Average Distance Between Consecutive Points of Uniformly Distributed Sequences

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Abstract

In this paper we give best possible lower and upper bounds on the average distance between consecutive points of uniformly distributed sequences. The upper bound is attained with the dyadic van der Corput sequence. Furthermore we give a constructive proof that any element from the interval $[0, 1/2]$ can be obtained as the average distance between consecutive points of some uniformly distributed sequence.

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1 Introduction

A sequence $(x_n)_{n\geq 0}$ in the unit-interval $[0, 1)$ is said to be *uniformly distributed* if for all intervals $I \subseteq [0, 1)$ we have

$$\lim_{N\to\infty} \frac{A(I, N, (x_n))}{N} = \lambda(I),$$

where $A(I, N, (x_n)) = \#\{0 \leq n < N : x_n \in I\}$, the number of elements among the first N elements of the sequence that belong to I, and $\lambda(I)$ is the length of the interval I. An excellent introduction into this topic can be found in the book of Kuipers and Niederreiter [5] or in the book of Drmota and Tichy [2]. Typical examples of uniformly distributed sequences are:

1. The *van der Corput sequence* in integer base $b \geq 2$ for which the n-th point is given by $x_n = \frac{n_0}{b} + \frac{n_1}{b^2} + \frac{n_2}{b^3} + \cdots$ for $n \in \mathbb{N}_0$ with b-adic expansion $n = n_0 + n_1 b + n_2 b^2 + \cdots$. It is well known that the van der Corput sequence in base b is uniformly distributed (see [5, Chapter 2, Theorem 3.5]).

2. The $(n\alpha)$-*sequences* where the n-th element is given by $x_n = \{n\alpha\}$, and $\{\cdot\}$ denotes the fractional part. From Weyl's criterion one obtains immediately, that the $(n\alpha)$-sequence is uniformly distributed if and only if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ (see [5, Chapter 1, Example 2.1]).

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It is well known (see [5, Chapter 1, Theorem 2.6]) that if a sequence $(x_n)_{n\geq 0}$ is uniformly distributed, then necessarily $\limsup_{n\to\infty} n|x_{n+1}-x_n| = \infty$. In this paper we consider the average of the distances between consecutive elements of a uniformly distributed sequence $(x_n)_{n\geq 0}$ in the unit-interval, i.e., we analyze the quantity

$$\sum_{n=0}^{N-1} |x_{n+1}-x_n|. \tag{1}$$

Trivially, this quantity is bounded above by N. On the other hand, it is also clear that for uniformly distributed sequences $(x_n)_{n\geq 0}$ in the unit-interval the series $\sum_{n=0}^{\infty} |x_{n+1}-x_n|$ is divergent, since otherwise we would have $\sum_{n>N} |x_{n+1}-x_n| < \frac{1}{4}$ for some $N \in \mathbb{N}$. Hence for all $m > N$ we would have

$$|x_m - x_N| \leq \sum_{n=N}^{m-1} |x_{n+1}-x_n| < \frac{1}{4},$$

which means that all elements x_m with $m > N$ are in the interval $(x_N - \frac{1}{4}, x_N + \frac{1}{4}) \cap [0,1)$. Obviously, this is a contradiction to the uniform distribution property of $(x_n)_{n\geq 0}$.

A sequence $(x_n)_{n\geq 0}$ is said to be *completely uniformly distributed* if for any $s \geq 1$ the s-dimensional sequence $(\boldsymbol{x}_n^{(s)})_{n\geq 0}$, where $\boldsymbol{x}_n^{(s)} = (x_n, x_{n+1}, \ldots, x_{n+s-1})$, is uniformly distributed in $[0,1)^s$ (the definition of uniform distribution for sequences in $[0,1)$ can be generalized to uniform distribution of s-dimensional sequences in the obvious way, see [5, Chapter 1, Section 6]). Examples for completely uniformly distributed sequences can be found in [10]. Hence, if $(x_n)_{n\geq 0}$ is completely uniformly distributed, then from [5, Chapter 1, Theorem 6.1] we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x_{n+1}-x_n| = \int_0^1 \int_0^1 |x-y| \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{3}.$$

Now it is well known that almost all random sequences in the unit interval $[0,1)$ are completely uniformly distributed (this follows, for example, from [3, Theorem 4]; see also [6]). Hence we obtain that almost all random sequences in the unit interval are uniformly distributed with average distance of consecutive elements equal to $\frac{1}{3}$ in the limit.

The paper is organized as follows: in Section 2 we present the main results. In particular, we give best possible (asymptotic) upper and lower bounds on (1). Furthermore, we determine the average distance among consecutive points of the two prototypes of uniformly distributed sequences given above, the van der Corput sequence and the $(n\alpha)$-sequence. These results lead for any $\gamma \in [0, \frac{1}{2}]$ to the construction of a uniformly distributed sequence whose average distance between consecutive points is in the limit equal to γ. The proofs of the results are presented in Section 3.

Throughout the paper we denote by $\lfloor x \rfloor$ the integer part of x and by $\{x\}$ the fractional part of x, i.e., $x = \lfloor x \rfloor + \{x\}$. Furthermore, by $\|x\|$ we denote the distance from x to the nearest integer, i.e., $\|x\| = \min\{x - \lfloor x \rfloor, 1-(x-\lfloor x \rfloor)\}$. By log we denote the natural logarithm.

2 The Main Results

In this section we present the main results of this paper. The proofs of these results will be given in Section 3.

The first theorem shows that the sum of the first N distances between consecutive points of a uniformly distributed sequence grows faster than any positive constant times the logarithm of N.

