Distribution properties of sequences generated by Q-additive functions with respect to Cantor representation of integers

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Abstract

In analogy to ordinary q-additive functions based on q-adic expansions one may use Cantor expansions with a Cantor base Q to define (strongly) Q-additive functions. This paper deals with distribution properties of multi-dimensional sequences which are generated by such Q-additive functions. If in each component we have the same Cantor base Q, then we show that uniform distribution already implies well distribution and we provide an *if and only if* condition under which such sequences are uniformly distributed modulo one. For different Cantor bases in the single coordinate directions the question for uniform distribution becomes much more involved. We give a criterion which is sufficient and, in the case of strongly Q-additive functions, also necessary.

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1 Introduction

A sequence $(\boldsymbol{x}_n)_{n\geq 0}$ in $\mathbb{R}^s$ is said to be *uniformly distributed modulo one* if for all intervals $[\boldsymbol{a},\boldsymbol{b}) \subseteq [0,1)^s$ we have

$$\lim_{N\to\infty} \frac{\#\{n\,:\,0\leq n<N, \{\boldsymbol{x}_n\}\in[\boldsymbol{a},\boldsymbol{b})\}}{N} = \lambda_s([\boldsymbol{a},\boldsymbol{b})), \tag{1}$$

where λ_s denotes the s-dimensional Lebesgue measure and $\{\boldsymbol{x}\}$ denotes the fractional part of a vector $\boldsymbol{x}$ applied component wise. Furthermore, a sequence $(\boldsymbol{x}_n)_{n\geq 0}$ in $\mathbb{R}^s$ is said to be *well distributed modulo one* if for all intervals $[\boldsymbol{a},\boldsymbol{b}) \subseteq [0,1)^s$ we have

$$\lim_{N\to\infty} \frac{\#\{n\,:\,\nu\leq n<\nu+N, \{\boldsymbol{x}_n\}\in[\boldsymbol{a},\boldsymbol{b})\}}{N} = \lambda_s([\boldsymbol{a},\boldsymbol{b})) \qquad \text{uniformly in } \nu\in\mathbb{N}_0. \tag{2}$$

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Of course, a sequence that is well distributed modulo one is also uniformly distributed modulo one but the converse is not true in general.

Quantitative versions of (1) resp. (2) are often stated in terms of discrepancy resp. uniform discrepancy. For a sequence $\omega = (\boldsymbol{x}_n)_{n\geq 0}$ in $\mathbb{R}^s$ the *discrepancy* is defined by

$$D_N(\omega) = \sup_{\boldsymbol{a}\leq\boldsymbol{b}} \left| \frac{\#\{n : 0 \leq n < N, \{\boldsymbol{x}_n\} \in [\boldsymbol{a},\boldsymbol{b})\}}{N} - \lambda_s([\boldsymbol{a},\boldsymbol{b}))\right|,$$

where the supremum is extended over all sub-intervals $[\boldsymbol{a},\boldsymbol{b})$ of the unit-cube $[0,1)^s$. The so-called *uniform discrepancy* is defined as

$$\widetilde{D}_N(\omega) = \sup_{\nu\in\mathbb{N}_0} D_N((\boldsymbol{x}_{n+\nu})_{n\geq 0}).$$

A sequence is uniformly distributed modulo one if and only if its discrepancy tends to zero as N goes to infinity and it is well distributed modulo one if and only if its uniform discrepancy tends to zero as N goes to infinity.

An excellent introduction into these and related topics can be found in the book of Kuipers and Niederreiter [14] or in the book of Drmota and Tichy [4]. See also [17].

In this paper we consider uniform and well distribution properties of special sequences which are generated by so-called *Q-additive functions*, with respect to a *Cantor digit expansion* with base $Q = \{q_0, q_1, \ldots\}$ where $q_i \geq 2$ are integers for all $i \in \mathbb{N}_0$.

Details about Cantor digit expansions (sometimes also called mixed-radix systems) in general can be found, e.g., in [13]. We will call $Q = \{q_0, q_1, \ldots\}$ with integers $q_i \geq 2$ for all $i \in \mathbb{N}_0$ a *Cantor base* and we set $Q_0 := 1$, $Q_k := q_0 \cdots q_{k-1}$ for $k \in \mathbb{N}$ (we can, e.g., take $Q_k = (k+1)!$). The special case of ordinary q-adic expansions, $q \geq 2$ an integer, is contained if we choose $q_0 = q_1 = \ldots = q$ and hence $Q_k = q^k$. The main difference between Q-adic and ordinary q-adic expansions is that in the general case the i-th digit can take values in $\{0, \ldots, q_i - 1\}$, which may vary for each i and even become arbitrarily large. Each integer n possesses a unique finite representation

$$n = n_0 + n_1 q_0 + n_2 q_0 q_1 + \cdots = \sum_{i\geq 0} n_i Q_i, \quad \text{with} \quad n_i \in \{0, \ldots, q_i - 1\} \quad \text{for} \quad i \in \mathbb{N}_0.$$

We will call this the *Q-adic expansion* or the *Cantor expansion* of n. Additionally, each real number $x \in [0,1)$ has a representation of the form

$$x = \frac{x_0}{q_0} + \frac{x_1}{q_0 q_1} + \frac{x_2}{q_0 q_1 q_2} + \cdots = \sum_{i\geq 0} \frac{x_i}{Q_{i+1}}, \quad \text{with} \quad x_i \in \{0, \ldots, q_i - 1\} \quad \text{for} \quad i \in \mathbb{N}_0.$$

Let $Q = \{q_0, q_1, \ldots\}$ be a Cantor base. A function $f : \mathbb{N}_0 \to \mathbb{R}$ is called *Q-additive* if for $n \in \mathbb{N}_0$ with Cantor expansion $n = n_0 + n_1 q_0 + n_2 q_0 q_1 + \cdots$ we have

$$f(n) = f^{(0)}(n_0) + f^{(1)}(n_1) + f^{(2)}(n_2) + \cdots,$$

with a sequence of functions $f^{(i)} : \mathbb{N}_0 \to \mathbb{R}$, $i \geq 0$. Because the domains of definition of the $f^{(i)}$ exceed the ranges of the n_i, the $f^{(i)}$ are not uniquely determined by f. If in addition there exist $f^{(i)}$ and an $f^* : \mathbb{N}_0 \to \mathbb{R}$ such that

$$f^{(0)} = f^{(1)} = f^{(2)} = \ldots = f^*,$$

then f is called *strongly Q-additive*. For the q-adic case see, for example, [4, 5, 10].

Remark 1 Note that we want the sum-of-digits function to be a strongly Q-additive function, so we can not simply define strong Q-additivity by the condition

$$f(n_0 + n_1Q_1 + \dots) = f(n_0) + f(n_1) + \dots \tag{3}$$

as would perhaps seem natural following the ordinary q-adic example. Indeed, consider the example $Q = \{3, 5, \dots\}$ and f equal to the sum-of-digits function, s_Q. Then $f^*(n) = n$ and $f(3) = f(0+1\cdot 3) = f^*(0)+f^*(1) = f^*(1) = 1$ and similarly $f(9) = f^*(3) = 3 \neq f(3)$, which would lead to contradictions under condition (3). Therefore, to avoid the recursivity which causes this contradiction we distinguish the function f from the 'digit function' f^*.

An example for a Q-additive function is the function $n \mapsto \alpha n$, or more general, the weighted sum-of-digits function of the Cantor expansion, defined for a sequence $\boldsymbol{\gamma} = (\gamma_i)_{i\geq 0}$ by $s_{Q,\boldsymbol{\gamma}}(n) = n_0\gamma_0 + n_1\gamma_1 + \cdots$ if $n \in \mathbb{N}_0$ has Cantor expansion $n = n_0 + n_1q_0 + \cdots$. If the weights γ_i are constant, then $s_{Q,\boldsymbol{\gamma}}$ is even strongly Q-additive. By choosing $\gamma_i = Q_{i+1}^{-1}$ we obtain the 'Cantor version' of the van der Corput radical inverse function. For $\gamma_i = \alpha Q_i$ we obtain the function $n \mapsto \alpha n$ and for $\gamma_i = \alpha$ we obtain the function $n \mapsto \alpha s_Q(n)$, where $s_Q(n)$ is the usual (unweighted) Cantor sum-of-digits function. Hence all these functions are examples for Q-additive functions.

For Cantor bases $Q^{(1)}, \dots, Q^{(s)}$ and $1 \leq i \leq s$, let f_i denote a $Q^{(i)}$-additive function and let $\boldsymbol{f} : \mathbb{N}_0 \to \mathbb{R}^s$, $\boldsymbol{f}(n) = (f_1(n), \dots, f_s(n))$. In the case of strongly Q-additive functions we write $\boldsymbol{f}^*$ for $(f_1^*, \dots, f_s^*)$. Now we consider the s-dimensional sequence

$$\omega_{\boldsymbol{f}} := (\boldsymbol{f}(n))_{n\geq 0}. \tag{4}$$

When f is a one-dimensional, ordinary q-additive function, then it is known, that if the sequence (4) is of uniform distribution modulo one, then it is already well distributed. In this paper we give a quantitative, multi-dimensional version of this fact for Q-additive functions in terms of discrepancy. It is then the aim of this paper to give an *if and only if* condition under which the sequence (4) is uniformly distributed modulo one in the case that $Q^{(1)} = \dots = Q^{(s)} =: Q$. Such a condition was given in the case of the weighted q-adic sum-of-digits function in [16]. For the one-dimensional q-additive case such conditions were proved in [11]. Further more, for strongly Q-additive functions we provide also quantitative results in terms of discrepancy.

