CYCLIC DIGITAL NETS, HYPERPLANE NETS AND MULTIVARIATE INTEGRATION IN SOBOLEV SPACES *

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Abstract. Cyclic nets are a special case of digital nets and were recently introduced by Niederreiter. Here we present a construction algorithm for such nets, where we use the root mean square worst-case error of a randomly digitally shifted point set in a weighted Sobolev space as a selection criterion. This yields a feasible construction algorithm since for a cyclic net with q^m points (with fixed bijections and fixed ground field) there are q^m possible choices.

Our results here match the convergence rate and strong tractability results for polynomial lattice rules, hence providing us with an alternative construction algorithm. Further we improve upon previous results by including constructions over arbitrary finite fields and an arbitrary choice of bijections.

Key words. Cyclic digital net, weighted $\mathcal{L}_2$-discrepancy, QMC-algorithm, component-by-component algorithm

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1. Introduction.

In quasi-Monte Carlo (QMC) one considers the approximation of an integral $\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ by the average of $f(\boldsymbol{x}_h)$ for sample points $\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{N-1}$. This approach might appear simple at first, but for high dimensions the question of how to choose a good point set $\{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{N-1}\}$ becomes a truly challenging problem, with many questions yet to be answered (see for example [5, 13]). Generally speaking one wants the points $\{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{N-1}\}$ to be evenly spread over the unit cube. To assess the quality of a point set $\{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{N-1}\}$, in other words, to measure the distribution properties of $P=\{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{N-1}\}$ one often uses a norm of the discrepancy function. The discrepancy function is given by

$$\Delta(P,\boldsymbol{z})=\frac{A_N(P,[\boldsymbol{0},\boldsymbol{z}))}{N}-|\boldsymbol{z}|,$$

where $\boldsymbol{z}=(z_1,\ldots,z_s)\in[0,1]^s$, $A_N(P,[\boldsymbol{0},\boldsymbol{z}))$ is the number of points of P in $[\boldsymbol{0},\boldsymbol{z}):=\prod_{j=1}^s[0,z_j)$ and $|\boldsymbol{z}|=z_1\cdots z_s$. By taking a norm of this function we obtain a quality measure of the point set P. In this paper we consider the so-called $\mathcal{L}_2$-discrepancy. In the classical case this corresponds to the 2-norm of the discrepancy function. Ever since the paper [20] by Sloan and Woźniakowski it has become popular to consider weighted discrepancies, specifically in our case this means we consider the *weighted $\mathcal{L}_{2,\boldsymbol{\gamma}}$-discrepancy*, which is given by

$$\mathcal{L}_{2,\boldsymbol{\gamma}}^2(P)=\sum_{\substack{u\subseteq\{1,\ldots,s\}\\ u\neq\emptyset}}\prod_{j\in u}\gamma_j\int_{[0,1]^{|u|}}|\Delta(P,(\boldsymbol{z}_u,1))|^2\,\mathrm{d}\boldsymbol{z}_u, \tag{1.1}$$

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where $\boldsymbol{z}_u$ denotes the vector from $[0,1]^{|u|}$ containing the components of $\boldsymbol{z}$ whose indices are in u and $(\boldsymbol{z}_u, 1)$ is the vector $\boldsymbol{z}$ from $[0,1]^s$ with all components whose indices are not in u replaced by 1. Here $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots)$ is a sequence of nonnegative real numbers γ_j, $j \geq 1$, the so-called *weights*. As is apparent from (1.1) the weights can be used to modify the importance of lower dimensional projections. We remark that in this paper we consider only product weights and not general weights.

In [20] it was also shown that the $\mathcal{L}_2$-discrepancy coincides with the worst-case error in certain weighted Sobolev spaces. Later on in the paper we prefer to state our results in terms of this worst-case error, as is usually done (for details see Section 3). On the other hand it also seems enlightening to understand the geometrical meaning of this worst-case error, hence we also described the measure in terms of the $\mathcal{L}_2$-discrepancy rather than the worst-case error.

With this measure at hand we are able to compare the distribution properties of point sets in the unit cube. The point sets which we consider here are so-called (t, m, s)-nets and were introduced by Niederreiter [10]. A special construction scheme of such nets goes by the name of digital nets. In order to construct a digital net over some finite field, one needs to find $m \times m$ matrices $C_1, \ldots, C_s$ with elements in the finite field $\mathbb{F}_q$. This provides us with a point set of q^m points. But the number of possible choices of generating matrices is q^{sm^2}, where s denotes the dimension. Hence using computer search to choose the best one is not feasible for a practically useful number of points. Niederreiter [12] also introduced a special subclass of digital nets, namely polynomial lattices. Thereby the number of choices is reduced to q^{ms}. Moreover, it was shown in [4] that it is sufficient to search with a special algorithm (component-by-component), which further reduces the number of polynomial lattices considered to sq^m.

An alternative to polynomial lattices are cyclic nets, which were recently introduced by Niederreiter in [14]. In this paper we show how we can also construct cyclic nets using algorithms similarly as in [4]. Indeed the construction algorithm used for cyclic nets matches the Korobov-type construction of polynomial lattice rules. Both of these have a search space of size at most q^m. Subsequently we also generalize the notion of cyclic nets whereby we succeed in introducing an analog to the component-by-component algorithm of polynomial lattice rules. In the following we will call this construction scheme hyperplane nets.

The upper bounds presented here are comparable, though they are more general in the sense that we now also allow arbitrary finite fields (formerly we only considered finite fields of prime order). In this situation one also needs bijections between the finite field and the digits $\{0, 1, \ldots, q-1\}$. The results presented here show that cyclic nets perform just as well as polynomial lattices, also achieving the best possible convergence rate and strong tractability results under appropriate conditions on the weights. Similar results have also been obtained for lattice rules, see [6, 18].

The paper is organized as follows. In the subsequent section we state the definition of (t, m, s)-nets, cyclic nets, hyperplane nets and Walsh functions. Walsh functions are characters over the group of digital nets and are hence very useful for analyzing digital nets (see [3] for more information). Section 3 is concerned with construction algorithms for cyclic nets and hyperplane nets. In that section we also prove upper bounds on the $\mathcal{L}_2$-discrepancy (or worst-case error), whereby the good performance of our construction algorithm is ensured. Finally, in an appendix we generalize the results in the appendix of [3], allowing now more general bijections which in turn allows us to obtain results for constructions of cyclic and hyperplane nets over arbitrary finite fields.

2. (t, m, s)**-nets in base** b**.** In this section we recall the definition of (digital) (t, m, s)-nets in base b and a special construction of such nets due to Niederreiter.

A detailed theory of (t, m, s)-nets was developed in [10] (see also [11, Chapter 4] for a survey of this theory). Those (t, m, s)-nets in a base b provide sets of b^m points in the s-dimensional unit cube $[0, 1)^s$, which are extremely well distributed if the quality parameter t is 'small'.

DEFINITION 2.1 ((t, m, s)-nets). Let $b \geq 2$, $s \geq 1$ and $0 \leq t \leq m$ be integers. Then a point set P consisting of b^m points in $[0, 1)^s$ forms a (t, m, s)-net in base b, if every subinterval $J = \prod_{j=1}^{s} [a_j \, b^{-d_j}, (a_j + 1) \, b^{-d_j})$ of $[0, 1)^s$, with integers $d_j \geq 0$ and integers $0 \leq a_j < b^{d_j}$ for $1 \leq j \leq s$ and of volume b^{t-m}, contains exactly b^t points of P.

In practice, all concrete constructions of (t, m, s)-nets in base b are based on the general construction scheme of digital nets. To avoid too many technical notions - and since we only deal with this case - in the following we restrict ourselves to digital nets defined over the finite field $\mathbb{F}_q$ of prime-power order q. For a more general definition (over arbitrary finite, commutative rings) see for example Niederreiter [11], Larcher [7], or Larcher, Niederreiter and Schmid [8].

DEFINITION 2.2 (digital (t, m, s)-nets). Let q be a prime-power and let $s \geq 1$ and $m \geq 1$ be integers. Let $C_1, \ldots, C_s$ be $m \times m$ matrices over $\mathbb{F}_q$. Now we construct q^m points in $[0, 1)^s$: for $0 \leq h \leq q^m - 1$ let $h = h_0 + h_1 q + \cdots + h_{m-1} q^{m-1}$ be the q-adic expansion of h. Consider an arbitrary but fixed bijection $\varphi : \{0, 1, \ldots, q - 1\} \longrightarrow \mathbb{F}_q$. Identify h with the vector $\vec{h} = (\varphi(h_0), \ldots, \varphi(h_{m-1}))^\top \in \mathbb{F}_q^m$, where $\top$ means the transpose of the vector. For $1 \leq j \leq s$ multiply the matrix C_j by $\vec{h}$, i.e.,

$$C_j \vec{h} =: (y_{j,1}(h), \ldots, y_{j,m}(h))^\top \in \mathbb{F}_q^m,$$

and set

$$x_{h,j} := \frac{\varphi^{-1}(y_{j,1}(h))}{q} + \cdots + \frac{\varphi^{-1}(y_{j,m}(h))}{q^m}.$$

If for some integer t with $0 \leq t \leq m$ the point set consisting of the points

$$\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$$

for $0 \leq h < q^m$, is a (t, m, s)-net in base q, then it is called a digital (t, m, s)-net over $\mathbb{F}_q$, or, in brief, a digital net (over $\mathbb{F}_q$). The C_j are called its *generating matrices.* Concerning the determination of the quality parameter t of digital nets we refer to Niederreiter [11, Theorem 4.28], see also [17].

An essential tool for the investigation of digital nets are Walsh functions. A very general definition, corresponding to the most general construction of digital nets over finite rings, was given in [8]. There, Walsh functions over a finite abelian group G, using some bijection φ, were defined. Here we restrict ourselves to the case of $G = \mathbb{F}_{p^r}, p$ prime. We restate the definitions for this special case here for the sake of convenience.

DEFINITION 2.3 (Walsh functions). Let $q = p^r$, p prime, $r \in \mathbb{N}_0$ and let $\mathbb{F}_q$ be the finite field with q elements. Let $\mathbb{Z}_q = \{0, 1, \ldots, q - 1\} \subset \mathbb{Z}$ with ring operations modulo q and let $\varphi : \mathbb{Z}_q \longrightarrow \mathbb{F}_q$ be a bijection such that $\varphi(0) = 0$, the neutral element of addition in $\mathbb{F}_q$. Moreover denote by ψ the isomorphism of additive groups $\psi : \mathbb{F}_q \longrightarrow \mathbb{Z}_p^r$ and define $\eta := \psi \circ \varphi$. For $1 \le i \le r$ denote by π_i the projection $\pi_i : \mathbb{Z}_p^r \longrightarrow \mathbb{Z}_p$, $\pi_i(x_1, \dots, x_r) = x_i$.

$$\begin{tikzcd} \mathbb{Z}_q \arrow[r, "\varphi"] \arrow[dr, "\eta"'] & \mathbb{F}_q \arrow[d, "\psi"] & \\ & \mathbb{Z}_p^r \arrow[r, "\pi_i"] & \mathbb{Z}_p \end{tikzcd}$$

Let now $k \in \mathbb{N}_0$ with base q representation $k = \kappa_0 + \kappa_1 q + \cdots + \kappa_{m-1} q^{m-1}$ where $\kappa_l \in \mathbb{Z}_q$ and let $x \in [0,1)$ with base q representation $x = x_1/q + x_2/q^2 + \cdots$. Then the k-th Walsh function over the finite field $\mathbb{F}_q$ with respect to the bijection φ is defined by

$$_{\mathbb{F}_q,\varphi}\mathrm{wal}_k(x) := \prod_{l=0}^{m-1} \prod_{i=1}^{r} \exp\Big(2\pi \mathtt{i} \frac{(\pi_i \circ \eta)(\kappa_l)(\pi_i \circ \eta)(x_l)}{p}\Big).$$

For convenience we will in the rest of the paper omit the subscript and simply write wal_k if there is no ambiguity.

Multivariate Walsh functions are defined by multiplication of the univariate components, i.e., for $\boldsymbol{x} = (x_1, \dots, x_s) \in [0,1)^s, \boldsymbol{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s, s > 1$, we set

$$\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{s} \mathrm{wal}_{k_j}(x_j).$$

We summarize some important properties of Walsh functions over finite fields which will be used throughout the paper. The proofs of the subsequent results can be found e.g. in [9, 15].