Theorem 1 *Let $(x_n)_{n\geq 0}$ be uniformly distributed in $[0,1)$. Then we have*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=0}^{N-1} |x_{n+1} - x_n| = \infty.$$

The result of this theorem becomes quite natural in the light of the following corollary stating that sequences generated by increasing functions with a very slow growth are *not* uniformly distributed. This result is well known and was first proved by Niederreiter [7] (see also [10, Subsection 2.2.8]).

Corollary 1 *Let $f : \mathbb{N} \to \mathbb{R}^+$ be an increasing function with $f(n) = O(\log n)$. Then the sequence $(\{f(n)\})_{n\geq 1}$ is not uniformly distributed.*

A further consequence of Theorem 1 is the result from [5, Chapter 1, Theorem 2.6] on the $\limsup_{n\to\infty} n|x_{n+1} - x_n|$ for uniformly distributed sequences $(x_n)_{n\geq 0}$ as mentioned in the Introduction to this paper. Assuming that $\limsup_{n\to\infty} n|x_{n+1} - x_n| < \infty$ we would obtain $\sum_{n=0}^{N-1} |x_{n+1} - x_n| = O(\log N)$ which contradicts Theorem 1. Hence we must have

$$\limsup_{n\to\infty} n|x_{n+1} - x_n| = \infty.$$

The result from Theorem 1 is best possible in the sense that there exist (arbitrary slowly) growing functions h which generate uniformly distributed sequences for which the first N distances between consecutive points of the sequences are bounded above by $h(N)\log N$.

Theorem 2 *Let $h : [1,\infty) \to \mathbb{R}^+$ be an increasing, continuously differentiable function such that:*

1. $\lim_{x\to\infty} h(x) = \infty$,
2. *$h(x)/x$ tends monotonically to 0 as $x \to \infty$ and*
3. *$h'(x)\log x$ tends monotonically to 0 as $x \to \infty$.*

Then there exists a uniformly distributed sequence $(x_n)_{n\geq 0}$ in $[0,1)$ for which we have

$$\sum_{n=0}^{N-1} |x_{n+1} - x_n| \leq h(N)\log N.$$

Such a sequence is, for example, given by $x_n = \left\{\frac{1}{2}h(n)\log n\right\}$ for $n \geq 1$.

Now we turn to an asymptotic upper bound on the average of distances between consecutive points of a uniformly distributed sequence.

Theorem 3 *Let $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ be two uniformly distributed sequences in $[0,1)$. Then we have*

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x_n-y_n|\leq\frac{1}{2}.$$

In particular, we have

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x_{n+1}-x_n|\leq\frac{1}{2}.$$

Again, this result is best possible. The value $\frac{1}{2}$ is obtained, for example, by the van der Corput sequence in base 2.

Theorem 4 *Let $(x_n)_{n\geq 0}$ be the van der Corput sequence in base 2. Then we have*

$$\sum_{n=0}^{N-1}|x_{n+1}-x_n|=\frac{N}{2}-\frac{1}{2}x_{\lfloor\frac{N}{2}\rfloor}.$$

In particular, $\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x_{n+1}-x_n|=\frac{1}{2}$.

Hence the van der Corput sequence in base 2 is an example for a uniformly distributed sequence with the largest possible average distance between consecutive points.

Remark 1 With a much simpler argumentation as in the proof of Theorem 4, but less accurate, we can show that for the van der Corput sequence in arbitrary base b we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x_{n+1}-x_n|=\frac{2(b-1)}{b^2}. \tag{2}$$

Namely, from the construction of the van der Corput sequence in base b we find that $x_{n+1}-x_n=\frac{1}{b}$ whenever $n\not\equiv b-1 \pmod b$ and $x_{n+1}-x_n=-\frac{b^k-b-1}{b^k}$ whenever n is of the form $n=mb^k+\beta b^{k-1}+b^{k-1}-1$ with $m\in\mathbb{N}_0$, $\beta\in\{0,\ldots,b-2\}$ and $k\geq 2$. Therefore we have

$$\begin{aligned}\sum_{n=0}^{N-1}|x_{n+1}-x_n| &= \frac{b-1}{b}\left\lfloor\frac{N}{b}\right\rfloor+\sum_{\substack{k\geq 2, m\geq 0\\ 0\leq\beta\leq b-2\\ mb^k+\beta b^{k-1}+b^{k-1}-1\leq N-1}}\frac{b^k-b-1}{b^k}\\ &= N\frac{b-1}{b^2}+N(b-1)\sum_{k=2}^{\lfloor\log N\rfloor}\frac{b^k-b-1}{b^{2k}}+O(\log N)\\ &= N\frac{b-1}{b^2}+N(b-1)\frac{(b^{\lfloor\log N\rfloor+1}-b^{\lfloor\log N\rfloor}-1)(b^{\lfloor\log N\rfloor}-b)}{(b-1)b^{2(\lfloor\log N\rfloor+1)}}+O(\log N).\end{aligned}$$

From this the result (2) follows, since $\lim_{N\to\infty}\frac{(b^{\lfloor\log N\rfloor+1}-b^{\lfloor\log N\rfloor}-1)(b^{\lfloor\log N\rfloor}-b)}{(b-1)b^{2(\lfloor\log N\rfloor+1)}}=\frac{1}{b^2}$.

The next theorem shows, that for any $(n\alpha)$-sequence we have that the average distance between consecutive points is in the limit $<\frac{1}{2}$ where again $\frac{1}{2}$ is best possible.