In the case of different but pairwise coprime Cantor bases $Q^{(1)}, \dots, Q^{(s)}$ (meaning that $\gcd(Q_k^{(i)}, Q_l^{(j)}) = 1$ for all $i, j \in \{1, \dots, s\}$, $k, l \geq 0$) we can give a sufficient condition for uniform distribution modulo one and, in case that for each $i \in \{1, \dots, s\}$ we have that f_i is strongly $Q^{(i)}$-additive, also a necessary one.

In [2] well distribution properties of one-dimensional sequences $(\alpha f(n))_{n\geq 0}$ for irrational α and strongly q-additive functions f attaining only non-negative integer values are studied in more detail. Of course, the sequences given by (4) contain such sequences as special case. Results on one-dimensional Q-additive functions that slightly improve ours and various special cases can be found in [9].

We close the introduction with some notation: throughout the paper let the dimension $s \in \mathbb{N}$ be fixed. By $\boldsymbol{x} \cdot \boldsymbol{y}$ we denote the usual inner product of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^s$, $\lfloor \cdot \rfloor$ denotes the *integer-part function* and $\| \cdot \|$ the *distance-to-the-nearest-integer function*. Finally, if $\boldsymbol{f}$ is an s-dimensional vector of Q-additive functions with the same base Q in each component, we set $\boldsymbol{f}^{(l)} := (f_1^{(l)}, \dots, f_s^{(l)})$, where $f_i^{(l)}(a) = f_i(aQ_l)$ (i.e., the upper indices have the same meaning as in the definition of Q-additivity) for $l \geq 0, a < q_{i,l}, i \in \{1, \dots, s\}$. Analogously in the case of strongly Q-additive functions for $\boldsymbol{f}^*$.

2 Results for equal Cantor bases

It was first shown by Coquet [1] that a one-dimensional uniformly distributed sequence which is generated by a q-additive function is already well distributed. Here we give a quantitative version of this fact in terms of discrepancy. We consider the more general multi-dimensional Cantor case.

Theorem 1 *Let Q be a Cantor base and let $\boldsymbol{f} : \mathbb{N}_0 \to \mathbb{R}^s$, $\boldsymbol{f}(n) = (f_1(n), \ldots, f_s(n))$, where each function f_i is Q-additive. Then we have*

$$\widetilde{D}_N(\omega_{\boldsymbol{f}}) \ll_s \left(q_{k_N} D_{\lfloor\sqrt{N}\rfloor}(\omega_{\boldsymbol{f}})\right)^{\frac{1}{s+1}},$$

where k_N is such that $Q_{k_N} \leq \sqrt{N} < q_{k_N} Q_{k_N} = Q_{k_N+1}$. (In the case of ordinary q-adic expansions we simply have $q_{k_N} = q$.)

Proof of Theorem 1. First we use a technique from [2]. Let $\nu \in \mathbb{N}_0$ be fixed. For $N \in \mathbb{N}$ choose k such that $Q_k \leq N$ and m_1, m_2 such that $(m_1 - 1)Q_k \leq \nu < m_1 Q_k$ and $m_2 Q_k \leq \nu + N - 1 < (m_2 + 1)Q_k - 1$. Then for $\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}$ we have

$$\begin{aligned}
\left|\sum_{n=\nu}^{\nu+N-1} \mathrm{e}^{2\pi \mathtt{i} \boldsymbol{h}\cdot\boldsymbol{f}(n)}\right| &\leq 2Q_k + \sum_{t=m_1}^{m_2-1} \left|\sum_{n=tQ_k}^{(t+1)Q_k-1} \mathrm{e}^{2\pi \mathtt{i} \boldsymbol{h}\cdot\boldsymbol{f}(n)}\right| \\
&= 2Q_k + \sum_{t=m_1}^{m_2-1} \left|\sum_{n=0}^{Q_k-1} \mathrm{e}^{2\pi \mathtt{i} \boldsymbol{h}\cdot\boldsymbol{f}(n+tQ_k)}\right| \\
&= 2Q_k + (m_2 - m_1)\left|\sum_{n=0}^{Q_k-1} \mathrm{e}^{2\pi \mathtt{i} \boldsymbol{h}\cdot\boldsymbol{f}(n)}\right|.
\end{aligned}$$

We have $N + m_1 Q_k - 1 \geq N + \nu - 1 \geq m_2 Q_k$ and hence $m_2 - m_1 \leq N/Q_k$. Let k_N be maximal such that $Q_{k_N} \leq \sqrt{N}$. Therefore we find that for all $\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}$,

$$\begin{aligned}
\left|\frac{1}{N}\sum_{n=\nu}^{\nu+N-1} \mathrm{e}^{2\pi \mathtt{i} \boldsymbol{h}\cdot\boldsymbol{f}(n)}\right| &\leq \min_{k \leq k_N} \left(\frac{2Q_k}{N} + \left|\frac{1}{Q_k}\sum_{n=0}^{Q_k-1} \mathrm{e}^{2\pi \mathtt{i} \boldsymbol{h}\cdot\boldsymbol{f}(n)}\right|\right) \\
&\ll_s \min_{k \leq k_N} \left(\frac{2Q_k}{N} + r(\boldsymbol{h}) D_{Q_k}(\omega_{\boldsymbol{f}})\right) \\
&\ll_s q_{k_N} r(\boldsymbol{h}) D_{\lfloor\sqrt{N}\rfloor}(\omega_{\boldsymbol{f}}),
\end{aligned}$$

where for the second inequality we used [15, Corollary 3.17] and where for $\boldsymbol{h} = (h_1, \ldots, h_s) \in \mathbb{Z}^s$ we define $r(\boldsymbol{h}) = \prod_{i=1}^s \max\{1, |h_i|\}$. Now we use the Erdős-Turán-Koksma inequality (see, for example, [4, Theorem 1.21]), from which we obtain for all $H \in \mathbb{N}$, that

$$D_N((\boldsymbol{f}(n+\nu))_{n\geq 0}) \ll_s \frac{1}{H} + \sum_{0 < \|\boldsymbol{h}\|_\infty \leq H} \frac{1}{r(\boldsymbol{h})}\left|\frac{1}{N}\sum_{n=\nu}^{\nu+N-1} \mathrm{e}^{2\pi \mathtt{i} \boldsymbol{h}\cdot\boldsymbol{f}(n)}\right| \ll_s \frac{1}{H} + H^s q_{k_N} D_{\lfloor\sqrt{N}\rfloor}(\omega_{\boldsymbol{f}}).$$

Choosing $H = \left\lfloor (q_{k_N} D_{\lfloor\sqrt{N}\rfloor}(\omega_{\boldsymbol{f}}))^{-1/(s+1)} \right\rfloor$ we get $D_N((\boldsymbol{f}(n+\nu))_{n\geq 0}) \ll_s \left(q_{k_N} D_{\lfloor\sqrt{N}\rfloor}(\omega_{\boldsymbol{f}})\right)^{\frac{1}{s+1}}$ uniformly in $\nu \in \mathbb{N}_0$ and hence the result follows. $\square$

We give a full characterization of Q-additive functions $\boldsymbol{f} : \mathbb{N}_0 \to \mathbb{R}^s$ for which the sequence (4) is uniformly (resp. well) distributed modulo one. The proof is based on estimates for exponential sums and Weyl's criterion for uniform distribution modulo one (see, for example, [4, 14]).

Theorem 2 *Let Q be a Cantor base and let $\boldsymbol{f} : \mathbb{N}_0 \to \mathbb{R}^s$, $\boldsymbol{f}(n) = (f_1(n), \ldots, f_s(n))$, where each function f_i is Q-additive. Then the sequence $\omega_{\boldsymbol{f}}$ is uniformly distributed modulo one if and only if for every $\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}$ one of the following properties holds:*

Either

$$\sum_{k=0}^{\infty} \frac{1}{q_k^2} \sum_{a=1}^{q_k-1} \|\boldsymbol{h} \cdot \boldsymbol{f}^{(k)}(a)\|^2 = \infty$$

or there exists at least one $k \in \mathbb{N}_0$ such that

$$\sum_{a=0}^{q_k-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h} \cdot \boldsymbol{f}^{(k)}(a)} = 0.$$

Before we give the proof of this result we state a corollary for strongly Q-additive functions and we give some examples.

