fixed in Section 4.1. Let $\mathcal{H}^i(C,N)$, $V^i(C,N)$ and $V_0^i(C,N)$, $i=1,2$, be as in Section 4.3 where $C=C(\sigma,x)$ is a cube with side length σ small enough such that $\xi(P,\sigma) < \sigma'/2$.

We want to elaborate how the σ'-a.i. and Lemma 10 imply that the number of partition sets in C induced by the elements of $\mathcal{H}^i(C,N)$ can be estimated by the number of elements of $V^i(C,N)$, $i=1,2$.

In the present section we omit the superscript i whenever we do not need to distinguish between the cases $i=1$ and $i=2$. Assume C is partitioned by the halfspaces induced by the elements of $\mathcal{H}(C,N)$. As in Section 4.2, let, for $d' \in \{1,\dots,d\}$, $\Pi_{d'}(C,N)$ ($\Pi_{d'}(C,N)^o$) denote the set of all d'-dimensional (inner) partition cells in $\Pi(C,N)$. Instead of $w(x,|\mathcal{H}(C,N)|,C)$ we write abbreviating $w(x,N,C)$ for the weight of a point x in C as defined in Section 4.2 and $G(C,N)=\{x\in C\colon w(x,N,C)\geqslant 1\}$.

Firstly we show that, due to the σ'-a.i., $V(C,N)\setminus V_0(C,N)$ is small. For this reason we count the over-determined vertices in $\mathbb{T}^d$ after an N-fold translation of $\partial P^{\sigma'}$ by $\vec\alpha$ if P and $\vec\alpha$ are σ'-a.i.

Let $U(N)=U(N,P,\vec\alpha)$ be the set of vertices in $\mathbb{T}^d$ after an N-fold translation of $\partial P^{\sigma'}$ by $\vec\alpha$, i.e. an $x\in\mathbb{T}^d$ is in $U(N)$ if there exist faces $F_{r_1},\dots,F_{r_d}$ of P in general position and numbers $n_1,\dots,n_d\in\{0,\dots,N-1\}$ such that $x+n_j\vec\alpha\in F_{r_j}^{\sigma'}$ for all $j\in\{1,\dots,d\}$. Let, for $x\in\mathbb{T}^d$ and $N\in\mathbb{N}$,

$$I(x,N)=\bigl\{(n,r)\in\{0,\dots,N-1\}\times\{1,\dots,L\}\colon x+n\vec\alpha\in F_r^{\sigma'}\bigr\}.$$

Lemma 12. *Let P and $\vec\alpha$ be σ'-a.i. Then*

$$\bigl|\bigl\{x\in U(N)\colon \bigl|I(x,N)\bigr|>d\bigr\}\bigr| = o\bigl(N^d\bigr).$$

Proof. For $x\in U(N)$, let $n(x)=\min\{n\in\mathbb{N}\colon \exists r\in\{1,\dots,L\}$ such that $(n,r)\in I(x,N)\}$. For every $x\in U(N)$ there is a unique $x_0\in\partial P^{\sigma'}$ such that $x+n(x)\vec\alpha=x_0$. Clearly $|I(x,N)|\leqslant |I(x_0,N)|$. By the σ'-a.i., on every enlarged face $F_i^{\sigma'}$ of $\partial P^{\sigma'}$ lie at most $o(N^{d-1})$ such points $x_0\in U(N)$ fulfilling $|I(x_0,N)|>d$. Hence, also on $\partial P^{\sigma'}$ lie at most $o(N^{d-1})$ elements of $U(N)$ satisfying $|I(x_0,N)|>d$. Therefore, for every $n'\in\{0,1,\dots,N-1\}$, there are $o(N^{d-1})$ points x such that $n(x)=n'$, $x\in U(N)$ and $|I(x,N)|>d$. Thus, there are at most $No(N^{d-1})$ points $x\in U(N)$ for which $|I(x,N)|>d$. $\square$

Lemma 13. *Let P and $\vec\alpha$ be σ'-a.i. Then*

$$\bigl|V(C,N)\bigr| = \bigl|V_0(C,N)\bigr| + o\bigl(N^d\bigr).$$

Proof. The sets $U(N)$, $V^2(C,N)$ and $V^1(C,N)$ correspond, according to their definitions, to $F_r^{\sigma'}$, $\Phi_{u_r}(\sigma)$ and $\phi_{u_r}(\sigma)$ and $F_r^{\sigma'} \supseteq \Phi_{u_r}(\sigma) \supseteq \phi_{u_r}(\sigma)$. Hence, $\{x \in U(N) \cap C\colon |I(x,N)| > d\} \supseteq \{x \in C\colon \exists (j_1,\dots,j_d) \in V^2(C,N) \setminus V_0^2(C,N)$ such that $\{x\} = \bigcap_{k=1}^d H_{j_k},\ H_{j_k} \in \mathcal{H}^2(C,N)\} \supseteq \{x \in C\colon \exists (j_1,\dots,j_d) \in V^1(C,N) \setminus V_0^1(C,N)$ such that $\{x\} = \bigcap_{k=1}^d H_{j_k},\ H_{j_k} \in \mathcal{H}^1(C,N)\}$. Thus, by Lemma 12, the cardinality of all three sets is at most $o(N^d)$. Moreover, the σ'-a.i. implies that each x is contained in at most L hyperplanes, where L is the number of faces of our polytope P (cf. condition (4) of Definition 2). Hence, every $x \in U(N)$ with $|I(x,N)| > d$ corresponds to at most $\binom{L}{d}$ d-tuples which are elements of $V(C,N) \setminus V_0(C,N)$. □

The next lemma connects $|V_0(C,N)|$ and the weight function.