Theorem 5 *Let $(x_n)_{n\geq 0}$ be the $(n\alpha)$-sequence with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x_{n+1} - x_n| = 2\{\alpha\}(1 - \{\alpha\}).$$

Hence for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ the values $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x_{n+1} - x_n|$ are dense in $\left[0, \frac{1}{2}\right]$. Especially, every real $\gamma \in (0, \frac{1}{2})$ which is of the form $\gamma = 2\alpha(1-\alpha)$ for some irrational α can be obtained as the average distance between consecutive points of the $(n\alpha)$-sequence. (Note also that choosing randomly a irrational $\alpha \in (0,1)$ leads to a $(n\alpha)$-sequence with expected average distance between consecutive points of $\frac{1}{3}$ in the limit.) By constructing a more general sequence, we finally obtain that even any $\gamma \in [0, \frac{1}{2}]$ can be obtained as the average distance between consecutive points of a uniformly distributed sequence. This result should be compared with the fact that almost all random sequences have average distance between consecutive points of $\frac{1}{3}$ in the limit (see Section 1).

Corollary 2 *For each $\gamma \in [0, \frac{1}{2}]$ there exists a uniformly distributed sequence $(x_n)_{n\geq 0}$ in $[0,1)$ such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} |x_{n+1} - x_n| = \gamma.$$

We remark here, that for each $\gamma \in [0, \frac{1}{2}]$ we can give an explicit example for a uniformly distributed sequence with average distance between consecutive points equal to γ in the limit (see the proof of Corollary 2 in the subsequent section).

3 The Proofs

In this section we provide the proofs of the results from Section 2. For the proof of Theorem 1 we need the following lemmas.

Lemma 1 *Let $f : \mathbb{N} \to \mathbb{N}$ be a function and let $(x_n)_{n\geq 0}$ be uniformly distributed in $[0,1)$. Assume there is a constant $c = c(f) > 0$ such that there exists a $y = y(c) \in \mathbb{R}$ with the property that for all $x > y$ we have $f(x)\left(\frac{1}{2} - 3c\right) > 2cx$. Then there is an integer $N \in \mathbb{N}$ such that for all $n > N$ we have*

$$\sum_{i=n}^{n+f(n)} |x_{i+1} - x_i| > \frac{1}{2}. \tag{3}$$

Proof. Let us assume that the inequality is false for infinitely many numbers $n \in \mathbb{N}$. We now choose an arbitrary $\varepsilon \in (0, c)$. Since the sequence $(x_n)_{n\geq 0}$ is uniformly distributed modulo one, there is a constant $M \in \mathbb{N}$ such that for all $m > M$ and for all intervals $I \subseteq [0,1)$ we have

$$\left| \frac{A(I, m, (x_n))}{m} - \lambda(I) \right| \leq \varepsilon. \tag{4}$$

By assumption, there are infinitely many numbers $m > M$ such that Eq. (3) does not hold. We choose one such $m > M$ for which the inequality $f(m)\left(\frac{1}{2} - 3\varepsilon\right) > 2\varepsilon m$ is true as well. Then for all $m \le j < k \le m + f(m) + 1$ we have

$$|x_k - x_j| \le \sum_{i=j}^{k-1} |x_{i+1} - x_i| \le \sum_{i=m}^{m+f(m)} |x_{i+1} - x_i| \le \frac{1}{2}$$

Hence the elements $x_m, x_{m+1}, ..., x_{f(m)+m+1}$ can be found within an interval $I^* \subseteq [0, 1)$ of length at most $\frac{1}{2}$. From (4) we conclude that

$$\left| \frac{A(I^*, m + f(m), (x_n))}{m + f(m)} - \frac{A(I^*, m, (x_n))}{m} \right| \le 2\varepsilon. \tag{5}$$

On the other hand

$$\frac{A(I^*, m + f(m), (x_n))}{m + f(m)} = \frac{A(I^*, m, (x_n))}{m + f(m)} + \frac{f(m)}{m + f(m)}$$

Inserting this into Eq. (5) yields

$$\left| \frac{f(m)}{m + f(m)} \left(1 - \frac{A(I^*, m, (x_n))}{m}\right) \right| \le 2\varepsilon$$

The expression within the brackets is strictly positive, therefore

$$2\varepsilon \ge \frac{f(m)}{m + f(m)} \left(1 - \frac{A(I^*, m, (x_n))}{m}\right) \ge \frac{f(m)}{m + f(m)} (1 - \lambda(I^*) - \varepsilon) \ge \frac{f(m)}{m + f(m)} \left(\frac{1}{2} - \varepsilon\right).$$

However, the inequality $f(m)\left(\frac{1}{2} - 3\varepsilon\right) > 2\varepsilon m$ gives

$$2\varepsilon \ge \frac{f(m)}{m + f(m)} \left(\frac{1}{2} - \varepsilon\right) > 2\varepsilon,$$

and this contradiction completes the proof. □

Lemma 2 *Let $m \in \mathbb{N}$ be arbitrary and let $T : \mathbb{R} \to \mathbb{R}$ be defined as $T(x) := x + \left\lfloor \frac{x}{m} \right\rfloor + 1$. For $n \in \mathbb{N}$ we define $T^{(n)} := T(T^{(n-1)})$ with $T^{(0)}(x) := x$. If $x > m$, then we have*

$$T^{(n)}(x) \le x \left(1 + \frac{2}{m}\right)^n$$

Proof. The result follows by induction on n. □

Proof of Theorem 1. Let $m > 2$ be a fixed integer. It is easy to see that the function $f(x) = \lfloor \frac{x}{m} \rfloor$ satisfies the requirements of Lemma 1. Therefore there exists a number $M \in \mathbb{N}$ such that for all $k > M$ we have