Corollary 1 *Let $Q = \{q_0, q_1, \ldots\}$ be a Cantor base such that $\sum_{k \geq 0} 1/q_k^2 = \infty$. Set q_{AP} equal to the maximal finite accumulation point of the sequence q_i if one exists and $q_{AP} := \infty$ else, i.e., if there are either zero or infinitely many finite accumulation points. Let*

$$q^* := \begin{cases} q_{AP} & \text{if } q_{AP} < \infty, \sum_{\substack{k \geq 0, \\ q_k > q_{AP}}} 1/q_k^2 < \infty, \\ \\ \infty & \text{if } q_{AP} < \infty, \sum_{\substack{k \geq 0, \\ q_k > q_{AP}}} 1/q_k^2 = \infty \text{ or if } q_{AP} = \infty. \end{cases} \tag{5}$$

Now let $\boldsymbol{f} : \mathbb{N}_0 \to \mathbb{R}^s$, $\boldsymbol{f}(n) = (f_1(n), \ldots, f_s(n))$, where each function f_i is strongly Q-additive. Then the sequence $\omega_{\boldsymbol{f}}$ is uniformly distributed modulo one if for every $\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}$ there is an a, $1 \leq a < q^*$, such that $\boldsymbol{h} \cdot \boldsymbol{f}^*(a) \notin \mathbb{Z}$.

For all Cantor bases Q such that

$$\text{either } q_k \text{ is bounded or} \qquad \forall a \geq 0 : \sum_{\substack{k \geq 0, \\ q_k > a}} \frac{1}{q_k^2} = \infty \qquad \text{holds,} \tag{6}$$

the statement can be sharpened to an equivalence. (Of the cases considered in the first part this excludes Q such that $q_{AP} < \infty, \limsup_{k \geq 0} q_k = \infty$. See also Example 3.)

The proof of Corollary 1 will be given subsequent to the proof of Theorem 2.

Example 1 Let Q be a Cantor base with $\sum_{k>0} 1/q_k^2 = \infty$. Consider the two-dimensional sequence $\omega_{Q,\alpha}$ where the first component is the Q-adic van der Corput sequence and the second component is the sequence $(\alpha s_Q(n))_{n \geq 0}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, where $s_Q(n)$ denotes the sum-of-digits function with respect to the Cantor expansion Q. Hence $f_1(n) = n_0/Q_1 + n_1/Q_2 + \cdots$ and $f_2(n) = n_0\alpha + n_1\alpha + \cdots$ whenever $n = n_0 + n_1Q_1 + n_2Q_2 + \cdots$.

Both functions are Q-additive and we have $\boldsymbol{f}^{(k)}(a) = \big(a/Q^{k+1}, a\alpha\big)$. For $\boldsymbol{h} = (h_1, h_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ we consider two cases. If $h_2 = 0$, then $h_1 \neq 0$. Choose $k \in \mathbb{N}_0$ maximal such that $Q_k | h_1$. Then we have $\sum_{a=0}^{q_k-1} \mathrm{e}^{2\pi \mathrm{i} h_1 a/Q_{k+1}} = 0$. If $h_2 \neq 0$ we have

$$\sum_{k=0}^{\infty} \frac{1}{q_k^2} \sum_{a=1}^{q_k-1} \left\| h_1 \frac{a}{Q_{k+1}} + h_2 a\alpha \right\|^2 = \infty.$$

Hence the sequence $\omega_{Q,\alpha}$ is uniformly distributed modulo one for irrational α.

Example 2 Let $\boldsymbol{f}, Q, q^*$ be as in Corollary 1. If there is an a, $1 \leq a < q^*$ such that $1, f_1^*(a), \ldots, f_s^*(a)$ are linearly independent over $\mathbb{Q}$, then the sequence $\omega_{\boldsymbol{f}}$ is uniformly distributed modulo one.

Example 3 Consider the Cantor base $Q = \{2, 4, 2, 8, 2, 16, 2, \ldots\}$ together with the strongly Q-additive one-dimensional function f given through f^* by

$$f^* = \left\langle 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \ldots \right\rangle, \text{ i.e., } f^*(n) := \begin{cases} 0 & \text{if } 0 \leq n < 2, \\ 2^{-\lfloor \log_2 n \rfloor} & \text{if } n \geq 2. \end{cases}$$

Then by the second condition of Theorem 2, $f(n)$ is uniformly distributed modulo 1, however there is no $a, 1 \leq a < q^* = 2$, such that $hf^*(a) \notin \mathbb{Z}$.

Note that this function is closely related to the *binary* van der Corput radical inverse function which itself is only q-additive but not strongly. Similar f^* and f can be constructed with respect to *arbitrary* Cantor bases Q' and any q^*.

Proof of Theorem 2. Let $\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}$. For fixed $k \in \mathbb{N}_0$ and $u \in \{0, \ldots, q_k - 1\}$ we have

$$\left| \sum_{a=0}^{q_k-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h} \cdot \boldsymbol{f}^{(k)}(a)} \right| \leq q_k - 4 \|\boldsymbol{h} \cdot \boldsymbol{f}^{(k)}(u)\|^2$$

and hence

$$\left| \sum_{a=0}^{q_k-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h} \cdot \boldsymbol{f}^{(k)}(a)} \right| \leq q_k - \frac{4}{q_k} \sum_{a=1}^{q_k-1} \|\boldsymbol{h} \cdot \boldsymbol{f}^{(k)}(a)\|^2 =: q_k - \nu_k(\boldsymbol{h}).$$

For $\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}$ and $k \in \mathbb{N}_0$ we say '$*_k$ holds', if $\sum_{a=0}^{q_k-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h} \cdot \boldsymbol{f}^{(k)}(a)} = 0$.

For $j \in \mathbb{N}_0$ we have

$$\left| \frac{1}{Q_j} \sum_{n=0}^{Q_j-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h} \cdot \boldsymbol{f}(n)} \right| = \frac{1}{Q_j} \prod_{k=0}^{j-1} \left| \sum_{a=0}^{q_k-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h} \cdot \boldsymbol{f}^{(k)}(a)} \right| \leq \prod_{k=0}^{j-1} \frac{q_k - \nu_k(\boldsymbol{h})}{q_k} \prod_{\substack{k=0 \\ *_k \text{ holds}}}^{j-1} 0,$$

where here and later on an empty product is considered to be one.

Let $N \in \mathbb{N}$ with Cantor base Q representation $N = N_0 + N_1 Q_1 + \cdots + N_m Q_m$ with $N_m \neq 0$. As in [16] for the special case of q-adic weighted sum-of-digits function we can show that

$$\left| \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h} \cdot \boldsymbol{f}(n)} \right| \leq \sum_{j=0}^{r-1} N_j Q_j + \sum_{j=r}^{m} N_j Q_j \prod_{k=0}^{j-1} \frac{q_k - \nu_k(\boldsymbol{h})}{q_k} \prod_{\substack{k=0 \\ *_k \text{ holds}}}^{j-1} 0.$$

for any $r \in \mathbb{N}_0$.

If there exists a $k \in \mathbb{N}_0$ such that $*_k$ holds, then let k_0 be minimal with this property. Then we have

$$\left|\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot \boldsymbol{f}(n)}\right| \leq \sum_{j=0}^{k_0} N_j Q_j \prod_{k=0}^{j-1} \frac{q_k - \nu_k(\boldsymbol{h})}{q_k} \leq \sum_{j=0}^{k_0} (q_j - 1) Q_j = Q_{k_0+1} - 1.$$

If for all $k \in \mathbb{N}_0$ the condition $*_k$ does not hold, then we have

$$\left|\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot \boldsymbol{f}(n)}\right| \leq Q_r + N \prod_{k=0}^{r-1} \frac{q_k - \nu_k(\boldsymbol{h})}{q_k}. \tag{7}$$

Define $x_r := Q_r / \left(\prod_{k=0}^{r-1} \frac{q_k - \nu_k(\boldsymbol{h})}{q_k}\right) \geq Q_r$ and choose r such that $x_r \leq N < x_{r+1}$. Then we have

$$Q_r \leq N \prod_{k=0}^{r-1} \frac{q_k - \nu_k(\boldsymbol{h})}{q_k}. \tag{8}$$

Since $\nu_k(\boldsymbol{h}) \leq \frac{4}{q_k} \frac{q_k - 1}{4} < 1$ we have on the other hand that

$$\prod_{k=0}^{r} \frac{q_k - \nu_k(\boldsymbol{h})}{q_k} \geq \prod_{k=0}^{r} \frac{1}{q_k} = \frac{1}{Q_{r+1}}$$

and hence

$$N < Q_{r+1} / \left(\prod_{k=0}^{r} \frac{q_k - \nu_k(\boldsymbol{h})}{q_k}\right) \leq Q_{r+1}^2.$$

Thus we have $r > r_N$, where r_N is minimal such that $Q_{r_N} \geq \lfloor \sqrt{N} \rfloor$. Hence

$$\prod_{k=0}^{r-1} \frac{q_k - \nu_k(\boldsymbol{h})}{q_k} \leq \prod_{k=0}^{r_N-1} \frac{q_k - \nu_k(\boldsymbol{h})}{q_k}. \tag{9}$$

From (7), (8) and (9) we find

$$\left|\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot \boldsymbol{f}(n)}\right| \leq 2N \exp\left(-\sum_{k=0}^{r_N-1} \frac{4}{q_k^2} \sum_{a=1}^{q_k-1} \|\boldsymbol{h} \cdot \boldsymbol{f}^{(k)}(a)\|^2\right). \tag{10}$$

In both of the above cases we obtain $\frac{1}{N}\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot \boldsymbol{f}(n)} \to 0$ as $N \to \infty$. Hence the result follows by Weyl's criterion.