PROPOSITION 2.4. *Let p, q, $\mathbb{F}_q$ and φ be as in Definition 2.3. For x, y with q-adic representations $x = \sum_{i=w}^{\infty} x_i q^{-i}$ and $y = \sum_{i=w}^{\infty} y_i q^{-i}$, $w \in \mathbb{Z}$ (hence the following operations are also defined for integers), define $x \oplus_\varphi y := \sum_{i=w}^{\infty} z_i q^{-i}$ where $z_i := \varphi^{-1}(\varphi(x_i) + \varphi(y_i))$ and $\ominus_\varphi x := \sum_{i=w}^{\infty} v_i q^{-i}$ where $v_i := \varphi^{-1}(-\varphi(x_i))$. Further we set $x \ominus_\varphi y := x \oplus_\varphi (\ominus_\varphi y)$. For vectors $\boldsymbol{x}, \boldsymbol{y}$ we define the operations component-wise. Then we have:*

1. *For all $k, l \in \mathbb{N}_0$ and all $x, y \in [0,1)$ we have*

$$\mathrm{wal}_k(x) \cdot \mathrm{wal}_l(x) = \mathrm{wal}_{k \oplus_\varphi l}(x), \quad \mathrm{wal}_k(x) \cdot \mathrm{wal}_k(y) = \mathrm{wal}_k(x \oplus_\varphi y)$$

and

$$\mathrm{wal}_k(x) \cdot \overline{\mathrm{wal}_l(x)} = \mathrm{wal}_{k \ominus_\varphi l}(x), \quad \mathrm{wal}_k(x) \cdot \overline{\mathrm{wal}_k(y)} = \mathrm{wal}_k(x \ominus_\varphi y).$$

2. *We have*

$$\sum_{k=0}^{q-1} \mathrm{wal}_l(k/q) = \begin{cases} 0 & \text{if } l \neq 0, \\ q & \text{if } l = 0. \end{cases}$$

3. *We have*

$$\int_0^1 \mathrm{wal}_0(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \int_0^1 \mathrm{wal}_k(x)\,\mathrm{d}x = 0 \text{ if } k > 0.$$

4. *For all $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}_0^s$ we have the following orthogonality properties:*

$$\int_{[0,1)^s} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{l}}(\boldsymbol{x})} \,\mathrm{d}\boldsymbol{x} = \begin{cases} 1 & \text{if } \boldsymbol{k} = \boldsymbol{l}, \\ 0 & \text{otherwise.} \end{cases}$$

5. *For any $f \in \mathcal{L}_2([0,1)^s)$ and any $\boldsymbol{\sigma} \in [0,1)^s$ we have*

$$\int_{[0,1)^s} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \int_{[0,1)^s} f(\boldsymbol{x} \oplus_\varphi \boldsymbol{\sigma}) \,\mathrm{d}\boldsymbol{x}.$$

6. *For any integer $s \geq 1$ the system $\{\mathrm{wal}_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{N}_0^s\}$ is a complete orthonormal system in $\mathcal{L}_2([0,1)^s)$.*

Let $\mathbb{F}_q = \mathbb{Z}_p[\theta]$, such that $\{1, \theta, \ldots, \theta^{r-1}\}$ is a basis of $\mathbb{F}_q$ over $\mathbb{Z}_p$ as a vector space. Then the isomorphism ψ between $\mathbb{F}_q$ and $\mathbb{Z}_p^r$ shall be given by

$$\psi(x) = (x_1, \ldots, x_r)^\top, \text{ for } x = \sum_{i=1}^{r} x_i \theta^{i-1}, x_i \in \mathbb{Z}_p.$$

Let ψ be extended to vectors over $\mathbb{F}_q$, i.e. such that, for arbitrary m, vectors in $\mathbb{F}_q^m$ get mapped to vectors in $\mathbb{Z}_p^{rm}$.

Also let φ be extended to nonnegative integers by setting

$$\varphi(k) := (\varphi(\kappa_0), \ldots, \varphi(\kappa_{m-1}))^\top, \text{ for } k = \sum_{i=0}^{m-1} \kappa_i q^i, k_i \in \{0, \ldots, q-1\}.$$

We will also use the concatenation $\eta(k) := \psi(\varphi(k))$. We have the following commutative diagram:

$$\begin{tikzcd} \mathbb{Z}_{q^m} \arrow[r, "\varphi"] \arrow[dr, "\eta"'] & \mathbb{F}_q^m \arrow[d, "{\psi,\Psi}"] \\ & \mathbb{Z}_p^{rm} \end{tikzcd}$$

We now define a map Ψ of the linear transformations over $\mathbb{F}_q$ into the linear transformations over $\mathbb{Z}_p$. Let the representation of the element θ^r in $\mathbb{F}_q$ be given by $\theta^r = \theta_0 + \theta_1\theta + \cdots + \theta_{r-1}\theta^{r-1}$, $\theta_i \in \mathbb{Z}_p, i = 0, \ldots, r-1$. By Θ we denote the matrix

$$\Theta := \begin{pmatrix} 0 & 0 & 0 & \cdots & \theta_0 \\ 1 & 0 & 0 & \cdots & \theta_1 \\ 0 & 1 & 0 & \cdots & \theta_2 \\ & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \theta_{r-1} \end{pmatrix}.$$

It is easy to see that Θ acts on a vector $(x_1, \ldots, x_r) \in \mathbb{Z}_p^r$ in the same way as the linear transformation $x \mapsto \theta x, x \in \mathbb{F}_q$ does on $x_1 + x_2\theta + \cdots + x_r\theta^{r-1}$, i.e. $\Theta\psi(x) = \psi(\theta x)$. If $\alpha = \sum_{i=0}^{r-1} a_i \theta^i$ is the representation of an arbitrary element, denote by $\Psi(\alpha)$ the matrix

$$\Psi(\alpha) := \sum_{i=0}^{r-1} a_i \Theta^i.$$

Clearly then $\Psi(\alpha)\psi(x) = \psi(\alpha x)$. By linearity the mapping Ψ can be extended to matrices by applying it to the matrix entries and letting the matrices run together, i.e. with some abuse of notation

$$\Psi(A) := (\Psi(a_{i,j}))_{i,j} \in \mathbb{Z}_p^{rm_1 \times rm_2}, \qquad \text{for } A = (a_{i,j})_{i,j} \in \mathbb{F}_q^{m_1 \times m_2}, a_{i,j} \in \mathbb{F}_q,$$

for arbitrary m_1, m_2. Again by linearity $\Psi(A)\psi(\boldsymbol{x}) = \psi(A\boldsymbol{x})$ holds as well (for $A \in \mathbb{F}_q^{m_1 \times m_2}, \boldsymbol{x} \in \mathbb{F}_q^{m_2}$, $m_1, m_2 \in \mathbb{N}$).

LEMMA 2.5. *Let $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{q^m-1}\}$ be a digital net over $\mathbb{F}_q$ with bijection φ, where $\varphi(0) = 0$, generated by the $m \times m$ matrices $C_1, \ldots, C_s$ over $\mathbb{F}_q$, $m > 0$. Then for any vector $\boldsymbol{k} = (k_1, \ldots, k_s)$ of nonnegative integers $0 \le k_1, \ldots, k_s < q^m$ we have*

$$\sum_{h=0}^{q^m-1} {}_{\mathbb{F}_q,\varphi}\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = \begin{cases} q^m & \text{if } C_1^\top \varphi(k_1) + \cdots + C_s^\top \varphi(k_s) = \boldsymbol{0}, \\ 0 & \text{else}, \end{cases}$$

where $\boldsymbol{0}$ is the zero vector in $\mathbb{F}_q^m$.

Proof. Denote by ω_p the p-th root of unity, i.e., $\omega_p = \exp(2\pi \mathtt{i}/p)$. For each k_j, $1 \le j \le s$ let $\kappa_{j,l}$ denote the l-th q-adic digit of k_j, i.e., $k_j = \kappa_{j,0} + \cdots + \kappa_{j,m-1}q^{m-1}$. For $0 \le h \le q^m - 1$ let $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$. Then we have

$$\begin{aligned} \Sigma &:= \sum_{h=0}^{q^m-1} {}_{\mathbb{F}_q,\varphi}\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = \sum_{h=0}^{q^m-1} \prod_{j=1}^{s} \prod_{l=0}^{m-1} \prod_{i=1}^{r} \omega_p^{(\pi_i \circ \eta)(\kappa_{j,l})(\pi_i \circ \eta)(x_{h,j,l})} \\ &= \sum_{h=0}^{q^m-1} \prod_{j=1}^{s} \prod_{l=0}^{m-1} \omega_p^{\sum_{i=1}^{r} (\pi_i \circ \eta)(\kappa_{j,l})(\pi_i \circ \eta)(x_{h,j,l})} = \sum_{h=0}^{q^m-1} \prod_{j=1}^{s} \prod_{l=0}^{m-1} \omega_p^{\langle \eta(\kappa_{j,l}), \eta(x_{h,j,l}) \rangle}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. By the definition of digital nets we have

$$x_{h,j,l} = \varphi^{-1}(\langle \boldsymbol{c}_{j,l}^\top, \vec{h} \rangle),$$

where $\boldsymbol{c}_{j,l}$ denotes the l-th row vector of the matrix C_j and where $\vec{h} = (\varphi(h_0), \ldots, \varphi(h_{m-1}))^\top$ if $h = h_0 + \cdots + h_{m-1}q^{m-1}$. Therefore we obtain

$$\eta(x_{h,j,l}) = \psi \circ \varphi(\varphi^{-1}(\langle \boldsymbol{c}_{j,l}^\top, \vec{h} \rangle)) = \psi(\langle \boldsymbol{c}_{j,l}^\top, \vec{h} \rangle).$$

Since φ is a bijection we get

$$\begin{aligned} \Sigma &= \sum_{h=0}^{q^m-1} \prod_{j=1}^{s} \prod_{l=0}^{m-1} \omega_p^{\langle \eta(\kappa_{j,l}), \psi(\langle \boldsymbol{c}_{j,l}^\top, \vec{h} \rangle) \rangle} = \sum_{\boldsymbol{h} \in \mathbb{F}_q^m} \prod_{j=1}^{s} \prod_{l=0}^{m-1} \omega_p^{\langle \eta(\kappa_{j,l}), \psi(\langle \boldsymbol{c}_{j,l}^\top, \boldsymbol{h} \rangle) \rangle} \\ &= \sum_{\boldsymbol{h}' \in \mathbb{Z}_p^{rm}} \prod_{j=1}^{s} \prod_{l=0}^{m-1} \omega_p^{\langle \eta(\kappa_{j,l}), \Psi(\boldsymbol{c}_{j,l}) \boldsymbol{h}' \rangle} = \sum_{\boldsymbol{h}' \in \mathbb{Z}_p^{rm}} \omega_p^{\langle \boldsymbol{h}', \sum_{j=1}^{s} \sum_{l=0}^{m-1} \Psi(\boldsymbol{c}_{j,l})^\top \eta(\kappa_{j,l}) \rangle}. \end{aligned}$$

We have

$$\sum_{l=0}^{m-1} \Psi(\boldsymbol{c}_{j,l})^\top \eta(\kappa_{j,l}) = \Psi(C_j^\top) \eta(k_j),$$

since, denoting by $c_{j,l,i}$ the components of $\boldsymbol{c}_{j,l}$,

$$\left(\Psi(\boldsymbol{c}_{j,1})^\top \cdots \Psi(\boldsymbol{c}_{j,m})^\top \right) = \begin{pmatrix} \Psi(c_{j,1,1})^\top & \cdots & \Psi(c_{j,m,1})^\top \\ \vdots & \ddots & \vdots \\ \Psi(c_{j,1,m})^\top & \cdots & \Psi(c_{j,m,m})^\top \end{pmatrix} = \Psi(C_j^\top).$$

So we obtain

$$\Sigma = \sum_{\boldsymbol{h}' \in \mathbb{Z}_p^{rm}} \omega_p^{\langle \boldsymbol{h}', \sum_{j=1}^s \Psi(C_j^\top)\eta(k_j)\rangle}.$$

Bringing this into a form where we can evaluate the exponential sums, we get

$$\begin{aligned}\Sigma &= \prod_{l=0}^{m-1}\prod_{i=0}^{r-1}\sum_{h=0}^{p-1}\left(\omega_p^{(i\ m\ +\ l)\text{-th component of } (\sum_{j=1}^s \Psi(C_j^\top)\eta(k_j))}\right)^h \\ &= \begin{cases} q^m & \text{if } \Psi(C_1^\top)\eta(k_1) + \cdots + \Psi(C_s^\top)\eta(k_s) = \mathbf{0} \in \mathbb{Z}_p^{rm}, \\ 0 & \text{else,}\end{cases}\end{aligned}$$

and the lemma is proved by noting that

$$\Psi(C_1^\top)\eta(k_1) + \cdots + \Psi(C_s^\top)\eta(k_s) = \psi(C_1^\top\varphi(k_1) + \cdots + C_s^\top\varphi(k_s)) = \mathbf{0}$$

iff

$$C_1^\top\varphi(k_1) + \cdots + C_s^\top\varphi(k_s) = \mathbf{0},$$

since $\psi(0) = 0$. □

We next define cyclic digital nets following Niederreiter's article in [14]. (See e.g. this article of Niederreiter for more about the background of this notion and exact definitions of some terms not explained further in this paper.)