$$\sum_{n=k}^{k+\lfloor \frac{k}{m} \rfloor} |x_{n+1} - x_n| > \frac{1}{2}.$$

Let now $N > \max(M, m)$, we then estimate

$$\sum_{n=0}^{N-1} |x_{n+1} - x_n| \geq \sum_{n=M}^{M+\lfloor \frac{M}{m} \rfloor} |x_{n+1} - x_n| + \sum_{n=M+\lfloor \frac{M}{m} \rfloor+1}^{M+\lfloor \frac{M}{m} \rfloor+1+\left\lfloor \frac{M+\lfloor \frac{M}{m} \rfloor+1}{m} \right\rfloor} |x_{n+1} - x_n| + \cdots,$$

where the sums are being added while the upper index is still smaller than N. From our Lemma 2 we know that the upper index of the kth sum will be $\leq M(1+\frac{2}{m})^k$. By making use of the elementary inequality $\log(1+x) \leq x$ we see that there are at least $\frac{m}{2}(\log N - \log M)$ sums on the right side of the inequality. Each sum contributes at least $\frac{1}{2}$, therefore

$$\sum_{n=0}^{N-1} |x_{n+1} - x_n| \geq \frac{m}{4}(\log N - \log M)$$

for N sufficiently large. Since m was an arbitrary integer, the result follows. $\square$

Proof of Corollary 1. The result follows from Theorem 1 together with the subsequent lemma which is also required for the proof of Theorem 2. $\square$

Lemma 3 *Let $f : \mathbb{N} \to \mathbb{R}^+$ be an increasing function. Then for any $N \in \mathbb{N}$ we have*

$$\sum_{n=0}^{N-1} |\{f(n+1)\} - \{f(n)\}| \leq 2f(N).$$

Proof. We have

$$|\{f(n+1)\} - \{f(n)\}| \leq \begin{cases} f(n+1) - f(n) & \text{if } \lfloor f(n) \rfloor = \lfloor f(n+1) \rfloor, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore we obtain

$$\begin{aligned} \sum_{n=0}^{N-1} |\{f(n+1)\} - \{f(n)\}| &\leq \sum_{\substack{n=0 \\ \lfloor f(n) \rfloor = \lfloor f(n+1) \rfloor}}^{N-1} (f(n+1) - f(n)) + \sum_{\substack{n=0 \\ \lfloor f(n) \rfloor \neq \lfloor f(n+1) \rfloor}}^{N-1} 1 \\ &\leq f(N) + \#\{0 \leq n < N : \lfloor f(n) \rfloor \neq \lfloor f(n+1) \rfloor\}. \end{aligned}$$

For the increasing sequence $f(n)$, $n = 0, 1, \ldots, N-1$, it can occur at most $f(N)$ times that $\lfloor f(n) \rfloor \neq \lfloor f(n+1) \rfloor$ (the worst case is that each interval $[i, i+1)$, $i \in \{0, \ldots, \lfloor f(N) \rfloor\}$, contains exactly one element $f(n)$). Hence the result follows. $\square$

Proof of Theorem 2. The proof is based on Fejér's theorem (see, for example, [5, Chapter 1, Corollary 2.1]). Let $f(x) = \frac{1}{2}h(x)\log x$, then $f : [1, \infty) \to \mathbb{R}^+$ is an increasing, differentiable function with

$$x|f'(x)| = \frac{x}{2}\left(h'(x)\log x + \frac{h(x)}{x}\right) \geq \frac{1}{2}h(x)$$

and hence $\lim_{x\to\infty} x|f'(x)| = \infty$.

Since by assumption $h(x)/x$ and $h'(x)\log x$ both converge to 0 monotonically as $x \to \infty$ we find that

$$f'(x) = \frac{1}{2}\left(h'(x)\log x + \frac{h(x)}{x}\right)$$

converges to 0 monotonically as $x \to \infty$.

Thus, by Fejér's theorem the sequence $(f(n))_{n\geq 0}$ is uniformly distributed. From Lemma 3 we obtain that

$$\sum_{n=0}^{N-1} |x_{n+1} - x_n| = \sum_{n=0}^{N-1} |\{f(n+1)\} - \{f(n)\}| \leq 2f(N) = h(N)\log N,$$

as claimed. □

Proof of Theorem 3. [1] We consider an arbitrary distribution function (d.f.) $g : [0,1]^2 \to [0,1]$ of the two-dimensional sequence $(x_n, y_n)_{n\geq 0}$. Hence there exists an increasing sequence of natural numbers $N_1, N_2, \ldots$ such that

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=0}^{N_k-1} |x_n - y_n| = \int_0^1 \int_0^1 |x-y| \, \mathrm{d}_x \, \mathrm{d}_y g(x,y).$$

Integration by parts yields

$$\int_0^1 \int_0^1 |x-y| \, \mathrm{d}_x \, \mathrm{d}_y g(x,y) = \int_0^1 g(1,y) \, \mathrm{d}y + \int_0^1 g(x,1) \, \mathrm{d}x - 2\int_0^1 g(x,x) \, \mathrm{d}x.$$

Since the sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ are both uniformly distributed, we have that $g(1,y) = y$ and $g(x,1) = x$ for all $x, y \in [0,1]$. Such a distribution function is called a *copula* (see [9, p. 55] for basic properties of copulas). Now we obtain

$$\int_0^1 \int_0^1 |x-y| \, \mathrm{d}_x \, \mathrm{d}_y g(x,y) = 1 - 2\int_0^1 g(x,x) \, \mathrm{d}x.$$

It is known (see [9, p. 56]) that for every copula $g(x,y)$ we have $\max(x+y-1, 0) \leq g(x,y) \leq \min(x,y)$ and from this we find that

$$\int_0^1 \int_0^1 |x-y| \, \mathrm{d}_x \, \mathrm{d}_y g(x,y) \leq \frac{1}{2}.$$

Since g is an arbitrary d.f. the result follows. □

The proof of Theorem 4 is based on the following result.