Lemma 14. *Let P and $\vec{\alpha}$ be σ'-a.i. Let C^o denote the interior of C. Then*

$$\bigl|V_0\bigl(C^o,N\bigr)\bigr| = \sum_{x \in G(C,N)} w(x,N,C) + o\bigl(N^d\bigr).$$

Proof. By Lemma 8, an $x \in C$ is an element of $G(C,N)$ if and only if there exists a d-tuple $(j_1,\dots,j_d) \in V(C,N)$ such that $\{x\} = \bigcap_{i=1}^d H_{j_i} \cap C^o$, $H_{j_i} \in \mathcal{H}(C,N)$. As remarked subsequent to Eq. (1), the weight of a point of the boundary of C is always 0. Thus we only take points in C^o in consideration. Let $G_0(C,N)$ be the set of all vertices in C^o determined by uniquely intersecting d-tuples, i.e., $x \in G_0(C,N)$ if there exists a $(j_1,\dots,j_d) \in V_0(C,N)$ such that $\{x\} = \bigcap_{i=1}^d H_{j_i} \cap C^o$, $H_{j_i} \in \mathcal{H}(C,N)$. Note that every $x \in G_0(C,N)$ corresponds to exactly one uniquely intersecting d-tuple and vice versa. By Lemma 8, $w(x,N,C) = 1$ if $x \in G_0(C,N)$. For all $N \in \mathbb{N}$, by the σ'-a.i., every x in C is also an element of at most L different hyperplanes in $\mathcal{H}(C,N)$. Again L is the number of faces of our polytope P. By Lemma 9, Eq. (2), $w(x,N,C) \leqslant u(L,d)$. Hence,

$$\sum_{x \in G(C,N)} w(x,N,C) = \underbrace{\sum_{x \in G_0(C,N)} \underbrace{w(x,N,C)}_{=1}}_{=|V_0(C^o,N)|} + \underbrace{\sum_{\substack{x \in G(C,N)\\ \setminus G_0(C,N)}} \underbrace{w(x,N,C)}_{\in\{1,\dots,u(L,d)\}}}_{o(N^d)\text{ terms}}. \qquad \square$$

Lemma 16 uses the following local version of Lemma 3.

Lemma 15. *For every $\varepsilon > 0$ there exists an $N(\varepsilon) = N(\varepsilon, C, \vec{\alpha}) \in \mathbb{N}$ such that* $\operatorname{diam}(\pi) < \varepsilon$ *for every $\pi \in \Pi(C,N)$ whenever $N \geqslant N(\varepsilon)$.*

Proof. We use the geometric background of the elements of $\mathcal{H}^1(C,N)$ to find such an $N(\varepsilon)$. Since $\mathcal{H}^2(C,N) \supseteq \mathcal{H}^1(C,N)$ this $N(\varepsilon)$ works for both partitions $\Pi^i(C,N)$, $i = 1,2$.

We refer to the sets defined in Section 4.3. Pick d faces $F_{r_j} \subseteq H_{u_{r_j},\lambda_{r_j}}$, $r_j \in \{1,\dots,L\}$, $j = 1,\dots,d$, of the original polytope P in general position. By condition (2) of Definition 2, the corresponding sets $\gamma_{u_{r_j}}(C)$ have a positive measure. Choose $\delta > 0$ sufficiently small such that $\operatorname{diam}(D) \leqslant \varepsilon$, where D is the parallelepiped spanned by the vectors δu_{r_j}. Divide the intervals

$J_{u_{r_j}}(C)$ in K_j subintervals $J^l_{u_{r_j}}(C), l = 1, \dots, K_j$, of length at most δ, $K_j \in \mathbb{N}$ sufficiently large. Similar to Section 4.3 set

$$\gamma^l_{u_{r_j}}(C) = \{z \in [0,1)^d\colon\ z = y + \lambda u_{r_j},\ y \in \phi_{u_{r_j}}(\sigma),\ \lambda \in J^l_{u_{r_j}}(C)\}.$$

These slices of $\gamma_{u_{r_j}}(C)$ yield the sets $N^l_{r_j}(C) = \{n \in \mathbb{N}\colon\ x + n\vec{\alpha} \in \gamma^l_{u_{r_j}}(C)\}$, where x, the center of C, is seen as an element of $\mathbb{T}^d$. Set $n^l_{r_j}(C) = \min(N^l_{r_j}(C))$ and $N(\varepsilon) = N(\varepsilon, C, \vec{\alpha}) = \max\{n^l_{r_j}(C)\colon\ l = 1, \dots, K_j,\ j = 1, \dots, d\}$, i.e., every set $N^l_{r_j}(C)$ contains (at least) one $n \leqslant N(\varepsilon)$, $l = 1, \dots, K_j$, $j = 1, \dots, d$. Such an $N(\varepsilon)$ exists due to the well-distribution of the sequence $(n\vec{\alpha})^\infty_{n=0}$. Two numbers $n^l_{r_j}(C)$ and $n^{l+1}_{r_j}(C)$, $l = 1, \dots, K_j - 1$, induce two parallel hyperplanes in $H^1(C, N, \{u_{r_j}\})$ whose distance is $\leqslant \delta$. This observation and the choice of δ guarantee that the elements of $\Pi(C, N)$ have a diameter less than ε whenever $N \geqslant N(\varepsilon)$. $\square$

Let $\varepsilon > 0$ with $\xi(P, \sigma + \varepsilon) < \sigma'/2$. Replacing $C = C(\sigma, x)$ by $C^\varepsilon = C(\sigma + \varepsilon, x)$ we can also define the sets $\mathcal{H}^i(C^\varepsilon, N)$, $i = 1, 2$, the corresponding induced partitions of C^ε and the corresponding weight function.