Definition 2.6. Let integers $m \geq 1, s \geq 2$ and a finite field $\mathbb{F}_q$ be given. Fix an element $\alpha \in \mathbb{F}_{q^m}$ and consider the set of polynomials

$$\mathcal{P}_\alpha := \{f \in \mathcal{P}, f(\alpha) = 0\} \quad \subseteq \quad \mathcal{P} := \{f \in \mathbb{F}_{q^m}[x], \deg(f) < s\}.$$

For each $j = 1, \ldots, s$ choose an ordered basis $\mathcal{B}_j$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ and define ϕ as the mapping

$$\phi : f(x) = \sum_{j=1}^s \gamma_j x^{j-1} \in \mathcal{P} \mapsto (\gamma_{1,1}, \ldots, \gamma_{1,m}, \ldots, \gamma_{s,1}, \ldots, \gamma_{s,m}) \in \mathbb{F}_q^{ms},$$

where $(\gamma_{j,1}, \ldots, \gamma_{j,m})$ is the coordinate vector of γ_j with respect to the chosen basis $\mathcal{B}_j$.

We denote by $\mathcal{C}_\alpha$ the orthogonal subspace in $\mathbb{F}_q^{ms}$ of the image $\mathcal{N}_\alpha := \phi(\mathcal{P}_\alpha)$. Let

$$C_\alpha = (C_1^\top \cdots C_s^\top) \in \mathbb{F}_q^{m \times sm}$$

be a matrix whose row space is $\mathcal{C}_\alpha$. Then the C_j are the generating matrices of a *cyclic digital net with respect to* $\mathcal{B}_1, \ldots, \mathcal{B}_s$ and C_α is its overall generating matrix. We shall from now on assume a fixed choice of bases $\mathcal{B}_j$ and will therefore not explicitly mention them anymore.

In the following we will again use the idea of employing linear representations (i.e. the mapping ψ), but with $\mathbb{F}_q$ in the role of $\mathbb{Z}_p$ and $\mathbb{F}_{q^m}$ in the role of $\mathbb{F}_q$. To be more precise, let $\mathbb{F}_{q^m} = \mathbb{F}_q[\omega]$, such that the powers of ω form a basis of $\mathbb{F}_{q^m}/\mathbb{F}_q$. Let $\omega^m = \beta_0 + \cdots + \beta_{m-1}\omega^{m-1}, \beta_l \in \mathbb{F}_q$ and P the matrix

$$P := \begin{pmatrix} 0 & 0 & \cdots & \beta_0 \\ 1 & 0 & \cdots & \beta_1 \\ \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & \beta_{m-1} \end{pmatrix}.$$

Now, if we have the representation of α in $\mathbb{F}_{q^m}$ as $\alpha = \sum_{l=0}^{m-1} a_l \omega^l, a_l \in \mathbb{F}_q$ define

$$\psi(\alpha) := (a_0, \ldots, a_{m-1}) \in \mathbb{F}_q^m, \qquad \Psi(\alpha) := \sum_{l=0}^{m-1} a_l P^l \in \mathbb{F}_q^{m \times m}.$$

Note that for any $\alpha, x \in \mathbb{F}_{q^m} \setminus \{0\}$ we have $\Psi(\alpha)\psi(x) = \psi(\alpha x) \neq \mathbf{0} \in \mathbb{F}_q^m$ as $\alpha x \neq 0 \in \mathbb{F}_{q^m}$. Hence it follows that for any $\alpha \in \mathbb{F}_{q^m} \setminus \{0\}$ we have that the matrix $\Psi(\alpha)$ is regular.

Furthermore, for $k = \sum_{l=0}^{m-1} \kappa_l q^l$, let

$$\varphi'(k) := \sum_{l=0}^{m-1} \varphi(\kappa_l)\omega^l, \qquad \psi'(k) := \psi(\varphi'(k))$$

and define all extensions to vectors and matrices as above. We have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}_{q^m} \arrow[r, "\varphi'"] \arrow[dr, "\psi'"'] & \mathbb{F}_{q^m} \arrow[d, "{\psi,\Psi}"] \\
 & \mathbb{F}_q^m
\end{tikzcd}$$

Note that we have $\psi' = \varphi$.

Using similar methods as in Lemma 2.5 we can give the generating matrices for $\mathcal{C}_\alpha$ in the following form.

THEOREM 2.7. *Let $m, s, \mathbb{F}_q$ and $\alpha \in \mathbb{F}_{q^m} = \mathbb{F}_q[\omega]$ be given and define s matrices $B_j = (\psi(b_{j,1}), \ldots, \psi(b_{j,m})))^{-1}$, where the $b_{j,l}$ constitute the chosen basis $\mathcal{B}_j$. Then the generating matrices of the net are given by $C_j = (\Psi(\alpha^{j-1})B_j)^\top = (\Psi(\alpha)^{j-1}B_j)^\top$, $j = 1, \ldots, s$. Furthermore it follows that C_j is regular for $j = 1, \ldots, s$.*

Proof. Let ϕ_1 be the (additive) isomorphism between $\mathcal{P} \subset \mathbb{F}_{q^m}[x]$ and $\mathbb{F}_{q^m}^s$. To arrive at the ϕ of Definition 2.6 we have to account for the choice of arbitrary bases $\mathcal{B}_j$. We do this by multiplying with the transformation matrix B^{-1}, where B is a square, block diagonal matrix with the matrices B_j of the statement of the theorem in its diagonal. Then $\phi(f) = B^{-1}\psi(\phi_1(f)), f \in \mathcal{P}$. We summarize these relations in the following diagrams.

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\phi_1"] \arrow[d, "\phi"'] & \mathbb{F}_{q^m}^s \arrow[d, "{\psi,\Psi}"] \\
\mathbb{F}_q^{ms} \arrow[r, "B"] & \mathbb{F}_q^{ms}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}_\alpha \arrow[r, "\phi_1"] \arrow[d, "\phi"'] & \phi_1(\mathcal{P}_\alpha) \arrow[d, "\psi"] \\
\mathcal{N}_\alpha \arrow[r, "B"] & \mathcal{N}_\alpha^\circ
\end{tikzcd}$$

Our first goal is to describe $\mathcal{N}_\alpha^\circ := \psi(\phi_1(\mathcal{P}_\alpha))$. Clearly, $\phi_1(\mathcal{P}_\alpha)$ is the space of all vectors orthogonal to $(1, \alpha, \ldots, \alpha^{s-1})^\top$. So $\boldsymbol{x} \in \phi_1(\mathcal{P}_\alpha)$ iff

$$\begin{aligned}
0 = (1, \alpha, \ldots, \alpha^{s-1})\boldsymbol{x} \iff \mathbf{0} &= \psi((1, \alpha, \ldots, \alpha^{s-1})\boldsymbol{x}) \\
&= \Psi((1, \alpha, \ldots, \alpha^{s-1}))\psi(\boldsymbol{x}),
\end{aligned}$$

hence $\mathcal{N}_\alpha^\circ$ is the orthogonal space to the row space of

$$C_\alpha^\circ := \Psi((1, \alpha, \ldots, \alpha^{s-1})) = (\Psi(1), \Psi(\alpha), \ldots, \Psi(\alpha^{s-1})).$$

If the $\mathcal{B}_j$ are again taken into account, we have that $\mathcal{N}_\alpha$ is the image of $\mathcal{N}_\alpha^\circ$ under the automorphism $\boldsymbol{x} \mapsto B^{-1}\boldsymbol{x}$, accordingly its orthogonal space is the image under $\boldsymbol{x} \mapsto \boldsymbol{x}B$. Thus $C_\alpha := C_\alpha^\circ B$ is the overall generating matrix of the cyclic digital net (i.e. its row space is said orthogonal space) and $C_j := (\Psi(\alpha^{j-1})B_j)^\top$ are its generating matrices by the duality theory of digital nets. By the considerations before Lemma 2.5, Ψ is a ring homomorphism, so $\Psi(\alpha^j) = \Psi(\alpha)^j$.

In order to show that the matrices C_j are regular recall that for any $\alpha \in \mathbb{F}_{q^m} \setminus \{0\}$ the matrix $\Psi(\alpha)$ is regular and as B_j is regular as well, it follows that C_j has to be regular. $\square$

REMARK 2.8. *Note that every digital net with regular generating matrices C_j is cyclic with respect to some choice of bases $\mathcal{B}_j$. However, the focus in this paper lies on the class of all cyclic nets, i.e. where α runs through all elements in $\mathbb{F}_{q^m} \setminus \{0\}$, for fixed bases $\mathcal{B}_j$ and we show that there is at least one good cyclic net (i.e., good choice of $\alpha \in \mathbb{F}_{q^m} \setminus \{0\}$) for each fixed choice of $\mathcal{B}_j$. Those classes of cyclic nets which we use in our search algorithms do depend on the choice of bases, but once chosen the bases remain fixed throughout the search algorithm. In particular the search space of our algorithm will still be of size q^m for any given choice of bases.*

REMARK 2.9. *It can be shown with a little calculation that Korobov polynomial lattice rules can be constructed (up to reordering of points) as cyclic nets. There, we have all B_j equal to the identity matrix. Note that with a suitably modified definition of cyclic nets (namely, if we consider arbitrary polynomial residue class rings) this also works for composite moduli f.*

With similar little difficulty, Schmid's constacyclic shift-nets can be realized as cyclic nets, using the construction for $\mathbb{F}_{q^m} = \mathbb{F}_q[\theta]$, where $\theta^m = k$, if $f(x) = x^m - k$ is irreducible, and k is the factor for the shifted elements. (Again, we could also extend the definition of cyclic nets to include arbitrary polynomial residue class rings instead of only $\mathbb{F}_{q^m}/\mathbb{F}_q$.) Here, all B_j are constant and equal to the first, unshifted matrix C_1, and α is always chosen equal to θ.

These two relations are considered in detail in [16].

In view of Theorem 2.7 we propose the following generalization of the cyclic net construction:

DEFINITION 2.10. Given a finite field $\mathbb{F}_q$, $\mathbb{F}_{q^m} = \mathbb{F}_q[\omega]$ as above, choose s elements $\alpha_1, \ldots, \alpha_s \in \mathbb{F}_{q^m}$, regular matrices $B_j \in \mathbb{F}_q^{m\times m}$ and let the generating matrices of a digital net be defined by the matrices $C_j = (\Psi(\alpha_j)B_j)^\top$. A digital net constructed in this manner shall be called a *hyperplane net with respect to* $\mathcal{B}_1, \ldots, \mathcal{B}_s$, where by $\mathcal{B}_j$ we denote the ordered bases corresponding to the matrices B_j as in Theorem 2.7. Again, we shall from now on assume a fixed choice of bases $\mathcal{B}_j$ and will therefore not explicitly mention them anymore.

REMARK 2.11. *Note that the generating matrices C_j of the hyperplane net are regular provided that $\alpha_j \neq 0 \in \mathbb{F}_{q^m}$. For $\alpha_j = 0$ we obtain that $C_j = \mathbf{0} \in \mathbb{F}_q^{m\times m}$, the matrix consisting only of the neutral element with respect to addition in $\mathbb{F}_q$.*

As a consequence, by Remark 2.8 a hyperplane net with regular generating matrices can also be considered as a cyclic net for some choice of bases $\mathcal{B}_j$. But the class of all hyperplane nets with fixed bases $\mathcal{B}_j$ is a proper superclass of the class of all cyclic nets with the same fixed bases $\mathcal{B}_j$. Hence the search space over all hyperplane nets is larger than the search space over all cyclic nets, as for the search we fix the bases in advance.

The name of the generalized construction is motivated by the following corollary of Lemma 2.5.