[1] We are geatful to Oto Strauch who proposed this proof which is much shorter and more general than our initial proof of Theorem 3.

Proposition 1 *Let $(x_n)_{n\geq 0}$ be the van der Corput sequence in base 2. For $2^m \leq N < 2^{m+1}$ we have*

$$\sum_{n=0}^{N-1} x_n = \frac{N}{2} - \frac{1}{2}\left(1 + \sum_{r=1}^{m} \left\| \frac{N}{2^r} \right\| \right) \tag{6}$$

and

$$\sum_{\substack{n=0 \\ n\equiv 0 \pmod 2}}^{N-1} x_n = \frac{N}{8} + \rho_N - \frac{1}{4}\sum_{i=1}^{m} \left\| \frac{N}{2^i} \right\| - \frac{1}{8}\sum_{i=1}^{m-1} \frac{\delta_{N,i}}{2^i}, \tag{7}$$

with $\rho_N = 0$ when N is odd and $\rho_N = -1/4$ when N is even and $\delta_{N,i} = 0$ when N is even and $\delta_{N,i} = (-1)^{N_i}$ when N is odd.

The following lemma is required for the proof of Proposition 1.

Lemma 4 *Let $0 \leq U < 2^m$ be an integer and for any integer $0 \leq n \leq U-1$ let $n = n_0 + n_1 2 + \cdots + n_{m-1}2^{m-1}$ be the binary representation of n. Then for any integer $0 \leq r < m$ we have*

$$\sum_{n=0}^{U-1} (-1)^{n_r} = 2^{r+1} \left\| \frac{U}{2^{r+1}} \right\|, \tag{8}$$

and for $1 \leq r < m$ we have

$$\sum_{\substack{n=0 \\ n\equiv 0 \pmod 2}}^{U-1} (-1)^{n_r} = 2^r \left\| \frac{U}{2^{r+1}} \right\| + \frac{\delta_{U,r}}{2}, \tag{9}$$

where $\delta_{U,r} = 0$ when U is even and $\delta_{U,r} = (-1)^{U_r}$ when U is odd.

Proof. Eq. (8) is a special case of [8, Lemma 4.1] (or [1, Lemma 3]). Hence we just deduce Eq. (9) from Eq. (8). We have

$$2^{r+1}\left\| \frac{U}{2^{r+1}} \right\| = \sum_{n=0}^{U-1}(-1)^{n_r} = \sum_{\substack{n=0 \\ n\equiv 0 \pmod 2}}^{U-1} (-1)^{n_r} + \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{U-1} (-1)^{n_r}.$$

If n is odd and $r \geq 1$ we have $n_r = (n-1)_r$ where $(n-1)_r$ is the r-th digit in the binary representation of $n-1$. Hence

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{U-1} (-1)^{n_r} = \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{U-1} (-1)^{(n-1)_r} = \sum_{\substack{n=0 \\ n\equiv 0 \pmod 2}}^{U-2} (-1)^{n_r} = \sum_{\substack{n=0 \\ n\equiv 0 \pmod 2}}^{U-1} (-1)^{n_r} - \delta_{U,r},$$

where $\delta_{U,r} = 0$ when U is even and $\delta_{U,r} = (-1)^{(U-1)_r} = (-1)^{U_r}$ when U is odd (note that $r \neq 0$). Together we obtain

$$2^{r+1}\left\| \frac{U}{2^{r+1}} \right\| = 2 \sum_{\substack{n=0 \\ n\equiv 0 \pmod 2}}^{U-1} (-1)^{n_r} - \delta_{U,r}$$

and the result follows. □

Now we give the proof of Proposition 1.

Proof of Proposition 1. We just give the (more involved) proof of Eq. (7) (Eq. (6) can be shown in the same way or, alternatively, follows from [1, Proposition 1]).

For $\xi \in \{0,1\}$ we have $\xi = \frac{1-(-1)^\xi}{2}$. Let $x \in [0,1)$ with canonical binary representation $x = \sum_{i=1}^\infty \frac{\xi_i}{2^i}$. Then we have

$$x = \sum_{i=1}^\infty \frac{1-(-1)^{\xi_i}}{2^{i+1}} = \frac{1}{2} - \sum_{i=1}^\infty \frac{(-1)^{\xi_i}}{2^{i+1}}. \tag{10}$$

For $n = n_0 + n_1 2 + n_2 2^2 + \cdots$ the n-th point of the van der Corput sequence is given by $x_n = \frac{n_0}{2} + \frac{n_1}{2^2} + \frac{n_2}{2^2} + \cdots$. Hence, using (10) we may write x_n as $x_n = \frac{1}{2} - \sum_{i=0}^\infty \frac{(-1)^{n_i}}{2^{i+2}}$. Now we have

$$\sum_{\substack{n=0\\ n\equiv 0 \pmod 2}}^{N-1} x_n = \sum_{\substack{n=0\\ n\equiv 0 \pmod 2}}^{N-1} \frac{1}{2} - \sum_{i=0}^\infty \frac{1}{2^{i+2}} \sum_{\substack{n=0\\ n\equiv 0 \pmod 2}}^{N-1} (-1)^{n_i}.$$

For any $m \in \mathbb{N}$ we have

$$\sum_{\substack{n=0\\ n\equiv 0 \pmod 2}}^{2^m-1} (-1)^{n_i} = \begin{cases} 0 & \text{if } 1 \le i < m, \\ 2^{m-1} & \text{if } i = 0 \text{ or if } i \ge m. \end{cases}$$