Lemma 16. *Let $\varepsilon > 0$. Then, for every $N \geqslant N(\varepsilon, C^\varepsilon, \vec{\alpha})$*

$$\sum_{x \in G(C,N)} w(x, N, C) \leqslant |\Pi_d(C, N)| \leqslant \sum_{x \in G(C^\varepsilon, N)} w(x, N, C^\varepsilon).$$

Proof. The asserted inequality is a direct consequence of Lemma 10 and the inequality $|\Pi_d(C, N)| \leqslant |\Pi_d(C^\varepsilon, N)^o|$ whenever $N \geqslant N(\varepsilon, C^\varepsilon, \vec{\alpha})$ due to Lemma 15. $\square$

As the subsequent lemma shows, the σ'-a.i. implies that, for asymptotic estimates, it suffices to count only the d-dimensional inner partition sets.

Lemma 17. *Let P and $\vec{\alpha}$ be σ'-a.i. Then*

$$|\Pi(C, N)^o| = |\Pi_d(C, N)^o| + o(N^d).$$

Proof. We show that the σ'-a.i. implies $|\Pi_{d'}(C, N)^o| = o(N^d)$, for all $d' \in \{0, \dots, d-1\}$. Let $d'' = d - d'$.

At first we claim that a set R in $\Pi_{d'}(C, N)^o$ is necessarily contained in $k > d''$ hyperplanes $H_{r_j} \in \mathcal{H}(C, N)$, $j = 1, \dots, k$.

Since R is an inner partition cell it is a convex polytope whose boundary lies in hyperplanes of $\mathcal{H}(C, N)$, i.e. there are numbers $r_1 < \cdots < r_K < r \in \mathbb{N}$ such that $R_1 = \bigcap^J_{j=1} H^-_{r_j} \cap \bigcap^K_{j=J+1} H^+_{r_j} \cap C$ is $d_1 > d'$-dimensional, $R_1 \cap H^{+/-}_r = R'$ and R' is d'-dimensional in C with $R' \supseteq R$, H_{r_j} and $H_r \in \mathcal{H}(C, N)$. But this immediately implies that R' must be a d'-dimensional face of R_1 contained in H_r. So, R (being an inner partition cell) and hence also R' must be contained in at least $d'' + 1$ hyperplanes $H_{r_{j_1}}, \dots, H_{r_{j_{d''}}}, H_r \in \mathcal{H}(C, N)$.

Hence all 0-dimensional inner partition sets are intersection points of at least $d + 1$ hyperplanes. By the σ'-a.i. and Lemma 12, there are $o(N^d)$ such over-determined intersection points in C. Moreover, condition (4) of the definition of the σ'-a.i. (Definition 2) implies that there are no $(d-1)$-dimensional inner partition cells. Thus, we can assume $d' \in \{1, \dots, d-2\}$.

We assign to each d'-dimensional inner partition set R the set $J(R) = \{r\colon\ R \subseteq H_r,\ H_r \in \mathcal{H}(C, N)\} \subseteq \mathbb{N}$ and call $\operatorname{lev}(R) = \min(J(R))$ the level of R.

Fix $l \in \mathbb{N}$. We estimate the number of d'-dimensional sets R in $\Pi(C, N)^o$ with $\operatorname{lev}(R) = l$ (lying in more than d'' hyperplanes of $\mathcal{H}(C, N)$). Related to each such set R is the d'-dimensional convex set $\operatorname{Cut}(R) = \bigcap_{r \in J(R)} H_r \cap C$ which is a subset of H_l. By Lemma 10, an upper bound for the number of inner d'-dimensional sets in $\operatorname{Cut}(R)$ is given by $\sum_{x \in G(\operatorname{Cut}(R), N)} w(x, \operatorname{Cut}(R), N)$. In other words, there is an injective mapping

$$\nu : T \mapsto \bigl(x(T), t\bigr), \quad x(T) \in G\bigl(\operatorname{Cut}(R), N\bigr),\ t \in \bigl\{1, \dots, u(L, d')\bigr\}.$$

ν assigns to each d'-dimensional inner partition cell $T \subseteq \operatorname{Cut}(R)$ a point $x(T) \in \operatorname{Cut}(R)$ with a positive weight. By Lemma 8, such an $x(T)$ is a vertex point in $\operatorname{Cut}(R)$ and hence also in C^o, i.e. $\{x(T)\} = \bigcap_{j \in S(x(T))} H_j$, $S(x(T)) = \{j\colon\ x(T) \in H_j,\ H_j \in \mathcal{H}(C, N)\}$. This works for all d'-dimensional sets $\operatorname{Cut}(R)$ with R in $\Pi_{d'}(C, N)^o$ of level l. Observe that for every fixed level l:

(a) All the assigned vertices $x(T)$ are contained in more than d hyperplanes since $\operatorname{Cut}(R)$ is already contained in more than d'' hyperplanes. Condition (3) of Definition 2 guarantees that there are at most $o(N^{d-1})$ such vertices in each $H_l \in \mathcal{H}(C, N)$.
(b) By condition (4) of Definition 2, such an assigned vertex $x(T)$ can be an element of at most $\binom{L-1}{d''}$ sets $\operatorname{Cut}(R)$ of dimension d'. (Once again L is the number of faces of our original polytope P.)

Combining (a) and (b) implies that there are at most $u(L, d')\binom{L-1}{d''} o(N^{d-1})$ sets in $\Pi'_d(C, N)^o$ of level l, for every $l \in \{0, \dots, N-1\}$. Hence, for any $d' < d$, $\Pi'_d(C, N)^o$ contains at most $o(N^d)$ elements. □

The (in-)equalities proved in Lemmata 13–17 imply the following corollary.