COROLLARY 2.12. *Let $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{q^m-1}\}$ be a digital net over $\mathbb{F}_q$ generated by the $m \times m$ matrices $C_1, \ldots, C_s$ over $\mathbb{F}_q$, $m > 0$, as given in Definition 2.10. Then for any vector $0 \leq k_1, \ldots, k_s < q^m$ of nonnegative integers we have*

$$\sum_{h=0}^{q^m-1} {}_{\mathbb{F}_q,\varphi}\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = \begin{cases} q^m & \text{if } \alpha_1\varphi'(\tau_1(k_1)) + \cdots + \alpha_s\varphi'(\tau_s(k_s)) = 0, \\ 0 & \text{else,} \end{cases}$$

with permutations $\tau_j(k) = \psi'^{-1}(B_j\psi'(k))$, and B_j as in Theorem 2.7.

Proof. By Definition 2.10 and Theorem 2.7, the generating matrices of the net are $C_j = (\Psi(\alpha_j)B_j)^\top$, so by Lemma 2.5 the sum equals q^m, iff (note that $\psi' = \varphi$)

$$\begin{aligned} \sum_{j=1}^{s} C_j^\top \psi'(k_j) \quad &= \quad \sum_{j=1}^{s} \Psi(\alpha_j)B_j\psi'(k_j) \\ = \sum_{j=1}^{s} \Psi(\alpha_j)\psi'(\tau_j(k_j)) = \mathbf{0} \iff &\sum_{j=1}^{s} \psi(\alpha_j\varphi'(\tau_j(k_j))) = \mathbf{0} \\ \iff \psi\Big(\sum_{j=1}^{s} \alpha_j\varphi'(\tau_j(k_j))\Big) = \mathbf{0} \iff &\sum_{j=1}^{s} \alpha_j\varphi'(\tau_j(k_j)) = 0, \end{aligned}$$

and vanishes otherwise, so the corollary follows. □

REMARK 2.13. *With this corollary it is not difficult to show that an equivalent definition in the spirit of Definition 2.6 exists: replace the $\mathcal{P}$ of Definition 2.6 by the space of linear forms*

$$\mathcal{P} = \{f(x_1, \ldots, x_s) = x_1\gamma_1 + \cdots + x_s\gamma_s, x_j \in \mathbb{F}_{q^m}\} \subset \mathbb{F}_{q^m}[x_1, \ldots, x_s]$$

and $\mathcal{P}_\alpha$ by $\mathcal{P}_{\boldsymbol{\alpha}} = \{f \in \mathcal{P}, f(\alpha_1, \ldots, \alpha_s) = 0\}$.

REMARK 2.14. *Similar as with cyclic nets, polynomial lattice rules can be regarded as a special case ($B_j = I$) of hyperplane nets. Note that we shall again need to modify the definition of hyperplane nets to account for composite moduli f. Together with Korobov polynomial lattice rules and cyclic digital nets etc. we get a hierarchy of nets that is dealt with specifically and in more detail in [16].*

3. Multivariate integration in weighted Sobolev spaces. In this section we consider multivariate integration in the weighted Sobolev space $H_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}$ induced by the reproducing kernel given by (see [2, 6, 18, 19])

$$K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{j=1}^{s}(1 + \gamma_j\varrho_{w_j}(x_j, y_j)),$$

where $\boldsymbol{w} = (w_1, \ldots, w_s) \in [0, 1]^s$ and

$$\begin{aligned} \varrho_w(x, y) &= \frac{|x - w| + |y - w| - |x - y|}{2} \\ &= \begin{cases} \min(|x - w|, |y - w|) & \text{if } (x - w)(y - w) \geq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The inner product in $H_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}$ is given by

$$\langle f,g\rangle_{H_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}} := f(\boldsymbol{w})g(\boldsymbol{w}) + \sum_{\substack{u\subseteq\{1,\ldots,s\}\\ u\neq\emptyset}} \prod_{j\in u}\gamma_j^{-1} \int_{[0,1)^{|u|}} \frac{\partial^{|u|} f}{\partial \boldsymbol{x}_u}(\boldsymbol{x}_u,\boldsymbol{w}) \frac{\partial^{|u|} g}{\partial \boldsymbol{x}_u}(\boldsymbol{x}_u,\boldsymbol{w}) \,\mathrm{d}\boldsymbol{x}_u,$$

where for $\boldsymbol{x}=(x_1,\ldots,x_s)$ and $u\subseteq\{1,\ldots,s\}$, $u\neq\emptyset$, we use the notation $\boldsymbol{x}_u=(x_j)_{j\in u}$ and $(\boldsymbol{x}_u,\boldsymbol{w})$ denotes the s-dimensional vector whose j-th component is x_j if $j\in u$ and w_j if $j\notin u$. The Sobolev space $H_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}$ can also be defined as the set of all square integrable functions where the norm induced by the above inner product is finite.

Choose a prime-power base $q=p^r$ and let $x=\frac{x_1}{q}+\frac{x_2}{q^2}+\cdots$ and $\sigma=\frac{\sigma_1}{q}+\frac{\sigma_2}{q^2}+\cdots$ be the base q representation of x and σ. Further choose a bijection $\varphi:\{0,1,\ldots,q-1\}\longrightarrow\mathbb{F}_q$ with $\varphi(0)=0$. Then the digitally shifted point (with respect to the bijection φ) $y=x\oplus_\varphi\sigma$ is given by $y=\frac{y_1}{q}+\frac{y_2}{q^2}+\cdots$, where $y_i=\varphi^{-1}(\varphi(x_i)+\varphi(\sigma_i))$. For vectors $\boldsymbol{x}$ and $\boldsymbol{\sigma}$ we define the digitally shifted point $\boldsymbol{x}\oplus_\varphi\boldsymbol{\sigma}$ component-wise. Obviously, the shift depends on the base q as well as on the bijection φ.

For a point set $P_N=\{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{N-1}\}$ and a $\boldsymbol{\sigma}\in[0,1)^s$ let $P_{N,\varphi,\boldsymbol{\sigma}}=\{\boldsymbol{x}_0\oplus_\varphi\boldsymbol{\sigma},\ldots,\boldsymbol{x}_{N-1}\oplus_\varphi\boldsymbol{\sigma}\}$ be the digitally shifted point set.

We recall that the *worst-case error* $e(P_N,K)$ for the integration of functions f from a reproducing kernel Hilbert space H with reproducing kernel K by means of a QMC-algorithm

$$Q_{N,s}(P_N,f)=\frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x}_n)$$

using a point set $P_N=\{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{N-1}\}$ is defined as

$$e(P_N,K):=\sup_{f\in H,\|f\|\le 1}\left|\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}-Q_{N,s}(P_N,f)\right|.$$

(In [20, Theorem 1] it is shown that for $\boldsymbol{w}=(1,\ldots,1)$ we have $e(P_N,K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}})=\mathcal{L}_{2,\boldsymbol{\gamma}}(P_N)$, the weighted $\mathcal{L}_{2,\boldsymbol{\gamma}}$-discrepancy of the point set P_N; see (1.1).)

Let the mean square worst-case error $\widehat{e}^2(P_N,K)$ be given by

$$\mathbb{E}[e^2(P_{N,\varphi,\boldsymbol{\sigma}},K)]=\int_{[0,1)^s} e^2(P_{N,\varphi,\boldsymbol{\sigma}},K)\,\mathrm{d}\boldsymbol{\sigma}.$$

Then we have $\widehat{e}^2(P_N,K)=e^2(P_N,K_{\mathrm{ds}})$, where

$$K_{\mathrm{ds}}(\boldsymbol{x},\boldsymbol{y}):=\int_{[0,1)^s} K(\boldsymbol{x}\oplus_\varphi\boldsymbol{\sigma},\boldsymbol{y}\oplus_\varphi\boldsymbol{\sigma})\,\mathrm{d}\boldsymbol{\sigma}$$

is the so-called *shift invariant kernel* of the kernel K. The proof of this result is similar to that of [3, Theorem 7].

In Appendix A of this paper it is shown that the shift invariant kernel $K_{\mathrm{ds},q,\boldsymbol{\gamma},\boldsymbol{w},\varphi}$ for the reproducing kernel $K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}$ is given by

$$K_{\mathrm{ds},q,\boldsymbol{\gamma},\boldsymbol{w},\varphi}(\boldsymbol{x},\boldsymbol{y})=\sum_{\boldsymbol{k}\in\mathbb{N}_0^s}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{k})\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{y})},$$

where $\boldsymbol{w} = (w_1, \ldots, w_s) \in [0,1]^s$ and $\widehat{r}_q(\boldsymbol{w}, \boldsymbol{\gamma}, \boldsymbol{k}) = \prod_{j=1}^s \widehat{r}_q(w_j, \gamma_j, k_j)$, where

$$\widehat{r}_q(w, \gamma, k) := \begin{cases} 1 + \gamma(w^2 - w + \frac{1}{3}) & \text{if } k = 0, \\ -\frac{\gamma}{2}\Big(\frac{1}{3q^{2a}} + \frac{2}{q^{2a}}\Re\Big(\sum_{\substack{u,v=0 \\ v \geq u}}^{q-1} \frac{(v-u)\mathrm{wal}_{\kappa_{a-1}}\left(\frac{u \ominus_\varphi v}{q}\right)}{q}\Big)\Big) & \text{if } k > 0. \end{cases}$$

Here for $k > 0$, κ_{a-1} denotes the most significant bit in the base q representation of k and $\Re$ is the real part function. Note that this result generalizes the result in [3, Appendix A], as we now also allow Walsh functions over arbitrary finite fields and arbitrary bijections ϕ between $\mathbb{Z}_q$ and $\mathbb{F}_q$ which satisfy $\phi(0) = 0$.

REMARK 3.1. *Since $K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}$ is a reproducing kernel, it is easy to see that the corresponding shift invariant kernel $K_{\mathrm{ds},q,\boldsymbol{\gamma},\boldsymbol{w},\varphi}$ is a reproducing kernel as well and from this one can see (by the properties of reproducing kernels; see [1]) that $\widehat{r}_q(w, \gamma, k)$ is nonnegative for any $k \in \mathbb{N}_0$.*

Further, for $x = \frac{x_1}{q} + \frac{x_2}{q^2} + \cdots$ and $y = \frac{y_1}{q} + \frac{y_2}{q^2} + \cdots$ we define $\rho_{\mathrm{ds},q,w}(x, x) := w^2 - w + \frac{1}{2}$ and if $x \neq y$,

$$\rho_{\mathrm{ds},q,w}(x, y) := w^2 - w + \frac{1}{2} - \frac{1}{2q^{i_0+1}} \times \\ \times \Big(\sum_{\substack{u=0 \\ u < u \oplus_\varphi x_{i_0} \ominus_\varphi y_{i_0}}}^{q-1} (u \oplus_\varphi x_{i_0} \ominus_\varphi y_{i_0} - u) + \sum_{\substack{u=0 \\ u < u \oplus_\varphi y_{i_0} \ominus_\varphi x_{i_0}}}^{q-1} (u \oplus_\varphi y_{i_0} \ominus_\varphi x_{i_0} - u)\Big), \quad (3.1)$$

where i_0 is the smallest index such that the digits of x and y differ. Note that we have $\rho_{\mathrm{ds},q,w}(x, y) = \rho_{\mathrm{ds},q,w}(x \ominus_\varphi y, 0)$. Using Lemma B.2 from Appendix B it can easily be checked that

$$K_{\mathrm{ds},q,\boldsymbol{\gamma},\boldsymbol{w},\varphi}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{j=1}^s (1 + \gamma_j \rho_{\mathrm{ds},q,w_j}(x_j, y_j)).$$