Assume now that there is a $\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}$ such that $\sum_{k=0}^{\infty} \frac{1}{q_k^2} \sum_{a=1}^{q_k-1} \|\boldsymbol{h} \cdot \boldsymbol{f}^{(k)}(a)\|^2 < \infty$ and for all $k \in \mathbb{N}_0$ we have that $*_k$ does not hold, i.e., $\sum_{a=0}^{q_k-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot \boldsymbol{f}^{(k)}(a)} \neq 0$. Then for $j \in \mathbb{N}_0$ we have

$$\left|\frac{1}{Q_j} \sum_{n=0}^{Q_j-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot \boldsymbol{f}(n)}\right| = \frac{1}{Q_j} \prod_{k=0}^{j-1} \left|\sum_{a=0}^{q_k-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot \boldsymbol{f}^{(k)}(a)}\right| \neq 0.$$

Using [16, Lemma 1] we obtain

$$\left|\sum_{a=0}^{q_k-1} \mathrm{e}^{2\pi \mathbf{i} \boldsymbol{h}\cdot \boldsymbol{f}^{(k)}(a)}\right| \geq q_k \left(1-\pi^2 \nu_k(\boldsymbol{h})\right).$$

Let $0<c<1$ and let $l \in \mathbb{N}$ be large enough such that $1-\pi^2 \sum_{k>l} \nu_k(\boldsymbol{h}) > c > 0$. For $j>l$ we have

$$\begin{aligned}
\left|\frac{1}{Q_j}\sum_{n=0}^{Q_j-1} \mathrm{e}^{2\pi \mathbf{i} \boldsymbol{h}\cdot \boldsymbol{f}(n)}\right| &\geq \prod_{k=0}^{l} \frac{1}{q_k}\left|\sum_{a=0}^{q_k-1} \mathrm{e}^{2\pi \mathbf{i} \boldsymbol{h}\cdot \boldsymbol{f}^{(k)}(a)}\right| \prod_{k=l+1}^{j-1} \left(1-\pi^2 \nu_k(\boldsymbol{h})\right) \\
&\geq c' \left(1-\pi^2 \sum_{k>l} \nu_k(\boldsymbol{h})\right) > c'\cdot c > 0.
\end{aligned}$$

and by Weyl's criterion $\omega_{\boldsymbol{f}}$ is not uniformly distributed modulo one. □

Proof of Corollary 1. If each f_i, $1 \leq i \leq s$, is strongly Q-additive, then the condition from Theorem 2 reads as follows: for every $\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}$ one of the following properties holds:

Either

$$\sum_{k=0}^{\infty} \frac{1}{q_k^2} \sum_{a=1}^{q_k-1} \|\boldsymbol{h}\cdot \boldsymbol{f}^*(a)\|^2 = \infty$$

or there exists a $k \in \mathbb{N}_0$ such that $\sum_{a=0}^{q_k-1} \mathrm{e}^{2\pi \mathbf{i} \boldsymbol{h}\cdot \boldsymbol{f}^*(a)} = 0$.

Assume that for every $\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}$ there exists an a', $1 \leq a' < q^*$, such that $\boldsymbol{h}\cdot \boldsymbol{f}^*(a') \notin \mathbb{Z}$. We want to show equidistribution and distinguish two cases:

1. $q^* = q_{AP} < +\infty$. Then

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{q_k^2} \sum_{a=1}^{q_k-1} \|\boldsymbol{h}\cdot \boldsymbol{f}^*(a)\|^2 &\geq \frac{1}{(q^*)^2} \sum_{\substack{k=0\\ q_k=q^*}}^{\infty} \sum_{a=1}^{q^*-1} \|\boldsymbol{h}\cdot \boldsymbol{f}^*(a)\|^2 \\
&\geq \frac{\|\boldsymbol{h}\cdot \boldsymbol{f}^*(a')\|^2}{(q^*)^2} \sum_{\substack{k=0\\ q_k=q^*}}^{\infty} 1
\end{aligned}$$

   and the last sum diverges since there are infinitely many values for $k \in \mathbb{N}_0$ such that $q_k = q^*$.

2. $q^* = \infty$. Note that in all cases, either if $q_{AP} < \infty$ and the required sum diverges or if $q_{AP} = \infty$, i.e., q_k has no or infinitely many accumulation points the second condition of (6), $\sum_{q_k>a} q_k^{-2} = \infty$ for all $a \geq 0$, holds. Hence

$$\sum_{k=1}^{\infty} \frac{1}{q_k^2} \sum_{a=1}^{q_k-1} \|\boldsymbol{h}\cdot \boldsymbol{f}^*(a)\| \geq \sum_{\substack{k=1\\ q_k>a'}}^{\infty} \frac{1}{q_k^2} \|\boldsymbol{h}\cdot \boldsymbol{f}^*(a')\| = \infty.$$

In any of the two cases the sequence $\omega_{\boldsymbol{f}}$ is uniformly distributed modulo one.

Now assume that $\omega_{\boldsymbol{f}}$ is uniformly distributed modulo one but there exists an $\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}$ such that for every a, $1 \leq a < q^*$, we have $\boldsymbol{h}\cdot \boldsymbol{f}^*(a) \in \mathbb{Z}$. We distinguish the same two cases, slightly enhancing the requirements in the first case for this direction:

1. $q^* = q_{AP} = \limsup_{k\geq 0} q_k < +\infty$, i.e., the case of bounded q_k remains. Since for a uniformly distributed sequence each coordinate sequence has to be uniformly distributed as well it is enough to consider the case $s = 1$ only. Fix the integer $h \neq 0$ such that for all a, $1 \leq a < q^*$ we have $h \cdot f^*(a) \in \mathbb{Z}$. W.l.o.g. we may assume that $h > 0$. Define the union of intervals

$$I := \bigcup_{z=0}^{h-1} \left[\frac{z}{h}, \frac{z}{h} + \epsilon\right)$$

with $\epsilon > 0$ small enough to be determined later. The set $J := \{k \in \mathbb{N}_0 : q_k > q^*\}$ is finite. We distinguish two cases:

   (a) If J is empty, then for any $n \geq 0$ with Cantor expansion $\sum_{i\geq 0} n_i Q_i$ we get $hf(n) = hf^*(n_0) + hf^*(n_1) + \cdots = z \in \mathbb{Z}$, hence $\{f(n)\} \in I$ for all $n \in \mathbb{N}_0$. But $\lambda(I) = h\epsilon < 1$ for $\epsilon > 0$ small enough it follows that $(f(n))_{n\geq 0}$ is not uniformly distributed modulo one.

   (b) If $1 \leq |J| < \infty$, then J contains a maximal element $\overline{k}$. For $l > \overline{k}$ we define $N_l = Q_l$. and will deduce

$$\#\{n : 0 \leq n < N_l, f(n) \in I\} \geq \frac{N_l}{\prod_{k\in J} q_k}. \tag{11}$$

   For any $n = \sum_{i\geq 0} n_i Q_i$ with $n_j = 0$ for all $j \in J$ we have $hf(n) \in \mathbb{Z}$ and $\{f(n)\} \in I$ as in the case above.
   Since $\#\{n : 0 \leq n < N_l : n_j = 0 \text{ for all } j \in J\} = \frac{N_l}{\prod_{k\in J} q_k}$ the inequality (11) holds true for all N_l with $l > \overline{k}$. So for ϵ chosen appropriately we have

$$\frac{\#\{n : 0 \leq n < N, \{f(n)\} \in I\}}{N} \geq \frac{1}{\prod_{k\in J} q_k} \neq h\epsilon = \lambda(I) \tag{12}$$

   for infinitely many $N \in \mathbb{N}$. Thus $(f(n))_{n\geq 0}$ is not uniformly distributed modulo one.

2. $q^* = \infty$. Then $\boldsymbol{h} \cdot \boldsymbol{f}^*(a) \in \mathbb{Z}$ for all $a \geq 1$. Hence we have

$$\sum_{k=0}^{\infty} \frac{1}{q_k^2} \sum_{a=1}^{q_k-1} \|\boldsymbol{h} \cdot \boldsymbol{f}^*(a)\|^2 = 0$$

and $\sum_{a=0}^{q_k-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot \boldsymbol{f}^*(a)} = q_k$ for all $k \in \mathbb{N}_0$. This contradicts the uniform distribution modulo one of the sequence $\omega_{\boldsymbol{f}}$ by Theorem 2.

In both cases we obtained a contradiction hence there exists an a', $1 \leq a' < q^*$, such that $\boldsymbol{h} \cdot \boldsymbol{f}^*(a') \notin \mathbb{Z}$. □

We close this section with a quantitative result for strongly Q-additive functions.

A vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s)$ with irrational components α_i is said to be of approximation type η, if η is the infimum over all reals σ for which there exists a positive constant $c = c(\sigma, \boldsymbol{\alpha})$ such that $\|\boldsymbol{h} \cdot \boldsymbol{\alpha}\| \geq \frac{c}{r(\boldsymbol{h})^\sigma}$ for all $\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}$. Here $r(\boldsymbol{h})$ is as in the proof of Theorem 1.