Choosing m such that $2^m \le N < 2^{m+1}$ we obtain

$$\sum_{\substack{n=0\\ n\equiv 0 \pmod 2}}^{N-1} x_n = \sum_{\substack{n=0\\ n\equiv 0 \pmod 2}}^{N-1} \frac{1}{2} - \sum_{i=0}^\infty \frac{1}{2^{i+2}} \sum_{\substack{n=0\\ n\equiv 0 \pmod 2}}^{2^m-1} (-1)^{n_i} - \sum_{i=0}^\infty \frac{1}{2^{i+2}} \sum_{\substack{n=2^m\\ n\equiv 0 \pmod 2}}^{N-1} (-1)^{n_i}.$$

We have

$$\sum_{i=0}^\infty \frac{1}{2^{i+2}} \sum_{\substack{n=0\\ n\equiv 0 \pmod 2}}^{2^m-1} (-1)^{n_i} = 2^{m-3} + \sum_{i=m}^\infty \frac{2^{m-1}}{2^{i+2}} = \frac{2^m}{8} + \frac{1}{4}$$

and (splitting up the summation over i and invoking Lemma 4)

$$\begin{aligned}
&\sum_{i=0}^\infty \frac{1}{2^{i+2}} \sum_{\substack{n=2^m\\ n\equiv 0 \pmod 2}}^{N-1} (-1)^{n_i} \\
&= \sum_{\substack{n=2^m\\ n\equiv 0 \pmod 2}}^{N-1} \frac{1}{4} + \sum_{i=1}^{m-1} \frac{1}{2^{i+2}} \sum_{\substack{n=2^m\\ n\equiv 0 \pmod 2}}^{N-1} (-1)^{n_i} + \frac{1}{2^{m+2}} \sum_{\substack{n=2^m\\ n\equiv 0 \pmod 2}}^{N-1} (-1) + \sum_{i=m+1}^\infty \frac{1}{2^{i+2}} \sum_{\substack{n=2^m\\ n\equiv 0 \pmod 2}}^{N-1} 1 \\
&= \sum_{\substack{n=2^m\\ n\equiv 0 \pmod 2}}^{N-1} \frac{1}{4} + \sum_{i=1}^{m-1} \frac{1}{2^{i+2}} \sum_{\substack{n=2^m\\ n\equiv 0 \pmod 2}}^{N-1} (-1)^{n_i} \\
&= \sum_{\substack{n=2^m\\ n\equiv 0 \pmod 2}}^{N-1} \frac{1}{4} + \sum_{i=1}^{m-1} \frac{1}{2^{i+2}} \sum_{\substack{n=0\\ n\equiv 0 \pmod 2}}^{N-2^m-1} (-1)^{n_i}
\end{aligned}$$

$$= \sum_{\substack{n=2^m \\ n\equiv 0 \pmod 2}}^{N-1} \frac{1}{4} + \sum_{i=1}^{m-1} \frac{1}{2^{i+2}} \left(2^i \left\| \frac{N-2^m}{2^{i+1}} \right\| + \frac{\delta_{N-2^m,i}}{2} \right)$$

$$= \sum_{\substack{n=2^m \\ n\equiv 0 \pmod 2}}^{N-1} \frac{1}{4} + \frac{1}{4} \sum_{i=2}^{m} \left\| \frac{N}{2^i} \right\| + \frac{1}{8} \sum_{i=1}^{m-1} \frac{\delta_{N,i}}{2^i}.$$

Now we obtain

$$\begin{aligned} \sum_{\substack{n=0 \\ n\equiv 0 \pmod 2}}^{N-1} x_n &= \sum_{\substack{n=0 \\ n\equiv 0 \pmod 2}}^{N-1} \frac{1}{2} - \sum_{\substack{n=2^m \\ n\equiv 0 \pmod 2}}^{N-1} \frac{1}{4} - \frac{2^m}{8} - \frac{1}{4} - \frac{1}{4} \sum_{i=2}^{m} \left\| \frac{N}{2^i} \right\| - \frac{1}{8} \sum_{i=1}^{m-1} \frac{\delta_{N,i}}{2^i} \\ &= \frac{N}{8} + \widetilde{\rho}_N - \frac{1}{4} - \frac{1}{4} \sum_{i=2}^{m} \left\| \frac{N}{2^i} \right\| - \frac{1}{8} \sum_{i=1}^{m-1} \frac{\delta_{N,i}}{2^i}, \end{aligned}$$

where $\widetilde{\rho}_N = 0$ when N is even and $\widetilde{\rho}_N = \frac{1}{8}$ when N is odd. For even N we have $\|N/2\| = 0$ and for odd N we have $\frac{1}{4}\|N/2\| = 1/8 = 1/4 - \widetilde{\rho}_N$. Hence we have

$$\sum_{\substack{n=0 \\ n\equiv 0 \pmod 2}}^{N-1} x_n = \frac{N}{8} + \rho_N - \frac{1}{4} \sum_{i=1}^{m} \left\| \frac{N}{2^i} \right\| - \frac{1}{8} \sum_{i=1}^{m-1} \frac{\delta_{N,i}}{2^i},$$

with $\rho_N = 0$ when N is odd and $\rho_N = -1/4$ when N is even. □

Proof of Theorem 4. For even n we have $x_{n+1} - x_n > 0$ and for odd n we have $x_{n+1} - x_n < 0$. Hence together with Proposition 1 we get