Now we obtain, as in [3], that the mean square worst-case error for integration in the weighted Sobolev space $H_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}$ by using a random digital shift in base q with respect to a bijection φ on the point set $P_N = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$, with $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$, is given by

$$\begin{aligned} &\widehat{e}^2(P_n, K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}) \\ &= \int_{[0,1)^{2s}} K_{\mathrm{ds},q,\boldsymbol{\gamma},\boldsymbol{w},\varphi}(\boldsymbol{x}, \boldsymbol{y}) \,\mathrm{d}\boldsymbol{x} \,\mathrm{d}\boldsymbol{y} - \frac{2}{N} \sum_{h=0}^{N-1} \int_{[0,1)^s} K_{\mathrm{ds},q,\boldsymbol{\gamma},\boldsymbol{w},\varphi}(\boldsymbol{x}_h, \boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \\ &\quad + \frac{1}{N^2} \sum_{h,n=0}^{N-1} K_{\mathrm{ds},q,\boldsymbol{\gamma},\boldsymbol{w},\varphi}(\boldsymbol{x}_h, \boldsymbol{x}_n) \\ &= -\prod_{j=1}^s \Big(1 + \gamma_j\Big(w_j^2 - w_j + \tfrac{1}{3}\Big)\Big) + \frac{1}{N^2} \sum_{h,n=0}^{N-1} \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{r}_q(\boldsymbol{w}, \boldsymbol{\gamma}, \boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n)} \\ &= -\prod_{j=1}^s \Big(1 + \gamma_j\Big(w_j^2 - w_j + \tfrac{1}{3}\Big)\Big) + \frac{1}{N^2} \sum_{h,n=0}^{N-1} \prod_{j=1}^s (1 + \gamma_j \rho_{\mathrm{ds},q,w_j}(x_{h,j}, x_{n,j})). \end{aligned}$$

For the special case where the point set $P_{q^m,\varphi}$ is a digital (t,m,s)-net over $\mathbb{F}_q$ with generating matrices $C_1,\ldots,C_s$ and bijection φ (the same bijection as for the digital shift) we obtain

$$\begin{aligned}\widehat{e}^2(P_{q^m,\varphi},K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}) &= -\prod_{j=1}^{s}\left(1+\gamma_j\left(w_j^2-w_j+\tfrac{1}{3}\right)\right)\\ &\quad+\frac{1}{q^{2m}}\sum_{n=0}^{q^m-1}\Big(\sum_{h=0}^{q^m-1}\sum_{\boldsymbol{k}\in\mathbb{N}_0^s}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{k})\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h\ominus_\varphi\boldsymbol{x}_n)\Big)\\ &= -\prod_{j=1}^{s}\left(1+\gamma_j\left(w_j^2-w_j+\tfrac{1}{3}\right)\right)\\ &\quad+\frac{1}{q^{2m}}\sum_{n=0}^{q^m-1}\Big(\sum_{h=0}^{q^m-1}\prod_{j=1}^{s}(1+\gamma_j\rho_{\mathrm{ds},q,w_j}(x_{h,j}\ominus_\varphi x_{n,j},0))\Big).\end{aligned}$$

It is easy to show that a digital net $P_{q^m,\varphi}$ over $\mathbb{F}_q$ generated by matrices $C_1,\ldots,C_s$ with bijection φ together with the addition $\oplus_\varphi$ becomes a group. Hence each term in the sum over n has the same value and therefore we obtain

$$\begin{aligned}&\widehat{e}^2(P_{q^m,\varphi},K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}})\\ &= -\prod_{j=1}^{s}\left(1+\gamma_j\left(w_j^2-w_j+\tfrac{1}{3}\right)\right)+\sum_{\boldsymbol{k}\in\mathbb{N}_0^s}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{k})\frac{1}{q^m}\sum_{h=0}^{q^m-1}\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h)\\ &= -\prod_{j=1}^{s}\left(1+\gamma_j\left(w_j^2-w_j+\tfrac{1}{3}\right)\right)+\frac{1}{q^m}\sum_{h=0}^{q^m-1}\prod_{j=1}^{s}(1+\gamma_j\rho_{\mathrm{ds},q,w_j}(x_{h,j},0)).\end{aligned}$$

For $k\in\mathbb{N}_0$, $k=\kappa_0+\kappa_1 q+\cdots$ denote an m-bit truncation by

$$\mathrm{tc}_{m,\varphi}(k)=(\varphi(\kappa_0),\ldots,\varphi(\kappa_{m-1}))^\top\in\mathbb{F}_q^m.$$

For vectors $\boldsymbol{k}\in\mathbb{N}_0^s$ the mapping $\mathrm{tc}_{m,\varphi}$ is defined component-wise. Further define

$$\mathcal{N}=\{\boldsymbol{k}=(k_1,\ldots,k_s)\in(\mathbb{F}_q^m)^s : C_1^\top k_1+\cdots+C_s^\top k_s=\mathbf{0}\},$$

where $\mathbf{0}$ is the zero vector in $\mathbb{F}_q^m$.

Using these definitions and Lemma 2.5 we obtain

THEOREM 3.2. *Let $P_{q^m,\varphi}=\{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{q^m-1}\}$ be a digital (t,m,s)-net over $\mathbb{F}_q$ generated by $C_1,\ldots,C_s$ and with respect to the bijection φ, where $\varphi(0)=0$.*

1. *The mean square worst-case error for integration in the weighted Sobolev space $H_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}$ by using the digital net $P_{q^m,\varphi}$ is given by*

$$\widehat{e}^2(P_{q^m,\varphi},K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}})=\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}\\ \mathrm{tc}_{m,\varphi}(\boldsymbol{k})\in\mathcal{N}}}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{k}).$$

2. *Let $\boldsymbol{x}_h=(x_{h,1},\ldots,x_{h,s})$ for $0\le h\le q^m-1$. Then we have*

$$\begin{aligned}&\widehat{e}^2(P_{q^m,\varphi},K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}) \qquad (3.2)\\ &= -\prod_{j=1}^{s}\left(1+\gamma_j\left(w_j^2-w_j+\tfrac{1}{3}\right)\right)+\frac{1}{q^m}\sum_{h=0}^{q^m-1}\prod_{j=1}^{s}(1+\gamma_j\rho_{\mathrm{ds},q,w_j}(x_{h,j},0)),\end{aligned}$$

where $\rho_{\mathrm{ds},q,w}$ is given by (3.1).

Note that formula (3.2) allows us to compute the mean square worst-case error for integration in the weighted Sobolev space $H_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}$ for any digital net over $\mathbb{F}_q$ in $O(q^m s)$ operations.

3.1. Integration in $H_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}$ with cyclic nets. In this subsection we consider the special case where the digital net used for the QMC-rule is a cyclic net.

Let $\alpha \in \mathbb{F}_{q^m}$, $\mathcal{B}_1, \ldots, \mathcal{B}_s$ be s ordered bases of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ and let $C_1, \ldots, C_s$ be given as in Theorem 2.7. Then we have

$$\mathcal{N} = \mathcal{N}_\alpha = \phi(\mathcal{P}_\alpha),$$

see Definition 2.6. Let φ be a bijection from $\mathbb{Z}_q$ to $\mathbb{F}_q$ with $\varphi(0) = 0$. The cyclic net generated by $C_1, \ldots, C_s$ with respect to φ will be denoted by $P_{\alpha,\varphi}$.

ALGORITHM 3.3. *Given a dimension $s \geq 2$, an integer $m \geq 1$ and weights $\boldsymbol{\gamma} = (\gamma_j)_{j\geq 1}$.*

1. *Choose a prime power q, a finite field $\mathbb{F}_q$ with q elements, a bijection $\varphi : \{0, 1, \ldots, q-1\} \longrightarrow \mathbb{F}_q$ with $\varphi(0) = 0$ and s ordered bases $\mathcal{B}_1, \ldots, \mathcal{B}_s$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$.*
2. *Find $\alpha \in \mathbb{F}_{q^m} \setminus \{0\}$ that minimizes $\widehat{e}^2(P_{\alpha,\varphi}, K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}})$.*

In the following theorem we show that this construction yields the same upper bound as the Korobov construction of polynomial lattice rules, which is of course not surprising as Korobov polynomial lattice rules are just a special case.

THEOREM 3.4. *Let $s \geq 2$, let q be a prime power, $\mathbb{F}_q$ be a finite field with q elements, $\varphi : \{0, 1, \ldots, q-1\} \longrightarrow \mathbb{F}_q$ a bijection with $\varphi(0) = 0$ and $\mathcal{B}_1, \ldots, \mathcal{B}_s$ be s ordered bases of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. Further let $m \geq 1$. Assume that $\alpha^* \in \mathbb{F}_{q^m} \setminus \{0\}$ is constructed by Algorithm 3.3. Then we have*

$$\widehat{e}^2(P_{\alpha^*,\varphi}, K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}) \leq \Big(\frac{s}{q^m - 1}\Big)^{\frac{1}{\lambda}} \prod_{j=1}^{s} \Big(\Big(1 + \gamma_j \left[w_j^2 - w_j + \tfrac{1}{3}\right]\Big)^{\lambda} + \gamma_j^{\lambda} \zeta_q(\lambda)\Big)^{\frac{1}{\lambda}}$$

for all $\frac{1}{2} < \lambda \leq 1$. Here for $\lambda = 1$, $\zeta_q(1) = \frac{1}{6}$ and for $\frac{1}{2} < \lambda < 1$ we have

$$\zeta_2(\lambda) = \frac{1}{3^\lambda(2^{2\lambda} - 2)} \quad \text{and} \quad \zeta_q(\lambda) = \frac{(q-1)q^{2\lambda}}{6^\lambda(q^{2\lambda} - q)} \quad \text{for } q \neq 2.$$

Proof. Since

$$\min_{\alpha \in \mathbb{F}_{q^m}\setminus\{0\}} \widehat{e}^2(P_{\alpha,\varphi}, K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}) \leq \Big(\frac{1}{q^m - 1} \sum_{\alpha \in \mathbb{F}_{q^m}\setminus\{0\}} \widehat{e}^2(P_{\alpha,\varphi}, K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}})^{\lambda}\Big)^{\frac{1}{\lambda}},$$

it is enough to show that the inequality

$$\frac{1}{q^m - 1} \sum_{\alpha \in \mathbb{F}_{q^m}\setminus\{0\}} \widehat{e}^2(P_{\alpha,\varphi}, K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}})^{\lambda} \leq \frac{s}{q^m - 1} \prod_{j=1}^{s} \Big(\Big(1 + \gamma_j \left[w_j^2 - w_j + \tfrac{1}{3}\right]\Big)^{\lambda} + \gamma_j^{\lambda} \zeta_q(\lambda)\Big)$$

holds. With Jensen's inequality, which states that for a sequence (a_k) of nonnegative reals we have

$$\Big(\sum a_k\Big)^{\lambda} \leq \sum a_k^{\lambda}$$

for any $0 < \lambda \le 1$, we obtain

$$\begin{aligned} M_{s,q} &:= \frac{1}{q^m-1} \sum_{\alpha \in \mathbb{F}_{q^m}\setminus\{0\}} \widehat{e}^2(P_\alpha, K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}})^\lambda \\ &\le \frac{1}{q^m-1} \sum_{\boldsymbol{l}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda A(\mathrm{tc}_{m,\varphi}(\boldsymbol{l})), \end{aligned}$$

where for $\boldsymbol{k} \in \mathbb{F}_q^{sm}$ we define

$$A(\boldsymbol{k}) := \#\{\alpha \in \mathbb{F}_{q^m} \setminus \{0\} : \boldsymbol{k} \in \mathcal{N}_\alpha\}.$$

Now $\boldsymbol{k} \in \mathbb{F}_q^{sm} \setminus \{\boldsymbol{0}\}$ is contained in $\mathcal{N}_\alpha$ iff α is a zero of the corresponding polynomial $\phi^{-1}(\boldsymbol{k})$. This polynomial has degree of at most $s-1$ and hence it has at most $s-1$ zeros. Thus $A(\boldsymbol{k}) \le s-1$. Further we have $A(\boldsymbol{0}) = q^m - 1$.