Theorem 3 *Let Q be a Cantor base and let $\boldsymbol{f} : \mathbb{N}_0 \to \mathbb{R}^s$, $\boldsymbol{f}(n) = (f_1(n), \ldots, f_s(n))$, where each function f_i is strongly Q-additive and $q^* := \liminf_{k\geq 0} q_k \leq \infty$. If there exists an integer a, $1 \leq a < q^*$, such that $\boldsymbol{f}^*(a)$ is of approximation type η, then for every $\varepsilon > 0$ we have*

$$D_N(\omega_{\boldsymbol{f}}) \ll_{s,\boldsymbol{f},\varepsilon} \frac{1}{L_N^{\frac{1}{s}\left(\frac{1}{2\eta}-\varepsilon\right)}} \quad \text{where} \quad L_N := 4\sum_{k=0}^{r_N-1} q_k^{-2}$$

and r_N is minimal such that $Q_{r_N} \geq \sqrt{N}$. (In the special case $q_0 = q_1 = \ldots = q$ we have $1/L_N \ll_q 1/\log N$.)

Proof. From the Erdős-Turán-Koksma inequality (see, for example, [4, Theorem 1.21]), we obtain for all $H \in \mathbb{N}$,

$$\begin{aligned} D_N(\omega_{\boldsymbol{f}}) &\ll_s \frac{1}{H} + \sum_{0<\|\boldsymbol{h}\|_\infty \leq H} \frac{1}{r(\boldsymbol{h})} \left| \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h}\cdot \boldsymbol{f}(n)} \right| \\ &\ll_{s,\boldsymbol{f}} \frac{1}{H} + \sum_{0<\|\boldsymbol{h}\|_\infty \leq H} \frac{1}{r(\boldsymbol{h})} \exp(-cL_N (r(\boldsymbol{h}))^{-2(\eta+\varepsilon)}) \\ &\ll_{s,\boldsymbol{f}} \frac{1}{H} + (1+\log H)^s \exp(-cL_N H^{-2s(\eta+\varepsilon)}), \end{aligned}$$

where we have used inequality (10). With the choice $H = \left\lfloor L_N^{\frac{1}{s}\left(\frac{1}{2\eta}-\varepsilon\right)} \right\rfloor$ we obtain

$$\begin{aligned} (1+\log H)^s \exp(-cL_N H^{-2s(\eta+\varepsilon)}) &\ll_s \left(\frac{1}{2s\eta}\log L_N\right)^s \exp\left(-cL_N^{2\varepsilon^2-\frac{\varepsilon}{\eta}+2\varepsilon\eta}\right) \\ &\ll_s \left(\frac{1}{2s\eta}\log L_N\right)^s \exp\left(-cL_N^{2\varepsilon^2+\varepsilon}\right) \ll_{s,\varepsilon} \frac{1}{H} \end{aligned}$$

and hence the result follows.

For the special case $q_1 = q_2 = \ldots = q$ we note that $r_N \geq \frac{\log\sqrt{N}}{\log q}$ and hence $L_N \geq \frac{2\log N}{q^2 \log q}$.
□

3 Results for different, pairwise coprime Cantor bases

Now we turn to the case that $Q^{(1)} = \{q_{1,0}, q_{1,1}\ldots\}, \ldots, Q^{(s)} = \{q_{s,0}, q_{s,1}, \ldots\}$ are different, but *pairwise coprime*, which we define for Cantor bases by the condition $\gcd(Q_k^{(u)}, Q_l^{(v)}) = 1$ for all $u, v \in \{1, \ldots, s\}$, $k, l \geq 0$. We provide an upper bound for Weyl sums, from which we deduce distribution properties of $\omega_{\boldsymbol{f}}$.

We need some further notations: for $u \in \{1, \ldots, s\}, l \geq 0, a \geq 0$ we define

$$\begin{aligned} \theta_u^{(l)}(a) &:= f_u^{(l)}(a+1) - f_u^{(l)}(a) - f_u^{(l)}(1), \\ \delta_u^{(l)}(h_u) &:= \max\{4\|h_u\theta_u^{(l)}(a)\|^2 : 1 \leq a \leq q_{u,l}-2\}, \quad (\text{we set } \delta_u^{(l)}(h_u) := 0 \text{ for } q_{u,l} = 2) \end{aligned}$$

and then

$$\tau_u^{(l)}(h_u) := \begin{cases} \max\{\delta_u^{(l)}(h_u)/q_{u,l}^2, \delta_u^{(l+1)}(h_u)/q_{u,l+1}^2\} & \text{if this expression is } \neq 0, \\ \frac{1}{4}\|h_u(f_u^{(l+1)}(1) - q_{u,l}f_u^{(l)}(1))\|^2 & \text{else.} \end{cases}$$

Note that unless Q reduces to the ordinary q-adic case we can not omit the superscript (l) for strongly Q-additive f_u in $\delta_u^{(l)}, \tau_u^{(l)}$ since the values over which a ranges may vary with l.

For strongly Q-additive functions we set in addition $\theta_u^*(a) := f_u^*(a+1) - f_u^*(a) - f_u^*(1)$.

Proposition 1 *Let $Q^{(1)}, \ldots, Q^{(s)}$ be pairwise coprime Cantor bases and let $\boldsymbol{f} : \mathbb{N}_0 \to \mathbb{R}^s$, $\boldsymbol{f}(n) = (f_1(n), \ldots, f_s(n))$, where each function f_u is $Q^{(u)}$-additive.*

For all $\boldsymbol{h} = (h_1, \ldots, h_s) \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$, if for all $1 \leq u \leq s$ with $h_u \neq 0$ we have

$$\sum_{l=0}^{\infty} \tau_u^{(l)}(h_u) = \infty, \qquad \textit{then} \qquad \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h} \cdot \boldsymbol{f}(n)} = o(1). \tag{13}$$

In particular, the sequence $\omega_{\boldsymbol{f}}$ is uniformly distributed modulo one.

Remark 2 In a way, the first line in the definition of $\tau_u^{(l)}(h)$ measures how much the functions f_u are locally additive, modulo $(1/h)\mathbb{Z}$: the first line covers the additivity local to the digit ranges while the second considers additivity with respect to the consecutive digit functions.

In view of Proposition 1 this means that for good equidistribution convergence we are looking for f_u that are Q-additive without being 'too much' additive overall.

Proposition 1 generalizes [7, Theorem 1], which deals with the special case of ordinary weighted q_u-ary sum-of-digits functions. We will prove the proposition at the end of this section. First, we use it to show the following theorem.

Theorem 4 *Let $Q^{(1)}, \ldots, Q^{(s)}$ be pairwise coprime Cantor bases and let $\boldsymbol{f} : \mathbb{N}_0 \to \mathbb{R}^s$, $\boldsymbol{f}(n) = (f_1(n), \ldots, f_s(n))$, where each function f_u is strongly $Q^{(u)}$-additive. For each $u \in \{1, \ldots, s\}$ assume for the Cantor base $Q^{(u)}$ that* (6) *holds and that there is at least one finite accumulation point. Then $\omega_{\boldsymbol{f}}$ is uniformly distributed modulo one if and only if for all $u \in \{1, \ldots, s\}$ the u-th coordinate sequence $(f_u(n))_{n \geq 0}$ is uniformly distributed modulo one.*

Proof. Necessity is obvious because each component of a uniformly distributed sequence has to be uniformly distributed.

Now assume, that for all $1 \leq u \leq s$ the sequence $(f_u(n))_{n \geq 0}$ is uniformly distributed modulo one. Set q^* as in (5) (infinity is allowed as a value). By Corollary 1, for all $1 \leq u \leq s$ and for all integers $h \neq 0$ there exists some j, $1 \leq j < q_u^*$, such that $h f_u^*(j) \notin \mathbb{Z}$. We will show that the divergence condition in (13) is fulfilled.

First we argue that for this it is sufficient that there exists some a, $1 < a + 1 < q_u^*$, such that $h\theta_u^*(a) \notin \mathbb{Z}$ or, alternatively that there is a finite accumulation point $q_u' \leq q_u^*$ with $h(q_u' - 1) f_u^*(1) \notin \mathbb{Z}$. Either of those two conditions consequently means there is an l_0 with $\tau_u^{(l_0)}(h_u) \neq 0$ and $q_{u,l_0} \leq q^*$.

Now in case the first condition holds, since $\delta_u^{(l)}(h_u)$ is increasing as a function in $q_{u,l}$ (though not necessarily as a function in l) there is a $q_u' = q_{u,l_0}$ such that for all l with $q_{u,l} \geq q_u'$ we get $\delta_u^{(l)}(h_u) \geq \delta_u^{(l_0)}(h_u)$, and by our assumption of (6),

$$\sum_{l \geq 0} \tau_u^{(l)}(h_u) \geq \sum_{\substack{l \geq l_0, \\ q_{u,l} \geq q_u'}} \max\left\{ \frac{\delta_u^{(l)}(h_u)}{q_{u,l}^2}, \frac{\delta_u^{(l+1)}(h_u)}{q_{u,l+1}^2} \right\} \geq \delta_u^{(l_0)}(h_u) \sum_{\substack{l \geq l_0, \\ q_{u,l} \geq q_u'}} \frac{1}{q_{u,l}^2} = \infty.$$

In the second case, we have

$$\sum_{l\geq 0} \tau_u^{(l)}(h_u) \geq \tau_u^{(l_0)}(h_u) \sum_{\substack{l\geq 0\\ q_{u,l}=q'_u}} 1 = \infty.$$

We are now going to prove that one of these two conditions is always true.