Corollary 18. *Let P and $\vec{\alpha}$ be σ'-a.i. Let $\sigma > 0$ and $\varepsilon > 0$ such that $\xi(P, \sigma + \varepsilon) < \sigma'/2$. Let $C = C(\sigma, x)$ and $C^\delta = C(\sigma + \delta, x)$, $\delta \in \mathbb{R}$. Use the notation introduced so far. Then*

$$\bigl|V\bigl(C^{-\varepsilon}, N\bigr)\bigr| + o\bigl(N^d\bigr) \leqslant \bigl|\Pi(C, N)\bigr| \leqslant \bigl|V\bigl(C^{\varepsilon}, N\bigr)\bigr| + o\bigl(N^d\bigr).$$

Proof. Let $N \geqslant N(\varepsilon/2, C^\varepsilon, \vec{\alpha})$. Then, writing $\simeq$ for equality up to $o(N^d)$,

$$\begin{aligned}
\bigl|V\bigl(C^{-\varepsilon}, N\bigr)\bigr| &\simeq \bigl|V_0\bigl(C^{-\varepsilon}, N\bigr)\bigr| && \text{(Lemma 13)}\\
&\leqslant \bigl|V_0\bigl(\bigl(C^{-\varepsilon/2}\bigr)^o, N\bigr)\bigr| && \text{(Lemma 15)}\\
&\simeq \textstyle\sum_{x \in G(C^{-\varepsilon/2}, N)} w\bigl(x, N, C^{-\varepsilon/2}\bigr) && \text{(Lemma 14)}\\
&\leqslant \bigl|\Pi_d\bigl(C^{-\varepsilon/2}, N\bigr)\bigr| && \text{(Lemma 16)}\\
&\leqslant \bigl|\Pi_d(C, N)^o\bigr| && \text{(Lemma 15)}\\
&\simeq \bigl|\Pi(C, N)^o\bigr| && \text{(Lemma 17)}\\
&\leqslant \bigl|\Pi(C, N)\bigr|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left|\Pi\left(C^{\varepsilon}, N\right)^{o}\right| && \text{(Lemma 15)}\\
&\simeq \left|\Pi_d\left(C^{\varepsilon}, N\right)^{o}\right| && \text{(Lemma 17)}\\
&\leqslant \textstyle\sum_{x\in G(C^{\varepsilon},N)} w\left(x, N, C^{\varepsilon}\right) && \text{(Lemma 10)}\\
&\simeq \left|V_0\left(\left(C^{\varepsilon}\right)^{o}, N\right)\right| && \text{(Lemma 14)}\\
&\leqslant \left|V_0\left(C^{\varepsilon}, N\right)\right| &&\\
&\simeq \left|V\left(C^{\varepsilon}, N\right)\right| && \text{(Lemma 13).} \qquad \square
\end{aligned}
$$

Corollary 18 together with Lemma 11 guarantee that, assuming σ'-a.i. of P and $\vec{\alpha}$, an asymptotic lower (upper) bound for the local complexity of a cube C is given by the number of intersecting d-tuples generated by the elements of $\mathcal{H}^1(C^{-\varepsilon}, N)$ in $C^{-\varepsilon}$ ($\mathcal{H}^2(C^{\varepsilon}, N)$ in C^{ε}). The goal of the next section is to compute estimates for this number.

4.5. Asymptotic growth rate of the number of intersecting d-tuples

In this section $W = \{w_1, \ldots, w_d\}$ denotes a d-element subset of $\{u_1, \ldots, u_L\} \subseteq \mathbb{S}^{d-1}$, the set of normal vectors of the faces of our polytope P. Let $C = C(\sigma, x) \subseteq \mathbb{T}^d$, $\sigma > 0$ sufficiently small such that $\xi(P, \sigma) < \sigma'/2$. Let $C_0 = [0, \sigma)^d$ be the translate of C rooted at 0 and $T_W(\sigma)$ the parallelepiped defined in Section 2.4.

Lemma 19. *Assume the elements of W are linearly independent. Then*

$$
\begin{aligned}
&\left|V^i\left(T_W(\sigma), N, W\right)\right|\\
&\quad= \begin{cases} N^d \prod_{j=1}^d \lambda^{d-1}(\phi_{w_j}(\sigma))|J_{w_j}(T_W(\sigma))| + o(N^d) & \text{if } i = 1,\\ N^d \prod_{j=1}^d \lambda^{d-1}(\Phi_{w_j}(\sigma))|J_{w_j}(T_W(\sigma))| + o(N^d) & \text{if } i = 2.\end{cases}
\end{aligned}
$$

Proof. For $i = 1, 2$, first observe that $\operatorname{diam}(T_W(\sigma)) \leqslant \xi(P, \sigma) < \sigma'/2$ implies that the sets $V^i(T_W(\sigma), N, W)$ can be defined as in Section 4.3. The elements of W are the normal vectors of $T_W(\sigma)$. Thus $|V^i((T_W(\sigma), N, W)| = \prod_{j=1}^d |\mathcal{H}^i(T_W(\sigma), N, \{w_j\})|$. Recall that, by Eqs. (6) and (8),

$$
\begin{aligned}
&\left|\mathcal{H}^i\left(T_W(\sigma), N, \{w_j\}\right)\right|\\
&\quad= \begin{cases} N\lambda^{d-1}(\phi_{w_j}(\sigma))|J_{w_j}(T_W(\sigma))| + o(N) & \text{if } i = 1,\\ N\lambda^{d-1}(\Phi_{w_j}(\sigma))|J_{w_j}(T_W(\sigma))| + o(N) & \text{if } i = 2.\end{cases} \qquad \square
\end{aligned}
$$

Lemma 20. *Assume the elements of W are linearly independent. Take any $\zeta > 0$ sufficiently small such that $\xi(P, \zeta + \sigma) < \sigma'/2$. Then*

$$
\begin{aligned}
&\left|V^1(C, N, W)\right| + o\left(N^d\right) \geqslant \left|V^1\left(T_W(\sigma - \zeta), N, W\right)\right| \quad \textit{and}\\
&\left|V^2(C, N, W)\right| \leqslant \left|V^2\left(T_W(\sigma + \zeta), N, W\right)\right| + o\left(N^d\right).
\end{aligned}
$$

Proof. C and $T_W(C)$, both Jordan measurable, can be approximated arbitrarily well (w.r.t. λ^d) by the union of disjoint copies of the small parallelepiped $T_W(\zeta)$, $\zeta > 0$. Due to the well-distribution of the sequence $(k\alpha)_{k=0}^{\infty}$ the cardinality of $V^i(T_W(\zeta) + x, N, W)$ is asymptotically independent of $x \in \mathbb{T}^d$, i.e., $|V^i(T_W(\zeta)+x,N,W)| - |V^i(T_W(\zeta)+y,N,W)| = o(N^d)$ for all $x, y \in \mathbb{T}^d$. Since $\lambda^d(C) = \lambda^d(T_W(C))$ the assertion follows. $\square$

Remark 21. Obviously $V^i(C,N,W)$ is empty if the elements of W do not span $\mathbb{E}^d$.