For $\boldsymbol{l} \in \mathbb{N}_0^s$ we have $\mathrm{tc}_{m,\varphi}(q^m\boldsymbol{l}) = \boldsymbol{0}$ and hence

$$\begin{aligned} M_{s,q} \le \frac{1}{q^m-1} \Bigg(& \sum_{\boldsymbol{l}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},q^m\boldsymbol{l})^\lambda A(\boldsymbol{0}) \\ & + \sum_{\boldsymbol{l}\in\mathbb{N}_0^s} \sum_{\substack{\boldsymbol{l}^*\in\mathbb{N}_0^s \\ 0<\|\boldsymbol{l}^*\|_\infty<q^m}} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l}^*+q^m\boldsymbol{l})^\lambda A(\mathrm{tc}_{m,\varphi}(\boldsymbol{l}^*)) \Bigg) \\ \le & \sum_{\boldsymbol{l}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},q^m\boldsymbol{l})^\lambda + \frac{s-1}{q^m-1} \sum_{\boldsymbol{l}\in\mathbb{N}_0^s} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda. \end{aligned}$$

The first sum in the last line in the inequality above can be estimated by

$$\begin{aligned} & \sum_{\boldsymbol{l}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},q^m\boldsymbol{l})^\lambda \\ & \quad = -\prod_{j=1}^s \widehat{r}_q(w_j,\gamma_j,0)^\lambda + \prod_{j=1}^s \sum_{k=0}^\infty \widehat{r}_q(w_j,\gamma_j,q^m k)^\lambda \\ & \quad = -\prod_{j=1}^s \left(1+\gamma_j\left(w_j^2 - w_j + \tfrac{1}{3}\right)\right)^\lambda \\ & \qquad + \prod_{j=1}^s \left(\left(1+\gamma_j\left(w_j^2 - w_j + \tfrac{1}{3}\right)\right)^\lambda + \sum_{k=1}^\infty \widehat{r}_q(w_j,\gamma_j,q^m k)^\lambda\right) \\ & \quad \le \frac{1}{q^{2\lambda m}} \prod_{j=1}^s \left(\left(1+\gamma_j\left[w_j^2 - w_j + \tfrac{1}{3}\right]\right)^\lambda + \sum_{k=1}^\infty \widehat{r}_q(w_j,\gamma_j,k)^\lambda\right). \end{aligned}$$

and the second sum is

$$\begin{aligned} \sum_{\boldsymbol{l}\in\mathbb{N}_0^s} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda &= \prod_{j=1}^s \sum_{k=0}^\infty \widehat{r}_q(w_j,\gamma_j,k)^\lambda \\ &= \prod_{j=1}^s \left(\left(1+\gamma_j\left[w_j^2 - w_j + \tfrac{1}{3}\right]\right)^\lambda + \sum_{k=1}^\infty \widehat{r}_q(w_j,\gamma_j,k)^\lambda\right). \end{aligned}$$

We have

$$\sum_{k=1}^{\infty} \widehat{r}_q(w_j,\gamma_j,k)^{\lambda} =: \gamma_j^{\lambda}\mu_q(\lambda).$$

First note that

$$\left|\frac{1}{6}+\Re\Big(\sum_{\substack{u,v=0\\ v\ge u}}^{q-1}\frac{(v-u)\mathrm{wal}_{\kappa_{a-1}}\left(\frac{u\ominus_{\varphi} v}{q}\right)}{q}\Big)\right| \le \frac{1}{6}+\frac{1}{q}\sum_{\substack{u,v=0\\ v\ge u}}^{q-1}(v-u)$$
$$=\frac{1}{6}+\frac{1}{q}\frac{q(q+1)(q-1)}{6}=\frac{q^2}{6}.$$

For $\frac{1}{2}<\lambda<1$ we have

$$\begin{aligned}\mu_q(\lambda) &= \sum_{k=1}^{\infty}\Big(-\frac{1}{2}\Big(\frac{1}{3q^{2a}}+\frac{2}{q^{2a}}\Re\Big(\sum_{\substack{u,v=0\\ v\ge u}}^{q-1}\frac{(v-u)\mathrm{wal}_{\kappa_{a-1}}\left(\frac{u\ominus_{\varphi} v}{q}\right)}{q}\Big)\Big)\Big)^{\lambda}\\ &\le \sum_{a=1}^{\infty}\sum_{k=q^{a-1}}^{q^a-1}\frac{1}{q^{2\lambda a}}\Big(\frac{q^2}{6}\Big)^{\lambda} = \frac{(q-1)q^{2\lambda}}{6^{\lambda}(q^{2\lambda}-q)} =: \zeta_q(\lambda)\end{aligned}$$

and for $\lambda=1$ we have

$$\begin{aligned}&\mu_q(1)\\ &= -\frac{1}{2}\sum_{a=1}^{\infty}\sum_{k=q^{a-1}}^{q^a-1}\Big(\frac{1}{3q^{2a}}+\frac{2}{q^{2a}}\Re\Big(\sum_{\substack{u,v=0\\ v\ge u}}^{q-1}\frac{(v-u)\mathrm{wal}_{\kappa_{a-1}}\left(\frac{u\ominus_{\varphi} v}{q}\right)}{q}\Big)\Big)\\ &= -\frac{1}{6}\sum_{a=1}^{\infty}\frac{1}{q^{2a}}(q^a-q^{a-1}) - \Re\Big(\sum_{a=1}^{\infty}\frac{1}{q^{2a}}\sum_{k=q^{a-1}}^{q^a-1}\sum_{\substack{u,v=0\\ v\ge u}}^{q-1}\frac{(v-u)\mathrm{wal}_{\kappa_{a-1}}\left(\frac{u\ominus_{\varphi} v}{q}\right)}{q}\Big).\end{aligned}$$

Now with Lemma B.1 in Appendix B we obtain

$$\begin{aligned}&\sum_{a=1}^{\infty}\frac{1}{q^{2a}}\sum_{k=q^{a-1}}^{q^a-1}\sum_{\substack{u,v=0\\ v\ge u}}^{q-1}\frac{(v-u)\mathrm{wal}_{\kappa_{a-1}}\left(\frac{u\ominus_{\varphi} v}{q}\right)}{q}\\ &= \sum_{a=1}^{\infty}\frac{1}{q^{a+2}}\sum_{\kappa_{a-1}=1}^{q-1}\sum_{\substack{u,v=0\\ v\ge u}}^{q-1}(v-u)\mathrm{wal}_{\kappa_{a-1}}\big(\tfrac{u\ominus_{\varphi} v}{q}\big) = \sum_{a=1}^{\infty}\frac{1}{q^{a+2}}q\frac{1-q^2}{6} = -\frac{q+1}{6q}.\end{aligned}$$

Therefore we have

$$\mu_q(1) = -\frac{1}{6q}+\frac{q+1}{6q} = \frac{1}{6} =: \zeta_q(1).$$

Let now $q=2$ and $\frac{1}{2}<\lambda<1$. Then

$$\mu_2(\lambda) = \sum_{a=1}^{\infty}\sum_{k=2^{a-1}}^{2^a-1}\Big(-\frac{1}{2}\Big(\frac{1}{3\cdot 2^{2a}}+\frac{2}{2^{2a}}\sum_{\substack{u,v=0\\ v\ge u}}^{1}\frac{(v-u)\mathrm{wal}_1\left(\frac{u\ominus_{\varphi} v}{2}\right)}{2}\Big)\Big)^{\lambda}$$

and by using

$$\sum_{\substack{u,v=0\\ v\geq u}}^{1} \frac{(v-u)\mathrm{wal}_1\left(\frac{u\ominus_\varphi v}{2}\right)}{2} = \frac{\mathrm{wal}_1\left(\frac{0\ominus_\varphi 1}{2}\right)}{2} = -\frac{1}{2},$$

we obtain

$$\mu_2(\lambda) = \frac{1}{3^\lambda(2^{2\lambda}-2)} =: \zeta_2(\lambda).$$

The theorem follows. □

The following corollary shows that under certain conditions on the weights we can obtain an upper bound which depends only polynomially on the dimension and thus proving tractability (see [20] for more information on tractability). An analogous result was shown for the Korobov construction of polynomial lattice rules, see [4].

COROLLARY 3.5. *Let $s \geq 2$, let q be a prime power, $\mathbb{F}_q$ be a finite field with q elements, $\varphi : \{0,1,\ldots,q-1\} \longrightarrow \mathbb{F}_q$ a bijection with $\varphi(0)=0$ and $\mathcal{B}_1,\ldots,\mathcal{B}_s$ be s ordered bases of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. Let $m \geq 1$ and suppose $\alpha^* \in \mathbb{F}_{q^m}\setminus\{0\}$ is constructed by Algorithm 3.3. Let $N = q^m$.*

1. *We have*

$$\widehat{e}(P_{\alpha^*,\varphi}, K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}) \leq c_{s,\boldsymbol{w},\boldsymbol{\gamma},\delta}\, s^{1-\delta} N^{-1+\delta} \qquad \text{for all } 0<\delta\leq \tfrac{1}{2},$$

where

$$c_{s,\boldsymbol{w},\boldsymbol{\gamma},\delta} := 2^{1-\delta}\prod_{j=1}^{s}\left(1+\gamma_j^{\frac{1}{2(1-\delta)}}\left[\left(w_j^2 - w_j + \tfrac{1}{3}\right)^{\frac{1}{2(1-\delta)}} + \zeta_q\left(\tfrac{1}{2(1-\delta)}\right)\right]\right)^{1-\delta}.$$

2. *Under the assumption*

$$A := \limsup_{s\to\infty}\frac{\sum_{j=1}^{s}\gamma_j}{\log s} < \infty$$

we obtain $c_{s,\boldsymbol{w},\boldsymbol{\gamma},1/2} \leq \bar{c}_\eta s^{(A+\eta)/2}$ and therefore

$$\widehat{e}(P_{\alpha^*,\varphi}, K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}) \leq \bar{c}_\eta s^{(1+(A+\eta))/2} N^{-\frac{1}{2}} \qquad \text{for all } \eta>0,$$

where the constant $\bar{c}_\eta$ depends only on the arbitrarily chosen parameter η. Thus the root mean square worst-case error of the cyclic net generated by α^* (with respect to the bijection φ) satisfies a bound which depends only polynomially on the dimension.

The result can be shown using the methods employed in the proof of [4, Corollary 4.8].

3.2. Integration in $H_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}$ with hyperplane nets. In this subsection we consider the special case where the digital net used for the QMC-rule is a hyperplane net.

Let $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_s) \in \mathbb{F}_{q^m}^s$, $\mathcal{B}_1,\ldots,\mathcal{B}_s$ be s ordered bases of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ and let $C_1,\ldots,C_s$ be given as in Definition 2.10. Let

$$\mathcal{N}_{\boldsymbol{\alpha}} = \{\boldsymbol{k} = (k_1,\ldots,k_s) \in (\mathbb{F}_q^m)^s : C_1^\top k_1 + \cdots + C_s^\top k_s = \boldsymbol{0}\},$$

where $\mathbf{0}$ is the zero vector in $\mathbb{F}_q^m$. Let φ be a bijection from $\mathbb{Z}_q$ to $\mathbb{F}_q$ with $\varphi(0)=0$. The hyperplane net generated by $C_1,\dots,C_s$ with respect to φ will be denoted by $P_{\boldsymbol{\alpha},\varphi}$.

From now on we write $\widehat{e}^2(P_{\boldsymbol{\alpha},\varphi},K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}})=\widehat{e}^2(\alpha_1,\dots,\alpha_s)$ to stress the dependence of the mean square worst-case error on $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_s)$.

ALGORITHM 3.6. *Given a dimension $s\ge 2$, an integer $m\ge 1$ and weights $\boldsymbol{\gamma}=(\gamma_j)_{j\ge 1}$.*

1. *Choose a prime power q, a finite field $\mathbb{F}_q$ with q elements, a bijection $\varphi:\{0,1,\dots,q-1\}\longrightarrow\mathbb{F}_q$ with $\varphi(0)=0$ and ordered bases $\mathcal{B}_1,\dots,\mathcal{B}_s$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$.*
2. *Choose $\alpha_1\in\mathbb{F}_{q^m}\setminus\{0\}$.*
3. *For $d=2,3,\dots,s$ find $\alpha_d\in\mathbb{F}_{q^m}\setminus\{0\}$ by minimizing the mean square worst-case error $\widehat{e}^2(\alpha_1,\dots,\alpha_d)$.*

In the following theorem we show that this construction yields the same upper bound as the component-by-component construction of polynomial lattice rules.

THEOREM 3.7. *Let $s\ge 2$, let q be a prime power, $\mathbb{F}_q$ be a finite field with q elements, $\varphi:\{0,1,\dots,q-1\}\longrightarrow\mathbb{F}_q$ a bijection with $\varphi(0)=0$ and $\mathcal{B}_1,\dots,\mathcal{B}_s$ be ordered bases of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. Further let $m\ge 1$. Assume that $(\alpha_1^*,\dots,\alpha_s^*)\in(\mathbb{F}_{q^m}\setminus\{0\})^s$ is constructed by Algorithm 3.6. Then for all $d=1,2,\dots,s$ we have*

$$\widehat{e}^2(\alpha_1^*,\dots,\alpha_d^*)\le(q^m-1)^{-\frac{1}{\lambda}}\prod_{j=1}^{d}\left(\left(1+\gamma_j\left[w_j^2-w_j+\tfrac{1}{3}\right]\right)^{\lambda}+\zeta_q(\lambda)\gamma_j^{\lambda}\right)^{\frac{1}{\lambda}}$$

for all $\frac{1}{2}<\lambda\le 1$. Here $\zeta_q(\lambda)$ is defined as in Theorem 3.4.