$$\begin{aligned} \sum_{n=0}^{N-1} |x_{n+1} - x_n| &= \sum_{\substack{n=0 \\ n\equiv 0 \pmod 2}}^{N-1} (x_{n+1} - x_n) + \sum_{\substack{n=0 \\ n\equiv 1 \pmod 2}}^{N-1} (x_n - x_{n+1}) \\ &= 2 \left(\sum_{\substack{n=0 \\ n\equiv 1 \pmod 2}}^{N-1} x_n - \sum_{\substack{n=0 \\ n\equiv 0 \pmod 2}}^{N-1} x_n \right) + (-1)^{N+1} x_N \\ &= 2 \left(\sum_{n=0}^{N-1} x_n - 2 \sum_{\substack{n=0 \\ n\equiv 0 \pmod 2}}^{N-1} x_n \right) + (-1)^{N+1} x_N \\ &= \frac{N}{2} - 1 - 4\rho_N + \frac{1}{2} \sum_{i=1}^{m-1} \frac{\delta_{N,i}}{2^i} + (-1)^{N+1} x_N. \end{aligned}$$

For even N we have $\rho_N = -1/4$ and $\delta_{N,i} = 0$ and $0 \le x_N < 1/2$, and hence

$$-1 - 4\rho_N + \frac{1}{2} \sum_{i=1}^{m-1} \frac{\delta_{N,i}}{2^i} + (-1)^{N+1} x_N = -x_N.$$

From this it follows that for even N

$$N - 2\sum_{n=0}^{N-1} |x_{n+1} - x_n| = 2x_N = x_{\frac{N}{2}}.$$

For odd N we know that $x_N = x_{N-1} + \frac{1}{2}$, therefore

$$N - 2\sum_{n=0}^{N-1} |x_{n+1} - x_n| = (N-1) - 2\sum_{n=0}^{N-2} |x_{n+1} - x_n| = x_{\frac{N-1}{2}}.$$

□

Finally, we give the proof of Theorem 5.

Proof of Theorem 5. W.l.o.g. we may assume that $\alpha \in (0,1)$. From the construction of the sequence one can see that

$$|x_{n+1} - x_n| = \begin{cases} \alpha & \text{if } x_n \in [0, 1-\alpha), \\ 1-\alpha & \text{if } x_n \in [1-\alpha, 1). \end{cases}$$

Therefore and since $(x_n)_{n\geq 0}$ is uniformly distributed, for $N \to \infty$ we obtain

$$\begin{aligned} \frac{1}{N}\sum_{n=0}^{N-1} |x_{n+1} - x_n| &= \frac{1}{N}\sum_{\substack{n=0 \\ x_n \in [0,1-\alpha)}}^{N-1} \alpha + \frac{1}{N}\sum_{\substack{n=0 \\ x_n \in [1-\alpha,1)}}^{N-1} (1-\alpha) \\ &= \alpha(1-\alpha+o(1)) + (1-\alpha)(\alpha+o(1)) \end{aligned}$$

and the result follows. □

Proof of Corollary 2. For $\gamma = \frac{1}{2}$ we can take the van der Corput sequence in base 2 and for $\gamma = 0$ we can take the sequence given in Theorem 2.

Each $\gamma \in (0, \frac{1}{2})$ can be written as $\gamma = 2\alpha(1-\alpha)$ for some $\alpha \in (0, \frac{1}{2})$. This is equivalent to $\alpha = \frac{1}{2}(1 - \sqrt{1-2\gamma})$. If α is irrational, then by Theorem 5 the $(n\alpha)$-sequence has the demanded properties.

Otherwise γ itself has to be rational. For a fixed irrational $0 < c < \min\{\gamma, \frac{1}{2} - \gamma\}$ the numbers $\gamma - c$ and $\gamma + c$ are irrational, and hence there exist irrational α and β such that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} |\{(n+1)\alpha\} - \{n\alpha\}| = \gamma - c \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} |\{(n+1)\beta\} - \{n\beta\}| = \gamma + c.$$

By Theorem 5 we have $2\alpha(1-\alpha) = \gamma - c$ and $2\beta(1-\beta) = \gamma + c$.

We now consider the sequence $(x_n)_{n\geq 0}$ given by $0, \{\alpha\}, \{\alpha+\beta\}, \{\alpha+2\beta\}, \{2\alpha+2\beta\}, \ldots$ where, beginning with the element 0 we add consecutively α once, then β twice, α three times and so forth. The resulting sequence can be written as $x_0 = 0$ and for $k \in \mathbb{N}$ and $\frac{k(k-1)}{2} < n \leq \frac{k(k+1)}{2}$ we have

$$x_n = \begin{cases} \left\{\left(n - \frac{k^2-1}{4}\right)\alpha + \frac{k^2-1}{4}\beta\right\} & \text{if } k \text{ is odd,} \\ \left\{\frac{k^2}{4}\alpha + \left(n - \frac{k^2}{4}\right)\beta\right\} & \text{if } k \text{ is even.} \end{cases}$$

Using Weyl's criterion we show that the sequence $(x_n)_{n\geq 0}$ is uniformly distributed whenever α and β are irrational. For $N\in\mathbb{N}$ choose $M\in\mathbb{N}$ such that $\frac{M(M-1)}{2}<N\leq\frac{M(M+1)}{2}$ and hence $M=O(\sqrt{N})$. Then for any integer $h\neq 0$ we have

$$\begin{aligned}\sum_{n=0}^{N-1}\mathrm{e}^{2\pi\mathtt{i}hx_n} &= \sum_{\substack{k=0\\ k\equiv 0\ (\bmod\ 2)}}^{M-1}\ \sum_{n=\frac{k(k-1)}{2}+1}^{\frac{k(k+1)}{2}} \mathrm{e}^{2\pi\mathtt{i}h\left(\frac{k^2}{4}\alpha+\left(n-\frac{k^2}{4}\right)\beta\right)}\\ &\quad+\sum_{\substack{k=0\\ k\equiv 1\ (\bmod\ 2)}}^{M-1}\ \sum_{n=\frac{k(k-1)}{2}+1}^{\frac{k(k+1)}{2}} \mathrm{e}^{2\pi\mathtt{i}h\left(\left(n-\frac{k^2-1}{4}\right)\alpha+\frac{k^2-1}{4}\beta\right)}+O(M). \end{aligned}\tag{11}$$