If $q_u^* = 2$ we have by Corollary 1 that $hf_u^*(1) \notin \mathbb{Z}$ for all nonzero integers h and we are done in view of the second condition.

On the other hand, if $q_u^* \geq 3$ we choose j, $1 \leq j < q_u^*$, minimal such that $hf_u^*(j) \notin \mathbb{Z}$ and distinguish the following cases:

- The case $j > 1$ yields $h\theta_u^*(j-1) = h(f_u^*(j) - f_u^*(j-1) - f_u^*(1)) \notin \mathbb{Z}$ since $hf_u^*(j-1) \in \mathbb{Z}$ and $hf_u^*(1) \in \mathbb{Z}$ and we are done as we fulfill the first condition.

- For the case $j = 1$ we assume that none of the two conditions holds, which implies

$$\begin{aligned} h\theta_u^*(a) = h(f_u^*(a+1) - f_u^*(a) - f_u^*(1)) \in \mathbb{Z} & \text{ for all } a,\ 1 < a+1 < q_u^*, \quad \text{and} \\ h(q'_u - 1)f_u^*(1) \in \mathbb{Z} & \text{ for all finite acc. points } q'_u \leq q_u^*. \end{aligned}$$

  Therefore we have $\exp(2\pi \mathtt{i} h(f_u^*(a+1) - f_u^*(a) - f_u^*(1))) = 1$, and, through induction, $\exp(2\pi\mathtt{i}hf_u^*(a)) = \exp(2\pi\mathtt{i}haf_u^*(1))$ for all a, $0 \leq a < q_u^*$. We now consider $h' = h(q'_u - 1)$, where q'_u is any of the finite accumulation points. Then also $h'\theta_u^*(a) \in \mathbb{Z}$ and hence again $\exp(2\pi\mathtt{i}h'f_u^*(a)) = \exp(2\pi\mathtt{i}ah'f_u^*(1))$, for all a, $0 \leq a < q_u^*$, which equals 1 in consequence of $h(q'_u - 1)f^*(1) \in \mathbb{Z}$. But this contradicts our assumption that for all nonzero integer h' there exists some j, $0 \leq j < q_u^*$, such that $h'f_u^*(j) \notin \mathbb{Z}$.

□

Remark 3 That one finite accumulation point is needed in the condition for the Cantor base can be seen with the following $f(n)$ as counterexample that sabotages the second case of the 'sufficient' direction. Consider $f(n) = s_Q(n)\lambda$, $\lambda = \sum_{k\geq 0} 2^{-k!}$, where Q is chosen such that it contains enough q_l of the form $2^{k!} + 1$ to fulfill the divergence condition in (13).

Now we give the *Proof of Proposition 1.*

We use a technique developed by Kim [12], advanced by Drmota and Larcher [3] and further generalized by Hofer [7, 8]. To present the proof in convenient units we will highlight the main steps in several lemmas.

Our goal is to prove the convergence to zero of the Weyl sum given in the proposition. We fix an $\boldsymbol{h} \in \mathbb{Z}^s \backslash \{\mathbf{0}\}$ and introduce the notations $g_u(n) := \exp(2\pi\mathtt{i}h_u f_u(n))$ for $1 \leq u \leq s$ and $g(n) := \prod_{u=1}^{s} g_u(n)$.

The first step is to apply the following lemma, a version of the Weyl-van der Corput inequality, to $g(n)$. The appropriate choice for the quantity K will be determined at the end of the proof.

Lemma 1 *For integers $N \geq K \geq 1$ and a sequence a_n of complex numbers with $|a_n| \leq 1$ we have*

$$\left| \sum_{n=0}^{N-1} a_n \right|^2 \leq \frac{2N^2}{K} + \frac{4N}{K} \sum_{k=1}^{K} \left| \sum_{n=0}^{N-k+1} \overline{a_n} a_{n+k} \right|.$$

Proof. A proof of the inequality can be found in [6, pp.10–11]. $\square$

Terms of the form $c(k) = \sum_n \overline{a_n} a_{n+k}$ as they appear in Lemma 1 are called correlation functions. We will use several of them, sometimes based on other correlation functions. (To relieve notation we will omit the bracketing of single upper indices of functions since a confusion with powers can be ruled out, i.e., $f^i(x)$ can be clearly distinguished from $f(x)^i$. We will keep the brackets for constants, however.) For every coordinate $u \in \{1,\ldots,s\}$ we set

$$\begin{aligned}
\Phi_{1,u}^{R}(k) &:= \frac{1}{R}\sum_{n=0}^{R-1} \overline{g_u(n)}\, g_u(n+k) \quad \text{for} \quad 0 \le k \le R \le N,\\
\Phi_{2,u}^{K,R}(r) &:= \frac{1}{K}\sum_{k=0}^{K-1} \overline{\Phi_{1,u}^{R}(k)}\, \Phi_{1,u}^{R}(k+r) \quad \text{for} \quad r \in \{0,1\} \text{ and } 0 \le K \le R \le N,
\end{aligned}$$

and
$$\Psi_R(k) := \sum_{n=1}^{R} \overline{g(n)}\, g(n+k) \quad \text{for} \quad 0 \le R \le N.$$

Furthermore, an additional upper index $l \ge 0$ shall denote a shift by l digits, e.g., $\Phi_{1,u}^{R,l}(k) := \Phi_{1,u}^{R}(k\, Q_l^{(u)})$.

Observe that in applying Lemma 1 to $g(n)$ the innermost sum ranges over terms of the form $\prod_u \overline{g_u(n)} g_u(n+k)$. Our aim will be to move the product over all $u \in \{1,\ldots,s\}$ outside of all sums. For this we will use recursions holding for the correlation functions $\Phi_{2,u}^{K,N}$. To formulate them we will define several more correlation type functions $\alpha_j^{(l)}, \beta_j^{(l)}$, which are simpler in that they only are local to a digit range $\{0,\ldots,q_{u,l}-1\}$ (for some $u, l \ge 0$). For any u, the actual coefficients of the recursion are then defined in terms of $\alpha_j^{(l)}$ and $\beta_j^{(l)}$ and also of a shape similar to correlation functions. With a fixed digit place $l \ge 0$ and fixed $u \in \{1,\ldots,s\}$, we set

$$\begin{aligned}
\alpha_j^{(l)} &:= \frac{1}{q_{u,l}} \sum_{i=0}^{q_{u,l}-j-1} \overline{g_u^l(i)}\, g_u^l(i+j),\\
\beta_j^{(l)} &:= \frac{1}{q_{u,l}} \sum_{i=q_{u,l}-j}^{q_{u,l}-1} \overline{g_u^l(i)}\, g_u^l(i+j-q_{u,l}), \qquad 0 \le j \le q_{u,l};
\end{aligned}$$

$$\begin{aligned}
\lambda_r^{(l)} &:= \frac{1}{q_{u,l}} \sum_{i=0}^{q_{u,l}-1} \left(\overline{\alpha_i^{(l)}}\, \alpha_{i+r}^{(l)} + \overline{\beta_i^{(l)}}\, \beta_{i+r}^{(l)} \right),\\
\mu_r^{(l)} &:= \frac{1}{q_{u,l}} \sum_{i=0}^{q_{u,l}-1} \overline{\alpha_i^{(l)}}\, \beta_{i+r}^{(l)}, \qquad \nu_r^{(l)} := \frac{1}{q_{u,l}} \sum_{i=0}^{q_{u,l}-1} \overline{\beta_i^{(l)}}\, \alpha_{i+r}^{(l)}, \qquad r \in \{0,1\}.
\end{aligned}$$

Lemma 2 *For fixed $u \in \{1,\ldots,s\}$, any $l \ge 0, r \in \{0,1\}$, and $q := q_{u,l}$ we have the recursion in l,*

$$\Phi_{2,u}^{qK,\,qR,l}(r) = \lambda_r^{(l)} \Phi_{2,u}^{K,R,l+1}(0) + \mu_r^{(l)} \Phi_{2,u}^{K,R,l+1}(0) + \nu_r^{(l)} \overline{\Phi_{2,u}^{K,R,l+1}(0)} \quad + E_{K,R}^{l+1}(r), \qquad (14)$$

where $|E_{K,R}^{l+1}(r)| \le 2/K$. *Furthermore, for the two-step recursion in* l, *with* $q' := q_{u,l}q_{u,l+1}$, *we get the bound*

$$\left|\Phi_{2,u}^{q'K,q'R,l}(r)\right| \le \rho_r^{(l)}|\Phi_{2,u}^{K,R,l+2}(0)| + \sigma_r^{(l)}|\Phi_{2,u}^{K,R,l+2}(1)| + \frac{7}{K},$$

where

$$\begin{aligned}
\rho_r^{(l)} &:= |\lambda_r^{(l)}\,\lambda_0^{(l+1)} + \mu_r^{(l)}\,\lambda_1^{(l+1)} + \nu_r^{(l)}\,\overline{\lambda_1^{(l+1)}}|,\\
\sigma_r^{(l)} &:= |\lambda_r^{(l)}\,\mu_0^{(l+1)} + \mu_r^{(l)}\,\mu_1^{(l+1)} + \nu_r^{(l)}\,\overline{\nu_1^{(l+1)}}| + |\lambda_r^{(l)}\,\nu_0^{(l+1)} + \mu_r^{(l)}\,\nu_1^{(l+1)} + \nu_r^{(l)}\,\overline{\mu_1^{(l+1)}}|,
\end{aligned}$$

and

$$\rho_r^{(l)} + \sigma_r^{(l)} \le 1 - \tau^{(l)}(h_u)/q_{u,l}^2.$$

Proof. In view of the locality of the correlation function Φ_2 to the digit range $\{0,\dots, q_{u,l}-1\}$ it is tedious but not difficult to convince oneself that the proofs of this recursions can be carried out with only minor adaption in the same way as the ones found in [7] for the ordinary q-adic case. In particular, this also applies to the two-step recursion.