Proof. First we show that the inequality

$$\widehat{e}^2(\alpha_1^*,\dots,\alpha_d^*)\le\Big(\frac{1}{q^m-1}\sum_{\boldsymbol{l}\in\mathbb{N}_0^d}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^{\lambda}\Big)^{\frac{1}{\lambda}} \tag{3.3}$$

holds for all $d=1,2,\dots,s$.

We start with $d=1$. The generating matrix C_1 is regular for all $\alpha_1\in\mathbb{F}_{q^m}\setminus\{0\}$ and hence $\mathcal{N}_{\alpha_1}=\{\mathbf{0}\}$. Hence we have

$$\widehat{e}^2(\alpha_1^*)=\sum_{\substack{k\in\mathbb{N}_0\setminus\{0\}\\ q^m|k}}\widehat{r}_q(\omega_1,\gamma_1,k)=\sum_{k=1}^{\infty}\widehat{r}_q(\omega_1,\gamma_1,q^mk)=\frac{1}{q^{2m}}\sum_{k=1}^{\infty}\widehat{r}_q(\omega_1,\gamma_1,k).$$

The result for $d=1$ now follows by applying Jensen's inequality to the infinite sum above.

Suppose for some $1\le d<s$ we have $\boldsymbol{\alpha}^*=(\alpha_1^*,\dots,\alpha_d^*)\in(\mathbb{F}_{q^m}\setminus\{0\})^d$ and

$$\widehat{e}^2(\boldsymbol{\alpha}^*)\le\Big(\frac{1}{q^m-1}\sum_{\boldsymbol{l}\in\mathbb{N}_0^d}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^{\lambda}\Big)^{\frac{1}{\lambda}}.$$

Now we have

$$\begin{aligned}\widehat{e}^2(\boldsymbol{\alpha},\alpha_{d+1})&=\sum_{\substack{(\boldsymbol{l},l_{d+1})\in\mathbb{N}_0^{d+1}\setminus\{\mathbf{0}\}\\ (\mathrm{tc}_{m,\varphi}(\boldsymbol{l}),\mathrm{tc}_{m,\varphi}(l_{d+1}))\in\mathcal{N}_{(\boldsymbol{\alpha},\alpha_{d+1})}}}\widehat{r}_q(w_{d+1},\gamma_{d+1},l_{d+1})\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})\\&=\widehat{r}_q(w_{d+1},\gamma_{d+1},0)\widehat{e}^2(\boldsymbol{\alpha})+\theta(\alpha_{d+1}),\end{aligned}$$

where

$$\theta(\alpha_{d+1}) = \sum_{l_{d+1}=1}^{\infty} \widehat{r}_q(w_{d+1},\gamma_{d+1},l_{d+1}) \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^d \\ (\mathrm{tc}_{m,\varphi}(\boldsymbol{l}),\mathrm{tc}_{m,\varphi}(l_{d+1}))\in\mathcal{N}_{(\boldsymbol{\alpha},\alpha_{d+1})}}} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l}).$$

In Algorithm 3.6, α_{d+1}^* is chosen such that the mean square worst-case error $\widehat{e}^2(\boldsymbol{\alpha}^*,\alpha_{d+1})$ is minimized. Since the only dependence on α_{d+1} is in $\theta(\alpha_{d+1})$, we have $\theta(\alpha_{d+1}^*) \le \theta(\alpha_{d+1})$ for all $\alpha_{d+1} \in \mathbb{F}_{q^m}\setminus\{0\}$. Hence for any $\lambda \le 1$ we obtain

$$\theta(\alpha_{d+1}^*) \le \Big(\frac{1}{q^m-1}\sum_{\alpha_{d+1}\in\mathbb{F}_{q^m}\setminus\{0\}} \theta(\alpha_{d+1})^\lambda\Big)^{\frac{1}{\lambda}}.$$

From Jensen's inequality it follows that

$$\theta(\alpha_{d+1})^\lambda \le \sum_{l_{d+1}=1}^{\infty} \widehat{r}_q(w_{d+1},\gamma_{d+1},l_{d+1})^\lambda \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^d \\ (\mathrm{tc}_{m,\varphi}(\boldsymbol{l}),\mathrm{tc}_{m,\varphi}(l_{d+1}))\in\mathcal{N}_{(\boldsymbol{\alpha},\alpha_{d+1})}}} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda.$$

If l_{d+1} is a multiple of q^m, then $\mathrm{tc}_{m,\varphi}(l_{d+1})=0$ and the sum is independent of α_{d+1}. Otherwise $\mathrm{tc}_{m,\varphi}(l_{d+1})\neq 0$. We obtain

$$\frac{1}{q^m-1}\sum_{\alpha_{d+1}\in\mathbb{F}_{q^m}\setminus\{0\}} \theta(\alpha_{d+1})^\lambda \le \sum_{\substack{l_{d+1}=1 \\ q^m | l_{d+1}}}^{\infty} \widehat{r}_q(w_{d+1},\gamma_{d+1},l_{d+1})^\lambda \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^d \\ \mathrm{tc}_{m,\varphi}(\boldsymbol{l})\in\mathcal{N}_{\boldsymbol{\alpha}}}} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda$$
$$+\frac{1}{q^m-1}\sum_{\substack{l_{d+1}=1 \\ q^m \nmid l_{d+1}}}^{\infty} \widehat{r}_q(w_{d+1},\gamma_{d+1},l_{d+1})^\lambda \sum_{\boldsymbol{l}\in\mathbb{N}_0^d} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda \sum_{\substack{\alpha_{d+1}\in\mathbb{F}_{q^m}\setminus\{0\} \\ (\mathrm{tc}_{m,\varphi}(\boldsymbol{l}),\mathrm{tc}_{m,\varphi}(l_{d+1}))\in\mathcal{N}_{(\boldsymbol{\alpha},\alpha_{d+1})}}} 1.$$

From $\mathrm{tc}_{m,\varphi}(l_{d+1}) \neq 0$ we obtain

$$\sum_{\boldsymbol{l}\in\mathbb{N}_0^d} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda \sum_{\substack{\alpha_{d+1}\in\mathbb{F}_{q^m}\setminus\{0\} \\ (\mathrm{tc}_{m,\varphi}(\boldsymbol{l}),\mathrm{tc}_{m,\varphi}(l_{d+1}))\in\mathcal{N}_{(\boldsymbol{\alpha},\alpha_{d+1})}}} 1$$
$$= \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^d \\ \mathrm{tc}_{m,\varphi}(\boldsymbol{l})\notin\mathcal{N}_{\boldsymbol{\alpha}}}} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda = \sum_{\boldsymbol{l}\in\mathbb{N}_0^d} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda - \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^d \\ \mathrm{tc}_{m,\varphi}(\boldsymbol{l})\in\mathcal{N}_{\boldsymbol{\alpha}}}} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda.$$

Therefore we can estimate

$$\frac{1}{q^m-1}\sum_{\alpha_{d+1}\in\mathbb{F}_{q^m}\setminus\{0\}} \theta(\alpha_{d+1})^\lambda \le \sum_{\substack{l_{d+1}=1 \\ q^m | l_{d+1}}}^{\infty} \widehat{r}_q(w_{d+1},\gamma_{d+1},l_{d+1})^\lambda \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^d \\ \mathrm{tc}_{m,\varphi}(\boldsymbol{l})\in\mathcal{N}_{\boldsymbol{\alpha}}}} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda$$
$$+\frac{1}{q^m-1}\sum_{\substack{l_{d+1}=1 \\ q^m \nmid l_{d+1}}}^{\infty} \widehat{r}_q(w_{d+1},\gamma_{d+1},l_{d+1})^\lambda \Big(\sum_{\boldsymbol{l}\in\mathbb{N}_0^d} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda - \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^d \\ \mathrm{tc}_{m,\varphi}(\boldsymbol{l})\in\mathcal{N}_{\boldsymbol{\alpha}}}} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda\Big)$$
$$\le \sum_{l=1}^{\infty} \widehat{r}_q(w_{d+1},\gamma_{d+1},q^m l)^\lambda \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^d \\ \mathrm{tc}_{m,\varphi}(\boldsymbol{l})\in\mathcal{N}_{\boldsymbol{\alpha}}}} \widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda$$

$$+\frac{1}{q^m-1}\sum_{l=1}^{\infty}\widehat{r}_q(w_{d+1},\gamma_{d+1},l)^\lambda\Big(\sum_{\boldsymbol{l}\in\mathbb{N}_0^d}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda-\sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^d\\ \mathrm{tc}_{m,\varphi}(\boldsymbol{l})\in\mathcal{N}_{\boldsymbol{\alpha}}}}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda\Big)$$

$$\le\frac{1}{q^m-1}\sum_{l=1}^{\infty}\widehat{r}_q(w_{d+1},\gamma_{d+1},l)^\lambda\sum_{\boldsymbol{l}\in\mathbb{N}_0^d}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda.$$

Here the last inequality follows from the fact that

$$\sum_{l=1}^{\infty}\widehat{r}_q(w_{d+1},\gamma_{d+1},q^m l)^\lambda-\frac{1}{q^m-1}\sum_{l=1}^{\infty}\widehat{r}_q(w_{d+1},\gamma_{d+1},l)^\lambda\le 0$$

which follows from the definition of $\widehat{r}_q$ and since $\lambda>\frac{1}{2}$. Now we have

$$\theta(\alpha_{d+1}^*)\le\Big(\frac{1}{q^m-1}\sum_{l=1}^{\infty}\widehat{r}_q(w_{d+1},\gamma_{d+1},l)^\lambda\sum_{\boldsymbol{l}\in\mathbb{N}_0^d}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda\Big)^{\frac{1}{\lambda}}.$$

From this it follows that

$$\begin{aligned}\widehat{e}^2(\boldsymbol{\alpha}^*,\alpha_{d+1}^*)&\le\widehat{r}_q(w_{d+1},\gamma_{d+1},0)\widehat{e}^2(\boldsymbol{\alpha}^*)\\&\quad+\Big(\frac{1}{q^m-1}\sum_{l=1}^{\infty}\widehat{r}_q(w_{d+1},\gamma_{d+1},l)^\lambda\sum_{\boldsymbol{l}\in\mathbb{N}_0^d}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda\Big)^{\frac{1}{\lambda}}\\&\le\Big(\frac{1}{q^m-1}\sum_{\boldsymbol{l}\in\mathbb{N}_0^d}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda\Big)^{\frac{1}{\lambda}}\Big(\widehat{r}_q(w_{d+1},\gamma_{d+1},0)+\Big(\sum_{l=1}^{\infty}\widehat{r}_q(w_{d+1},\gamma_{d+1},l)^\lambda\Big)^{\frac{1}{\lambda}}\Big).\end{aligned}$$

Again from Jensen's inequality we obtain

$$\widehat{r}_q(w_{d+1},\gamma_{d+1},0)+\Big(\sum_{l=1}^{\infty}\widehat{r}_q(w_{d+1},\gamma_{d+1},l)^\lambda\Big)^{\frac{1}{\lambda}}\le\Big(\sum_{l=0}^{\infty}\widehat{r}_q(w_{d+1},\gamma_{d+1},l)^\lambda\Big)^{\frac{1}{\lambda}}$$

and hence

$$\widehat{e}^2(\boldsymbol{\alpha}^*,\alpha_{d+1}^*)\le\Big(\frac{1}{q^m-1}\sum_{\boldsymbol{l}\in\mathbb{N}_0^{d+1}}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda\Big)^{\frac{1}{\lambda}}.$$

This finishes our induction proof of inequality (3.3).

Finally we have

$$\begin{aligned}\sum_{\boldsymbol{l}\in\mathbb{N}_0^d}\widehat{r}_q(\boldsymbol{w},\boldsymbol{\gamma},\boldsymbol{l})^\lambda&=\prod_{j=1}^{d}\sum_{l=0}^{\infty}\widehat{r}_q(w_j,\gamma_j,l)^\lambda\\&=\prod_{j=1}^{d}\Big((1+\gamma_j[w_j^2-w_j+\tfrac{1}{3}])^\lambda+\sum_{l=1}^{\infty}\widehat{r}_q(w_j,\gamma_j,l)^\lambda\Big).\end{aligned}$$

As in the proof of Theorem 3.4 we obtain

$$\sum_{l=1}^{\infty}\widehat{r}_q(w_j,\gamma_j,l)^\lambda\le\gamma_j^\lambda\zeta_q(\lambda)$$

and the result follows. □

The following corollary shows that under certain conditions on the weights we can obtain an upper bound which depends only polynomially on the dimension, and, with stronger conditions on the weights, we can also obtain an upper bound which is independent of the dimension, thus proving strong tractability (see [20] for more information on (strong) tractability). An analogous result was shown for the component-by-component construction of polynomial lattice rules, see [4].