We conclude this subsection with the subsequent formula.

Lemma 22. *Let $W = \{w_1, \ldots, w_d\} \subseteq \{u_1, \ldots, u_L\}$. Then*

$$\prod_{j=1}^{d} \left| J_{w_j}\big(T_W(\sigma)\big)\right| = \sigma^d \left|\mathrm{Det}(w_1, \ldots, w_d)\right|.$$

Proof. According to Section 2.4, $|J_{w_j}(T_W(\sigma))| = \sigma |\mathrm{Det}(w_1, \ldots, w_d)|^{1/d}$. $\square$

4.6. *Finalizing the proof of Proposition 6*

Let, according to the assumptions of Proposition 6, P be a polytope in $\mathbb{T}^d$ with faces F_r and normal vectors u_r, $r = 1, \ldots, L$. Let $\vec{\alpha} \in \mathbb{T}^d$ be strongly irrational and $\sigma' > 0$ sufficiently small such that P and $\vec{\alpha}$ are σ'-a.i. Let $C = C(\sigma, x)$ be an arbitrary cube with side length $\sigma > 0$ and center x. Let σ be small enough such that $\xi(P, \sigma) < \sigma'/2$.

Let $W = \{w_1, \ldots, w_d\} \subseteq \{u_1, \ldots, u_L\}$. Combining Lemmata 19 and 22 yields, for $i = 1, 2$,

$$\begin{aligned}
&\left|V^i\big(T_W(\sigma), N, W\big)\right| \\
&\quad = \sigma^d N^d \left|\mathrm{Det}(w_1, \ldots, w_d)\right| \prod_{j=1}^{d} \begin{matrix} \lambda^{d-1}\big(\phi_{w_j}(\sigma)\big) & (i=1) \\ \lambda^{d-1}\big(\Phi_{w_j}(\sigma)\big) & (i=2) \end{matrix} + o\big(N^d\big).
\end{aligned}$$

$|V^i(T_W(\sigma), N, W)| = 0$ if the elements of W are linearly dependent. Lemma 11, Eq. (9), Corollary 18 and the results gathered in Section 4.5 imply, for every ε and $\zeta > 0$ sufficiently small such that $\rho = \sigma - (\varepsilon + \zeta) > 0$,

$$\begin{aligned}
\mathcal{P}(C,N) &\geqslant \left|\Pi^1(C,N)\right| \geqslant \left|V^1\big(C^{-\varepsilon}, N\big)\right| + o\big(N^d\big) \\
&= \sum_{\substack{W=\{w_1,\ldots,w_d\} \\ \subseteq\{u_1,\ldots,u_L\}}} \left|V^1\big(C^{-\varepsilon}, N, W\big)\right| + o\big(N^d\big) \\
&\geqslant \sum_{\substack{W=\{w_1,\ldots,w_d\} \\ \subseteq\{u_1,\ldots,u_L\}}} \left|V^1\big(T_W\big(\sigma-(\varepsilon+\zeta)\big), N, W\big)\right| + o\big(N^d\big) \\
&= \sum_{\substack{W=\{w_1,\ldots,w_d\} \\ \subseteq\{u_1,\ldots,u_L\}}} \rho^d N^d \left|\mathrm{Det}(w_1, \ldots, w_d)\right| \prod_{j=1}^{d} \lambda^{d-1}\big(\phi_{w_j}(\rho)\big) + o\big(N^d\big) \\
&= \frac{N^d \rho^d}{d!} \left(\sum_{r_1=1}^{L} \cdots \sum_{r_d=1}^{L} \left(\left|\mathrm{Det}(u_{r_1}, \ldots, u_{r_d})\right| \prod_{j=1}^{d} \lambda^{d-1}\big(\phi_{u_{r_j}}(\rho)\big) \right)\right) + o\big(N^d\big)
\end{aligned}$$

and analogously, for any ζ and ε sufficiently small such that, for $\rho' = \sigma + \varepsilon + \zeta$, $\xi(P, \rho') < \sigma'/2$,

$$\mathcal{P}(C, N) \leqslant \frac{N^d \rho'^d}{d!} \left(\sum_{r_1=1}^{L} \cdots \sum_{r_d=1}^{L} \left(\left|\mathrm{Det}(u_{r_1}, \ldots, u_{r_d})\right| \prod_{j=1}^{d} \lambda^{d-1}\big(\Phi_{u_{r_j}}(\rho')\big) \right) \right) + o\big(N^d\big).$$

Since, for any small $\varepsilon, \zeta > 0$, the last two inequalities hold whenever $N \in \mathbb{N}$ is sufficiently large and $\lambda^{d-1}(\Phi_{u_r}(\rho')) \leqslant \lambda^{d-1}(F_r^{+2\xi(P,\sigma)})$ as well as $\lambda^{d-1}(\phi_{u_r}(\rho)) \geqslant \lambda^{d-1}(F_r^{-2\xi(P,\sigma)})$ for all $r \in \{1, \ldots, L\}$ we are done.