Since $\beta\in\mathbb{R}\setminus\mathbb{Q}$ and $h\neq 0$ for even $k\in\mathbb{N}$ we have

$$\left|\sum_{n=\frac{k(k-1)}{2}+1}^{\frac{k(k+1)}{2}} \mathrm{e}^{2\pi\mathtt{i}h\left(\frac{k^2}{4}\alpha+\left(n-\frac{k^2}{4}\right)\beta\right)}\right| = \left|\sum_{n=\frac{k^2}{4}-\frac{k}{2}+1}^{\frac{k^2}{4}+\frac{k}{2}}\mathrm{e}^{2\pi\mathtt{i}hn\beta}\right|\leq\frac{2}{|\mathrm{e}^{2\pi\mathtt{i}h\beta}-1|}.$$

An analogous bound holds for the second sum in (11) over n with odd k. Hence it follows with Eq. (11) that for any $h\neq 0$ we have $\sum_{n=0}^{N-1}\mathrm{e}^{2\pi\mathtt{i}hx_n}=O(M)=O(\sqrt{N})$ and hence Weyl's criterion implies that the sequence $(x_n)_{n\geq 0}$ is uniformly distributed.

Now we turn to the sum of the distances between consecutive points of $(x_n)_{n\geq 0}$. For $N\in\mathbb{N}$ choose again $M\in\mathbb{N}$ such that $\frac{M(M-1)}{2}<N\leq\frac{M(M+1)}{2}$ and hence $M=O(\sqrt{N})$ and $\frac{M^2}{2}\sim N$ for $N\to\infty$. We write

$$\begin{aligned}\sum_{n=0}^{N-1}|x_{n+1}-x_n| &= \sum_{r\in\{0,1\}}\ \sum_{\substack{k=0\\ k\equiv r\ (\bmod\ 2)}}^{M-1}\ \sum_{n=\frac{k(k-1)}{2}}^{\frac{k(k+1)}{2}-1}|x_{n+1}-x_n|+O(M)\\ &= \sum_{\substack{k=0\\ k\equiv 0\ (\bmod\ 2)}}^{M-1}\ \sum_{\substack{n=\frac{k(k-1)}{2}\\ x_n\in[0,1-\alpha)}}^{\frac{k(k+1)}{2}-1}\alpha+\sum_{\substack{k=0\\ k\equiv 0\ (\bmod\ 2)}}^{M-1}\ \sum_{\substack{n=\frac{k(k-1)}{2}\\ x_n\in[1-\alpha,1)}}^{\frac{k(k+1)}{2}-1}(1-\alpha)\\ &\quad+\sum_{\substack{k=0\\ k\equiv 1\ (\bmod\ 2)}}^{M-1}\ \sum_{\substack{n=\frac{k(k-1)}{2}\\ x_n\in[0,1-\beta)}}^{\frac{k(k+1)}{2}-1}\beta+\sum_{\substack{k=0\\ k\equiv 1\ (\bmod\ 2)}}^{M-1}\ \sum_{\substack{n=\frac{k(k-1)}{2}\\ x_n\in[1-\beta,1)}}^{\frac{k(k+1)}{2}-1}(1-\beta)+O(\sqrt{N}).\end{aligned}$$

For any interval $J\subseteq[0,1)$ for $k\to\infty$ we have

$$\begin{aligned}\sum_{\substack{n=\frac{k(k-1)}{2}\\ x_n\in J}}^{\frac{k(k+1)}{2}-1}1 &= \#\left\{\frac{k(k-1)}{2}\leq n<\frac{k(k+1)}{2}\ :\ x_n\in J\right\}\\ &= \left(\frac{k(k+1)}{2}(\lambda(J)+o(1))-\frac{k(k-1)}{2}(\lambda(J)+o(1))\right)=k\lambda(J)+o(k).\end{aligned}$$

Hence we obtain

$$\sum_{n=0}^{N-1}|x_{n+1}-x_n|=2\alpha(1-\alpha)\sum_{\substack{k=0\\ k\equiv 0\ (\bmod\ 2)}}^{M-1}k+2\beta(1-\beta)\sum_{\substack{k=0\\ k\equiv 1\ (\bmod\ 2)}}^{M-1}k+\sum_{k=0}^{M-1}o(k)+O(\sqrt{N}).$$

Since $\sum_{\substack{k=0 \\ k \equiv r \pmod 2}}^{M-1} k = \frac{M^2}{4} + O(M)$ for $r \in \{0, 1\}$ and since $\sum_{k=0}^{M-1} o(k) = o(M^2) = o(N)$ and since $\frac{M^2}{2} \sim N$ for $N \to \infty$ we obtain

$$\sum_{n=0}^{N-1} |x_{n+1} - x_n| = (\alpha(1-\alpha) + \beta(1-\beta))N + o(N).$$

Now by construction, $2\alpha(1-\alpha) = \gamma - c$ and $2\beta(1-\beta) = \gamma + c$ and the result follows. □

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