Since the very last inequality, the estimate of $\rho_r^{(l)} + \sigma_r^{(l)}$, is crucial to the proof we go our aim is now here. We have

$$\begin{aligned}
\rho_r^{(l)} + \sigma_r^{(l)} \le{}& |\lambda_r^{(l)}|(|\lambda_0^{(l+1)}| + |\mu_0^{(l+1)}| + |\nu_0^{(l+1)}|)+\\
&+ (|\mu_r^{(l)}| + |\nu_r^{(l)}|)(|\lambda_1^{(l+1)}| + |\mu_1^{(l+1)}| + |\nu_1^{(l+1)}|)
\end{aligned}$$

and

$$|\lambda_r^{(l)}| + |\mu_r^{(l)}| + |\nu_r^{(l)}| \le \frac{1}{q_{u,l}} \sum_{i=0}^{q_{u,l}-1} \big(|\alpha_i^{(l)}| + |\beta_i^{(l)}|\big)\big(|\alpha_{i+r}^{(l)}| + |\beta_{i+r}^{(l)}|\big) \;\le\; 1, \qquad \text{for } r = 0,1.$$

There are two cases to distinguish, either at least one of $\delta_u^{(l)}(h_u) \ne 0$, $\delta_u^{(l+1)}(h_u) \ne 0$ holds, or both quantities are zero. We are assuming the former — without limitation, $\delta_u^{(l)}(h_u) \ne 0$ — here, so there exists at least one a, $1 \le a \le q_{u,l} - 2$ such that $h\theta_u^{(l)}(a) \notin \mathbb{Z}$. This, together with the inequality

$$|r + s\mathrm{e}^{2\pi \mathtt{i}\theta}| \le r + s - 4s\|\theta\|^2 \quad \text{for } 0 \le s \le r,$$

leads to a bound on $|\alpha_1^{(l)}|$.

$$\begin{aligned}
|\alpha_1^{(l)}| &= \frac{1}{q_{u,l}} \left| \sum_{i=0}^{q_{u,l}-2} \mathrm{e}^{2\pi \mathtt{i} h(f_u^{(l)}(i+1) - f_u^{(l)}(i))} \right|\\
&\le \frac{1}{q_{u,l}} \left| \mathrm{e}^{2\pi \mathtt{i} h(f_u^{(l)}(1) - f_u^{(l)}(0))} + \mathrm{e}^{2\pi \mathtt{i} h(f_u^{(l)}(a+1) - f_u^{(l)}(a))} \right| + \frac{q_{u,l}-3}{q_{u,l}}\\
&= \frac{1}{q_{u,l}} \left| 1 + \mathrm{e}^{2\pi \mathtt{i} h(f_u^{(l)}(a+1) - f_u^{(l)}(a) - f_u^{(l)}(1))} \right| + \frac{q_{u,l}-3}{q_{u,l}}\\
&\le \frac{q_{u,l}-1}{q_{u,l}} - 4\frac{\|h\theta_u^{(l)}(a)\|^2}{q_{u,l}}.
\end{aligned}$$

Inserting this into the above formula and using trivial estimates for the other exponential sums α_i, β_i gives

$$\left|\lambda_r^{(l)}\right| + \left|\mu_r^{(l)}\right| + \left|\nu_r^{(l)}\right| \le 1 - 4\frac{\|h\theta_u^{(l)}(a)\|^2}{q_{u,l}^2}$$

so that after minimizing over $l, l+1$ and all $a \in \{0, \dots, q_{u,l} - 1\}$ we get

$$\rho_r^{(l)} + \sigma_r^{(l)} \le 1 - \frac{\max\{\delta_u^{(l)}(h_u), \delta_u^{(l+1)}(h_u)\}}{q_{u,l}^2} = 1 - \frac{\tau_u^{(l)}(h_u)}{q_{u,l}^2}$$

for this case of $\delta^{(l)}(h_u)$. For the second case, proceeding analogously to [7, p.42] finishes the proof. $\square$

In order to be able to apply the recursions in Lemma 2, the next result shows that we can replace K, R by their nearest multiples of $Q_z^{(u)}$ for any $z \ge 0$, introducing an error term.

Lemma 3 *Let $R \ge K$, fix $u \in \{1, \dots, s\}, z \ge 0$, so that $Q_z^{(u)} \le K$. Then, setting $Q := Q_z^{(u)}$, $L := \lfloor K/Q \rfloor$, $M := \lfloor R/Q \rfloor$, we have*

$$\Phi_{2,u}^{K,R}(0) = \Phi_{2,u}^{QL,QM}(0) + O\left(\frac{Q}{K}\right).$$

Proof. This can be shown quite easily by applying the triangle inequality and trivial estimates to $|\Phi_{1,u}^{R}(k) - \Phi_{1,u}^{QM}(k)|$ and $|\Phi_{2,u}^{K,R}(k) - \Phi_{2,u}^{QL,QM}(k)|$ (cf. the first part of the proof of [12, Prop.1]). $\square$

At the end of the proof we will for each $u \in \{1, \dots, s\}$ choose appropriate $Q_t^{(u)} = R_u$ for $\Phi_{1,u}^{R_u}$, etc. Dependent on them and K (which we will also determine at that time) we set

$$F_1 := \prod_{u=1}^{s} R_u, \qquad F_2 := \sum_{u=1}^{s} \frac{K}{R_u}.$$

We now return to the Weyl sum of $f(n)$. Using Lemma 1 and our notation we obtain the inequality

$$K\left|\sum_{n=0}^{N-1} g(n)\right|^2 \le 2N^2 + 4N\sum_{k=1}^{K} |\Psi_{N-k-1}(k)|.$$

Lemma 4 makes the connection to the correlation functions Φ_1.

Lemma 4 *For arbitrary $R_u > K$, $1 \le u \le s$ of the form $R_u = Q_t^{(u)}$, we have*

$$|\Psi_{N-k-1}(k)| = N\prod_{u=1}^{s} |\Phi_{1,u}^{R_u}(k)| + O\big(NF_2 + F_1(1 + F_2)\big).$$

Proof. (Cf. [12, Prop.2]) We start by observing that, for $r_u := n \bmod R_u$ (i.e., $r_u \equiv n \pmod{R_u}, 0 \le r_u < R_u$), whenever $r_u + k < R_u$ we can reduce the argument in the following expression to its remainder modulo R_u (cf. [12, Lemma 6], this is the place where we use the Q_u-additivity of f_u). We have

$$\begin{aligned}\overline{g(n)}g(n+k) &= \exp\left(2\pi\mathtt{i}\sum_{u=1}^{s} f_u(n+k)-f_u(n)\right)\\ &= \exp\left(2\pi\mathtt{i}\sum_{u=1}^{s} f_u(r_u+k)-f_u(r_u)\right) = \prod_{u=1}^{s}\overline{g_u(r_u)}g_u(r_u+k) =: G(\boldsymbol{r}),\end{aligned}$$

with $\boldsymbol{r}=(r_1,\dots,r_s)$. Our aim is now to bound the terms in $\Psi_{N-k-1}(k)$ where this is not possible. We define

$$\begin{aligned}\mathcal{R} &:= \{\boldsymbol{r} : 0\le r_j < R_j \text{ for all } 1\le j\le s\},\\ \mathcal{R}_0 &:= \{\boldsymbol{r} : 0\le r_j < R_j - K \text{ for all } 1\le j\le s\},\quad \mathcal{R}_1 := \mathcal{R}\setminus\mathcal{R}_0.\end{aligned}$$

Then

$$\begin{aligned}\Psi_{N-k-1}(k) &= \sum_{n=1}^{N-k-1}\overline{g(n)}g(n+k)\\ &= \sum_{\boldsymbol{r}\in\mathcal{R}_0}\sideset{}{'}\sum_{n=1}^{N-k-1}\overline{g(n)}g(n+k) + \sum_{\boldsymbol{r}\in\mathcal{R}_1}\sideset{}{'}\sum_{n=1}^{N-k-1}\overline{g(n)}g(n+k)\end{aligned}$$

(here and in the following the primed sums denote summation over those n, where $r_u = n \bmod R_u$, for all $u\in\{1,\dots,s\}$)

$$\begin{aligned}&= \sum_{\boldsymbol{r}\in\mathcal{R}} G(\boldsymbol{r})\sideset{}{'}\sum_{n=0}^{N-k-1} 1 + \sum_{\boldsymbol{r}\in\mathcal{R}_1}\sideset{}{'}\sum_{n=0}^{N-k-1}\left(\overline{g(n)}g(n+k)-G(\boldsymbol{r})\right)\\ &=: \Sigma_1+\Sigma_2.\end{aligned}$$