5. The asymptotic independence of P and $\vec{\alpha}$

Let, still, P be a polytope in $[0, 1)^d$ with L faces $F_r \subseteq H_{u_r,\lambda_r}$, $r = 1, \ldots, L$. Let $\sigma' > 0$ be sufficiently small such that $P \tilde{+} B_{\sigma'} \subseteq [0, 1)^d$ and $F_r^{-\sigma'}$ is $(d-1)$-dimensional. Let $\vec{\alpha} \in \mathbb{T}^d$ be strongly irrational.

How natural is our Definition 2? Recall from Section 2.3 the two conditions of the σ'-a.i. involving P and $\vec{\alpha}$:

(1) For all $r \in \{1, \ldots, L\}$, the number $|Q_r(N, \sigma')|$ of over-determined vertices on the face $F_r^{\sigma'}$ induced by the N-fold translation of $\partial P^{\sigma'}$ by $\vec{\alpha}$ is of size $o(N^{d-1})$.
(2) There exists no $n \in \mathbb{N} \setminus \{0\}$ such that $(\partial P^{\sigma'} - n\vec{\alpha}) \cap \partial P^{\sigma'}$ contains a $(d-1)$-dimensional set.

We can show the following.

Proposition 23. *Let P be a polytope in $\mathbb{T}^d$ with L faces F_r and normal vectors u_r, $r = 1, \ldots, L$, and σ' sufficiently small such that $P \tilde{+} B_{\sigma'} \subseteq \big[0, 1)^d$ and every $F_r^{-\sigma'}$ is a $(d-1)$-dimensional set. Then there exists a meager zero set $Z \subseteq \mathbb{T}^d$ such that $\vec{\alpha} \in \mathbb{T}^d \setminus Z$ implies that $\vec{\alpha}$ is strongly irrational and P and $\vec{\alpha}$ are σ'-asymptotically independent.*

Proof. It is an easy and well-known fact that the set of nonstrongly irrational $\vec{\alpha} \in \mathbb{T}^d$ is a meager zero set.

Let us prove that condition (1) typically holds: Fix a face $F_r^{\sigma'}$, $r \in \{1, \ldots, L\}$, and assume that there exist $x \neq y$ in $F_r^{\sigma'}$, integers $n_j(x), n_j(y) \in \{0, \ldots, N-1\}$ and faces F_{r_j}, $r_j \in \{1, \ldots, L\} \setminus \{r\}$, $j = 1, \ldots, d$, in general position such that $x + n_j(x)\vec{\alpha} \in F_{r_j}^{\sigma'}$ and $y + n_j(y)\vec{\alpha} \in F_{r_j}^{\sigma'}$, for all $j = 1, \ldots, d$. Interpreting P as a polytope in $\mathbb{E}^d$ with faces $F_r \subseteq H_{u_r,\lambda_r}$, $r = 1, \ldots, L$, this translates to $x + n_j(x)\vec{\alpha} - m_j(x) \in H_{u_{r_j},\lambda_j}$, or, equivalently, $x \in H_{u_{r_j},\lambda_j - (n_j(x)\vec{\alpha} - m_j(x)) \cdot u_{r_j}}$, $j = 1, \ldots, d$, $m_j(x) \in \mathbb{Z}^d$ suitable. Analogously, $y \in H_{u_{r_j},\lambda_j - (n_j(y)\vec{\alpha} - m_j(y)) \cdot u_{r_j}}$, for all $j = 1, \ldots, d$ and suitable $m_j(y) \in \mathbb{Z}^d$. Hence, setting $z = x - y$, $\{z\} = \bigcap_{j=1}^d H_{u_{r_j},\nu_j}$, where $\nu_j = (n_j(x) - n_j(y))\vec{\alpha} - (m_j(x) - m_j(y)) \cdot u_{r_j}$. Observe that $z \neq 0$ implies that at least one of the involved hyperplanes $H_{u_{r_j},\nu_j}$ does not contain 0. Summing up, if two distinct points x and y lie on the same enlarged face $F_r^{\sigma'}$ and are intersection points of the same d-tuple of translates by $\vec{\alpha}$ multiples of faces F_{r_j}, $j = 1, \ldots, d$, then there exist $n_j \in \mathbb{Z}$ and $m_j \in \mathbb{Z}^d$ (depending on $\vec{\alpha}$) such that $\bigcap_{j=0}^d H_{u_{r_j},(n_j\vec{\alpha} - m_j) \cdot u_{r_j}} \in H_{u_r,0} \setminus \{0\}$.

Motivated by this, we define, for linearly independent u_{r_j}, $r_j \in \{1, \dots, L\}$, integers $n_j \in \mathbb{Z}$ and $m_j \in \mathbb{Z}^d$, $j = 1, \dots, d$, the set

$$\begin{aligned} A &= A_r\big((u_{r_j}, n_j, m_j)_{j=1}^d\big) \\ &= \left\{ \vec{\alpha} \in [0,1)^d \colon \bigcap_{j=0}^{d} H_{u_{r_j}, (n_j\vec{\alpha} - m_j)\cdot u_{r_j}} \subseteq H_{u_r,0} \setminus \{0\} \right\}. \end{aligned}$$

It is easily checked that A is either empty or contained in a hyperplane in $\mathbb{E}^d$ intersecting $[0,1)^d$. Hence, it is a set of measure 0 (w.r.t. λ^d) that does not have inner points. So, the sets

$$A(r) = \bigcup_{\substack{u_{r_1},\dots,u_{r_d} \\ \text{linearly independent}}} \bigcup_{n_1,\dots,n_d \in \mathbb{Z}} \bigcup_{m_1,\dots,m_d \in \mathbb{Z}^d} A_r\big((u_{r_j}, n_j, m_j)_{j=1}^d\big)$$

are meager zero sets in $[0,1)^d$ for all $r \in \{1, \dots, L\}$.

If $\vec{\alpha}$ in $[0,1)^d \setminus A(r)$ then, for faces $F_{r_1}, \dots, F_{r_d}$, $r_j \in \{1, \dots, L\} \setminus \{r\}$, with linearly independent u_{r_j}, there exists at most one d-tuple $(n_1, \dots, n_d) \in \mathbb{N}^d$ such that

$$\bigcap_{j=1}^{d} \big(F_{r_j}^{\sigma'} - n_j\vec{\alpha}\big) \subseteq F_r^{\sigma'}.$$

Hence, for every $r \in \{1, \dots, L\}$ the set $F_r^{\sigma'}$ contains at most one intersection point $\bigcap_{j=1}^d (F_{r_j}^{\sigma'} - n_j\vec{\alpha})$ for every choice of faces F_{r_j} with linearly independent u_{r_j}, $j = 1, \dots, d$. Therefore for almost all $\vec{\alpha}$ there are altogether at most $\binom{L-1}{d}$ intersection points in $F_r^{\sigma'}$ proving that condition (1) typically holds.

Likewise we show that the set of $\vec{\alpha} \in \mathbb{T}^d$ for which condition (2) fails is small:

For $n \in \mathbb{N} \setminus \{0\}$, the set $(\partial P^{\sigma'} - n\vec{\alpha}) \cap \partial P^{\sigma'}$ contains a $(d-1)$-dimensional set only if there are two faces F_r and F_s of P, $1, \leqslant r, s \leqslant L$, such that $(F_r^{\sigma'} - n\vec{\alpha}) \cap F_s^{\sigma'}$ contains a $(d-1)$-dimensional set. Again we interpret P as polytope in $\mathbb{E}^d$. Then, for $F_r \subseteq H_{u_r,\lambda_r}$ and $F_s \subseteq H_{u_s,\lambda_s}$, such a $(d-1)$-dimensional set can only occur if $u_r = \pm u_s$ and if for the normal distance δ of these hyperplanes holds

$$\big|(n\vec{\alpha} - k) \cdot u_r\big| = \delta,$$

for suitable $n \in \mathbb{N} \setminus \{0\}$ and $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$. Let $k \in \mathbb{Z}^d$, $n \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{R}$ and $u \in \mathbb{S}^{d-1}$. Then

$$B(u, \delta, k, n) = \big\{ \vec{\alpha} \in [0,1)^d \colon (n\vec{\alpha} - k) \cdot u = \delta \big\}$$

is either empty of defines a part of a hyperplane. Thus

$$B(u, \delta) = \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{Z}^d} B(u, \delta, k, n)$$

is a meager zero set. Since P is a polytope only finitely many choices for $u \in \mathbb{S}^{d-1}$ and $\delta \in \mathbb{R}$ are possible. Obviously $\delta = 0$ has to be respected for every u_r, $r \in \{1, \dots, L\}$. However, this shows that the set of $\vec{\alpha} \in \mathbb{T}^d$ for which condition (2) fails is meager and of zero measure. □

6. Final remarks and open questions

The following question is evident: Is the identity

$$\lim_{N \to \infty} \mathcal{P}_{(K,\vec{\alpha})}(N)/N^d = \lambda^d(\Pi K) \tag{10}$$

true for general $K \in \mathcal{K}^d$ in $[0, 1)^d$ and almost all $\vec{\alpha} \in \mathbb{T}^d$? The volume of the projection body is continuous w.r.t. the Hausdorff metric. Let us remark that one cannot hope for continuity of the asymptotic complexity w.r.t. the Hausdorff metric. One always has to respect dependencies of K and $\vec{\alpha}$. In the present work these dependencies are controlled via the σ'-asymptotic independence. If P and $\vec{\alpha}$ are not σ'-a.i. the asymptotic value of the complexity changes—a well known example is given by the Sturmian sequences: For $I = [a, b) \subseteq [0, 1)$ and $\alpha \in \mathbb{T}$, $\mathcal{P}_{(I,\alpha)}(N) = N + 1$ iff $|I| = \alpha$ and α is irrational (i.e. in the Sturmian case) while typically $\mathcal{P}_{(I,\alpha)}(N) = 2N$ (see e.g. [3]).

However, if the boundary of $K \in \mathcal{K}^d$ is sufficiently smooth it seems that our method allows to verify Eq. (10) in the special case $d = 2$. A detailed investigation of this and the general d-dimensional case is an interesting task for future research.

Highly desirable is a deeper understanding of the interplay between the complexity and the projection body itself. This might lead to a better understanding of the volume of the projection body and help, for instance, to obtain extremality results. Concluding we remark that our result allows to express the important product $(\lambda^d(P))^{1-d}\lambda^d(\Pi P)$, P a convex polytope in $[0, 1)^d$, in terms of $(h(P, \vec{\alpha})_k)_{k=-\infty}^{\infty} \in \{0, 1\}^{\mathbb{Z}}$ namely

$$\begin{aligned}&\big(\lambda^d(P)\big)^{1-d}\lambda^d(\Pi P)\\&\quad= \lim_{N \to \infty} \frac{(|\{k \in \{0, 1, \dots, N-1\}\colon h(P, \vec{\alpha})_k = 1\}|)^{1-d}\mathcal{P}_{(P,\vec{\alpha})}(N)}{N},\end{aligned}$$

for almost all $\vec{\alpha} \in \mathbb{T}^d$.

Acknowledgments

The author is much obliged to Franziska Berger, Monika Ludwig and Franz E. Schuster for introducing him to the field of convex geometry and for many valuable remarks. Moreover he thanks the Austrian Science Foundation FWF for the financial support through the projects No. P16547–N12, S8312 and S9612.

References

[1] B. Adamczewski, Y. Bugeaud, On the complexity of algebraic numbers, II. Continued fractions, Acta Math. 195 (2005) 1–20.