Now by the Chinese remainder theorem, using the condition that the Cantor bases are coprime in the sense given previously, the number of summands of the primed sums is $(N-k-1)/F_1+O(1)$ so that

$$\begin{aligned}|\Sigma_1| &\le \sum_{\boldsymbol{r}\in\mathcal{R}}\prod_{u=1}^{s}|\overline{g_u(r_u)}g_u(r_u+k)|\left(\frac{N}{F_1}+O(1)\right)\\ &= \prod_{u=1}^{s} R_u|\Phi_{1,u}^{R_u}(r_u)|\left(\frac{N}{F_1}+O(1)\right) = N\prod_{u=1}^{s}|\Phi_{1,u}^{R_u}(r_u)|+O(F_1).\end{aligned}$$

It remains to estimate $|\Sigma_2|$, for which we need a bound on the size of $|\mathcal{R}_1|$. We have

$$|\mathcal{R}_1| \le \sum_{u=1}^{s}|\{\boldsymbol{r} : 0\le r_u<R_u, R_j-K\le r_j<R_j\}| \le \sum_{u=1}^{s} K\prod_{\substack{j=1\\ j\ne u}}^{s} R_j = F_1F_2,$$

so, using trivial estimates,

$$|\Sigma_2| \le \sum_{\boldsymbol{r}\in\mathcal{R}_1}\sideset{}{'}\sum_{n=0}^{N-k-1} 2 \le 2|\mathcal{R}_1|\left(\frac{N}{F_1}+O(1)\right) \le 2F_2N+O(F_1F_2).$$

Altogether,

$$|\Psi_{N-k-1}(k)| \leq N \prod_{u=1}^{s} |\Phi_{1,u}^{R_u}(r_u)| + 2NF_2 + O(F_1(1+F_2)),$$

which concludes the proof. $\square$

We have now arrived at an inequality of the form

$$\begin{aligned}\left|\sum_{n=0}^{N-1} g(n)\right|^2 &\leq \frac{2N^2}{K} + \frac{4N^2}{K}\sum_{k=1}^{K}\prod_{u=1}^{s} |\Phi_{1,u}^{R_u}(k)| + O(N^2F_2 + NF_1(1+F_2)) \\ &=: \frac{4N^2}{K}\left(\Sigma_3 + \frac{1}{2}\right) + O(N^2F_2 + NF_1(1+F_2)).\end{aligned}$$

Lemma 4 brought the product in front of the inner sum, we now bring it in front of the outer sum using Hölder's inequality.

$$\begin{aligned}\Sigma_3 &\leq K^{1/(s+1)} \prod_{u=1}^{s}\left(\sum_{k=1}^{K} |\Phi_{1,u}^{R_u}(k)|^{s+1}\right)^{1/(s+1)} \\ &\leq K \prod_{u=1}^{s}\left(\frac{1}{K}\sum_{k=1}^{K} |\Phi_{1,u}^{R_u}(k)|^{2}\right)^{1/(s+1)} \qquad \text{(since } |\Phi_1| \leq 1\text{)} \\ &\leq K \prod_{u=1}^{s}\left(\left|\Phi_{2,u}^{K,R_u}(0)\right| + \frac{2}{K}\right)^{1/(s+1)}.\end{aligned}$$

The final lemma of the proof will use the recursions of Lemma 2 to give the asymptotics of $|\Phi_2(0)|$.

Lemma 5 *Let $u \in \{1,\ldots,s\}$ be fixed and set*

$$s(m) = s_u(m) := \frac{1}{2}\sum_{l=0}^{m-1} \frac{\tau_u^{(l)}(h_u)}{q_{u,l}^2}.$$

Then, for any $K, R_u, t \geq 0$, $Q_{2t}^{(u)} =: Q \leq K \leq R_u$, we have

$$|\Phi_{2,u}^{K,R_u}(0)| = O(\mathrm{e}^{-s_u(t)}) + O(Q/K).$$

Proof. Setting

$$s^{(i)}(m) := \sum_{\substack{l=0 \\ l\equiv i(2)}}^{m-1} \frac{\tau_u^{(l)}(h_u)}{q_{u,l}^2}, \quad i \in \{0,1\},$$

at least one of $\exp(-s^{(i)}(t)) \leq \exp(-s(t)), i = 0,1$ holds. We first assume it is $s^{(0)}(t)$.

Let $t \geq 0$. First we apply Lemma 3 to reduce the expression to $|\Phi_{2,u}^{QL,QM}(0)| + O(Q/K)$, with $L, M \geq 1$ as in Lemma 3.

Now, with $S_{2t} := |\Phi_{2,u}^{QL,QM}(0)|$, $T_{2t} := |\Phi_{2,u}^{QL,QM}(1)|$, the two-step recursion of Lemma 2 can be written in matrix form as

$$\begin{pmatrix} S_{2t} \\ T_{2t} \end{pmatrix} \leq \begin{pmatrix} \rho_0^{(2t)} & \sigma_0^{(2t)} \\ \rho_1^{(2t)} & \sigma_1^{(2t)} \end{pmatrix} \begin{pmatrix} S_{2t-2}^{(2)} \\ T_{2t-2}^{(2)} \end{pmatrix} + \frac{7}{Q_{2t-2}^{(u)} L} \begin{pmatrix} 1 \\ 1 \end{pmatrix} =: \mathcal{M}^{(2t)} \begin{pmatrix} S_{2t-2}^{(2)} \\ T_{2t-2}^{(2)} \end{pmatrix} + \frac{7}{Q_{2t-2}^{(u)} L} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(and by applying the recursion repeatedly:)

$$\leq \prod_{l=0}^{t-1} \mathcal{M}^{(2l)} \begin{pmatrix} S_0^{(2t)} \\ T_0^{(2t)} \end{pmatrix} + \sum_{j=1}^{t} \frac{7}{Q_{2j-2}^{(u)} L} \prod_{l=0}^{t-j-1} \mathcal{M}^{(2l)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By the bound on $\rho_r + \sigma_r$ in Lemma 2, by [12, Lemma 5] and $1 - x \leq \exp(-x)$ and also the trivial bounds $S, T \leq 1$ we altogether get

$$S_{2t} \leq \mathrm{e}^{-s(t)} + \sum_{j=1}^{t} \frac{7}{Q_{2(j-1)}^{(u)} L} \mathrm{e}^{-s(t-j)} \leq \mathrm{e}^{-s(t)} \left(1 + \frac{7}{L} \sum_{j=1}^{t} \frac{\mathrm{e}^{s(t)-s(t-j)}}{Q_{2(j-1)}^{(u)}} \right)$$

and so $S_{2t} = O\left(\mathrm{e}^{-s(t)}\right)$, since

$$\left| \frac{\mathrm{e}^{s(t)-s(t-j)}}{Q_{2j}^{(u)}} \right| \leq \left(\frac{\mathrm{e}^{1/4}}{\min_{u,l} q_{u,l}^2} \right)^j < \frac{1}{3^j},$$

which proves the claim.

In the case that $\exp(-s^{(1)}(t)) \leq \exp(-s(t))$, we can proceed analogously, after initially applying a one-step recursion from Lemma 3. This does not change the asymptotics. $\square$

Collecting all the results, altogether we get, for $t_0 > 0$,

$$\left| \sum_{n=0}^{N-1} g(n) \right|^2 = O\left(N^2 \left[\min_u \left(\mathrm{e}^{-s_u(t_0)} + \frac{Q_{2t_0}^{(u)} + 2}{K} \right)^{1/(s+1)} + \frac{1}{2K} + F_2 \right] + N(F_1(1 + F_2)) \right),$$

where we can take the minimum over all u since we can use the trivial bound 1 for the remaining factors in Σ_3.

Now we return to fixing the quantities K, R_u and t_0. Since the goal is to have $o(N^2)$ on the right side of the last equation, F_2 should be $o(1)$, considering the N^2 term, hence $F_1 = o(N)$. This can be achieved by choosing $R_u = o(N^{1/s})$ and $K = o(\min_u R_u)$, e.g., by setting

$$R_u := \max\{Q_t^{(u)} : Q_t^{(u)} \leq N^{1/s-\varepsilon}, t \geq 0\} \text{ and } K := \min_u \lfloor R_u^{1-\varepsilon} \rfloor,$$

for some fixed $\varepsilon > 0$. Finally, t_0 is determined by $Q_{2t_0}^{(u)}/K = o(1)$ for all u, e.g., we can set

$$t_0 := \max\{t : \max_u Q_{2t}^{(u)} \leq K^{1-\varepsilon}\}.$$

Since t_0 is ultimately an increasing function in N and $s_u(t)$ diverges, the sum $N^{-1} \sum_{n=0}^{N-1} \exp(2\pi \mathtt{i} \boldsymbol{h} \cdot \boldsymbol{f}(n))$ is $o(1)$ and thus $f(n)$ is uniformly distributed modulo one by Weyl's criterion. This closes the proof of Proposition 1. $\square$

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