[2] B. Adamczewski, Y. Bugeaud, On the complexity of algebraic numbers, I. Expansions in integer bases, Ann. of Math. 165 (2007) 547–565.

[3] P. Alessandri, V. Berthé, Three distance theorems and combinatorics on words, Enseign. Math. 44 (1998) 103–132.
[4] V. Berthé, Sequences of low complexity: Automatic and Sturmian sequences, in: F. Blanchard, A. Maas, A. Nogueira (Eds.), Symbolic Dynamics and Its Applications, in: London Math. Soc. Lecture Note Ser., vol. 279, Cambridge University Press, 2000, pp. 1–34.
[5] K. Böröczky Jr., The stability of the Rogers–Shephard inequality and of some related inequalities, Adv. Math. 190 (2005) 47–76.
[6] N.S. Brannen, Volumes of projection bodies, Mathematika 43 (1996) 255–264.
[7] R. Gardner, Geometric Tomography, Cambridge Univ. Press, Cambridge, 1995.
[8] R. Gardner, G. Zhang, Affine inequalities and radial mean bodies, Amer. J. Math. 120 (1998) 505–528.
[9] E. Grinberg, G. Zhang, Convolutions, transforms, and convex bodies, Proc. London Math. Soc. (3) 78 (1999) 77–115.
[10] K. Kiener, Extremalität von Ellipsoiden und die Faltungsungleichung von Sobolev, Arch. Math. (Basel) 46 (1986) 162–168.
[11] L. Kuipers, H. Niederreiter, Uniform Distribution of Sequences, A. Wiley–Interscience Publ., New York, 1974.
[12] M. Lothaire, Algebraic Combinatorics on Words, Encyclopaedia Math. Appl., vol. 90, Cambridge Univ. Press, Cambridge, 2002.
[13] M. Ludwig, Projection Bodies and Valuations, Adv. Math. 172 (2002) 158–168.
[14] M. Ludwig, Minkowski valuations, Trans. Amer. Math. Soc. 357 (2005) 4191–4213.
[15] E. Lutwak, Volume of mixed bodies, Trans. Amer. Math. Soc. 294 (1986) 487–500.
[16] E. Lutwak, Centroid bodies and dual mixed volumes, Proc. London Math. Soc. (3) 60 (1990) 365–391.
[17] E. Lutwak, On a conjectured projection inequality of Petty, in: Integral Geometry and Tomography, Arcata, CA, 1989, in: Contemp. Math., vol. 113, Amer. Math. Soc., Providence, RI, 1990, pp. 171–182.
[18] E. Lutwak, Inequalities for mixed projection bodies, Trans. Amer. Math. Soc. 339 (1993) 901–916.
[19] E. Lutwak, Selected affine isoperimetric inequalities, in: P.M. Gruber, J. Wills (Eds.), Handbook of Convex Geometry, Vol. A, North-Holland, Amsterdam, 1993, pp. 151–176.
[20] E. Lutwak, D. Yang, G. Zhang, L_p affine isoperimetric inequalities, J. Differential Geom. 56 (2000) 111–132.
[21] E. Lutwak, D. Yang, G. Zhang, A new affine invariant for polytopes and Schneider's projection problem, Trans. Amer. Math. Soc. 353 (2001) 1767–1779.
[22] E. Lutwak, D. Yang, G. Zhang, Sharp L_p affine Sobolev inequalities, J. Differential Geom. 62 (2002) 17–38.
[23] M. Morse, G.A. Hedlund, Symbolic dynamics I, Amer. J. Math. 60 (1938) 815–866.
[24] M. Morse, G.A. Hedlund, Symbolic dynamics II, Amer. J. Math. 62 (1940) 1–42.
[25] C.M. Petty, Isoperimetric problems, in: Proceedings of the Conference on Convexity and Combinatorial Geometry, Univ. Oklahoma, Norman, OK, 1971, Dept. Math., Univ. Oklahoma, Norman, OK, 1971, pp. 26–41.
[26] N. Pytheas Fogg, Substitutions in Dynamics, Arithmetics and Combinatorics, Lecture Notes in Math., vol. 1794, Springer, New York, 2002.
[27] J. Schmeling, E. Szabó, R. Winkler, Hartman and Beatty bisequences, in: Algebraic Number Theory and Diophantine Analysis, Walter de Gruyter, Berlin, 2000, pp. 405–421.
[28] M. Schmuckenschläger, Petty's projection inequality and Santalo's affine isoperimetric inequality, Geom. Dedicata 57 (1995) 285–295.
[29] R. Schneider, Convex Bodies: The Brunn–Minkovski Theory, Cambridge Univ. Press, 1993.
[30] R. Schneider, Random hyperplanes meeting a convex body, Z. Wahrscheinlichkeitsth. verw. Geb. 61 (1981) 379–387.
[31] R. Schneider, F.E. Schuster, Rotation equivariant Minkowski valuations, Int. Math. Res. Not. (2006) 1–20, Art. ID 72894.
[32] F.E. Schuster, Convolutions and multiplier transformations of convex bodies, Trans. Amer. Math. Soc. 359 (2007) 5567–5591.
[33] C. Steineder, R. Winkler, Complexity of Hartman sequences, J. Theor. Nombres Bordeaux 17 (2005) 347–357.
[34] P. Walters, An Introduction to Ergodic Theory, Grad. Texts in Math., Springer-Verlag, New York, 2000.
[35] H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins, Math. Ann. 77 (1916) 313–352.
[36] R. Winkler, Ergodic group rotations, Hartman sets and Kronecker sequences, Monatsh. Math. 135 (2002) 333–343.
[37] G. Zhang, Geometric inequalities and inclusion measures of convex bodies, Mathematika 41 (1994) 95–116.
[38] G. Zhang, The affine Sobolev inequality, J. Differential Geom. 53 (1999) 183–202.