COROLLARY 3.8. *Let $s \geq 2$, let q be prime power, $\mathbb{F}_q$ be a finite field with q elements, $\varphi : \{0, 1, \ldots, q-1\} \longrightarrow \mathbb{F}_q$ a bijection with $\varphi(0) = 0$ and $\mathcal{B}_1, \ldots \mathcal{B}_s$ be s ordered bases of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. Further let $m \geq 1$. Suppose $\boldsymbol{\alpha}^* \in (\mathbb{F}_{q^m} \setminus \{0\})^s$ is constructed by Algorithm 3.6. Let $N = q^m$.*

1. *We have*

$$\widehat{e}(\boldsymbol{\alpha}^*) \leq c_{s,\boldsymbol{w},\boldsymbol{\gamma},\delta} N^{-1+\delta} \qquad \text{for all } 0 < \delta \leq \tfrac{1}{2},$$

where

$$c_{s,\boldsymbol{w},\boldsymbol{\gamma},\delta} := 2^{1-\delta} \prod_{j=1}^{s} \left(1 + \gamma_j^{\frac{1}{2(1-\delta)}} \left[\left(w_j^2 - w_j + \tfrac{1}{3}\right)^{\frac{1}{2(1-\delta)}} + \zeta_q\left(\tfrac{1}{2(1-\delta)}\right)\right]\right)^{1-\delta}.$$

2. *Suppose*

$$\sum_{j=1}^{\infty} \gamma_j^{\frac{1}{2(1-\delta)}} < \infty.$$

Then $c_{s,\boldsymbol{w},\boldsymbol{\gamma},\delta} \leq c_{\infty,\boldsymbol{w},\boldsymbol{\gamma},\delta} < \infty$ and we have

$$\widehat{e}(\boldsymbol{\alpha}^*) \leq c_{\infty,\boldsymbol{w},\boldsymbol{\gamma},\delta} N^{-1+\delta} \qquad \text{for all } 0 < \delta \leq \tfrac{1}{2}.$$

Thus the root mean square worst-case error of the hyperplane net generated by $\boldsymbol{\alpha}^*$ (with respect to the bijection φ) is bounded independently of the dimension.

3. *Under the assumption*

$$A := \limsup_{s \to \infty} \frac{\sum_{j=1}^{s} \gamma_j}{\log s} < \infty$$

we obtain $c_{s,\boldsymbol{w},\boldsymbol{\gamma},1/2} \leq \widetilde{c}_\eta s^{(A+\eta)/2}$ and therefore

$$\widehat{e}(\boldsymbol{\alpha}^*) \leq \widetilde{c}_\eta s^{(A+\eta)/2} N^{-\frac{1}{2}} \qquad \text{for all } \eta > 0,$$

where the constant $\widetilde{c}_\eta$ depends only on the arbitrarily chosen parameter η. Thus the root mean square worst-case error of the hyperplane net generated by $\boldsymbol{\alpha}^*$ (with respect to the bijection φ) satisfies a bound which depends only polynomially on the dimension.

The result can be shown using the methods employed in the proof of [4, Corollary 4.5].

Appendix A. Computation of the digital shift invariant kernel.

Here we compute the digital shift invariant kernel for the reproducing kernel

$$K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{j=1}^{s} (1 + \gamma_j \varrho_{w_j}(x_j, y_j)),$$

where $\boldsymbol{w} = (w_1, \ldots, w_s) \in [0,1]^s$ and

$$\begin{aligned}\varrho_w(x,y) &= \frac{|x-w|+|y-w|-|x-y|}{2} \\ &= \begin{cases} \min(|x-w|,|y-w|) & \text{if } (x-w)(y-w) \geq 0, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

We have

$$\begin{aligned}K_{\mathrm{ds},q,\boldsymbol{\gamma},\boldsymbol{w},\varphi}(\boldsymbol{x},\boldsymbol{y}) &:= \int_{[0,1)^s} K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}(\boldsymbol{x} \oplus_\varphi \boldsymbol{\sigma}, \boldsymbol{y} \oplus_\varphi \boldsymbol{\sigma}) \,\mathrm{d}\boldsymbol{\sigma} \\ &= \prod_{j=1}^s \int_0^1 K_{\mathrm{sob},1,w_j,\gamma_j}(x_j \oplus_\varphi \sigma, y_j \oplus_\varphi \sigma) \,\mathrm{d}\sigma,\end{aligned}$$

where $K_{\mathrm{sob},1,w,\gamma}(x,y) := 1 + \gamma\rho_w(x,y)$. So it suffices to compute

$$K_{\mathrm{ds},q,\gamma,w,\varphi}(x,y) := \int_0^1 K_{\mathrm{sob},1,w,\gamma}(x \oplus_\varphi \sigma, y \oplus_\varphi \sigma) \,\mathrm{d}\sigma.$$

It can easily be seen that the function $K_{\mathrm{sob},1,w,\gamma}(x,y)$ is in $\mathcal{L}_2([0,1)^2)$ and therefore we can apply [3, Lemma 5] from which we find that

$$K_{\mathrm{ds},q,\gamma,w,\varphi}(x,y) = \sum_{k=0}^{\infty} \widehat{K}(k) \mathrm{wal}_k(x) \overline{\mathrm{wal}_k(y)},$$

where

$$\widehat{K}(k) = \int_0^1 \int_0^1 K_{\mathrm{sob},1,w,\gamma}(x,y) \overline{\mathrm{wal}_k(x)} \mathrm{wal}_k(y) \,\mathrm{d}x \,\mathrm{d}y.$$

By Proposition 2.4.3 it follows easily that

$$\widehat{K}(k) = \begin{cases} 1 + \gamma(w^2 - w + \frac{1}{3}) & \text{if } k = 0, \\ -\frac{\gamma}{2} \int_0^1 \int_0^1 |x-y| \mathrm{wal}_k(y \ominus_\varphi x) \,\mathrm{d}x \,\mathrm{d}y & \text{if } k > 0.\end{cases}$$

Now we evaluate the last integral.

Lemma A.1. *Let* $q^{a-1} \leq k < q^a, \kappa_{a-1} = \lfloor k/q^{a-1} \rfloor > 0$. *Then*

$$\begin{aligned}\tau(k) &:= \int_{[0,1)^2} |x-y| \mathrm{wal}_k(y \ominus_\varphi x) \,\mathrm{d}x \,\mathrm{d}y \\ &= \frac{1}{3q^{2a}} + \frac{2}{q^{2a}} \Re\Big(\sum_{\substack{u,v=0 \\ v \geq u}}^{q-1} \frac{(v-u)\mathrm{wal}_{\kappa_{a-1}}\left(\frac{u \ominus_\varphi v}{q}\right)}{q} \Big),\end{aligned}$$

where $\Re$ *is the real part function.*

Proof. First we remark that

$$\int_{u/q^a}^{(u+1)/q^a} \int_{v/q^a}^{(v+1)/q^a} |x-y| \,\mathrm{d}x \,\mathrm{d}y = \begin{cases} 1/(3q^{3a}) & \text{if } u = v, \\ |v-u|/(q^{3a}) & \text{otherwise.}\end{cases}$$

In the same way as in [3, Appendix A], we partition the unit square in sub-squares where the Walsh function is constant and get

$$\begin{aligned}\tau(k) &= \sum_{u,v=0}^{q^a-1} \mathrm{wal}_k\left(\frac{u\ominus_\varphi v}{q^a}\right)\int_{u/q^a}^{(u+1)/q^a}\int_{v/q^a}^{(v+1)/q^a}|x-y|\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{3q^{2a}}+\frac{1}{q^{3a}}\sum_{\substack{u,v=0\\ v>u}}^{q^a-1}(v-u)\Big(\mathrm{wal}_k\left(\frac{u\ominus_\varphi v}{q^a}\right)+\overline{\mathrm{wal}_k\left(\frac{u\ominus_\varphi v}{q^a}\right)}\Big)\\ &= \frac{1}{3q^{2a}}+\frac{2}{q^{3a}}\Re\Big(\sum_{\substack{u,v=0\\ v>u}}^{q^a-1}(v-u)\mathrm{wal}_k\left(\frac{u\ominus_\varphi v}{q^a}\right)\Big). \qquad (A.1)\end{aligned}$$

Let

$$0\le u = qu'+u_0 < v = qv'+v_0 < q^a, \qquad v'>u', \qquad 0\le u_0,v_0<q.$$

Since $|\mathrm{wal}_k((u'\ominus_\varphi v')/q^{a-1})| = 1$ and $\sum_{u_0,v_0}(v'-u')\mathrm{wal}_{\kappa_{a-1}}((u_0\ominus_\varphi v_0)/q) = 0$, we have

$$\left|\sum_{u_0,v_0=0}^{q-1}((qv'+v_0)-(qu'+u_0))\mathrm{wal}_k\left(\frac{(qu'+u_0)\ominus_\varphi(qv'+v_0)}{q^a}\right)\right| = \left|\sum_{u_0,v_0=0}^{q-1}T_{\kappa_{a-1}}(u_0,v_0)\right|,$$

where $T_\kappa(u,v) := (v-u)\mathrm{wal}_\kappa((u\ominus_\varphi v)/q)$. By the character properties of the Walsh function system

$$\sum_{u_0,v_0=0}^{q-1}T_{\kappa_{a-1}}(u_0,v_0) = \sum_{u_0,v_0=0}^{q-1}(v_0-u_0)\mathrm{wal}_{\kappa_{a-1}}\left(\frac{u_0\ominus_\varphi v_0}{q}\right) \qquad (A.2)$$

$$= \sum_{v_0=0}^{q-1}v_0\overline{\mathrm{wal}}_{\kappa_{a-1}}\left(\frac{v_0}{q}\right)\sum_{u_0=0}^{q-1}\mathrm{wal}_{\kappa_{a-1}}\left(\frac{u_0}{q}\right)-\sum_{u_0=0}^{q-1}u_0\mathrm{wal}_{\kappa_{a-1}}\left(\frac{u_0}{q}\right)\sum_{v_0=0}^{q-1}\overline{\mathrm{wal}}_{\kappa_{a-1}}\left(\frac{v_0}{q}\right)=0$$

So the only (u,v) left in the sum (A.1) are $(qu'+u_0, qu'+v_0)$ for $0\le u'<q^{a-1}$. In this case the summands in the sum of (A.1) equal $T_{\kappa_{a-1}}(u_0,v_0)$ for fixed u', we get

$$\begin{aligned}\tau(k) &= \frac{1}{3q^{2a}}+\frac{2}{q^{3a}}\Re\Big(\sum_{u'=0}^{q^{a-1}-1}\sum_{\substack{u_0,v_0=0\\ v_0>u_0}}^{q-1}T_{\kappa_{a-1}}(u_0,v_0)\Big)\\ &= \frac{1}{3q^{2a}}+\frac{2}{q^{2a+1}}\Re\Big(\sum_{\substack{u_0,v_0=0\\ v_0\ge u_0}}^{q-1}(v_0-u_0)\mathrm{wal}_{\kappa_{a-1}}\left(\frac{u_0\ominus_\varphi v_0}{q}\right)\Big),\end{aligned}$$

as claimed. (Note the difference to (A.2) in the summation range of v_0.) □

Now define

$$\widehat{r}_q(w,\gamma,k) := \begin{cases} 1+\gamma(w^2-w+\frac{1}{3}) & \text{if } k=0,\\ -\frac{\gamma}{2}\Big(\frac{1}{3q^{2a}}+\frac{2}{q^{2a}}\Re\Big(\sum\limits_{\substack{u,v=0\\ v\ge u}}^{q-1}\frac{(v-u)\mathrm{wal}_{\kappa_{a-1}}\left(\frac{u\ominus_\varphi v}{q}\right)}{q}\Big)\Big) & \text{if } k>0,\end{cases}$$

and for $\boldsymbol{w} = (w_1, \ldots, w_s)$, $\boldsymbol{k} = (k_1, \ldots, k_s)$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots)$ define

$$\widehat{r}_q(\boldsymbol{w}, \boldsymbol{\gamma}, \boldsymbol{k}) := \prod_{j=1}^{s} \widehat{r}_q(w_j, \gamma_j, k_j).$$

Then we obtain

THEOREM A.2. *The digital shift invariant kernel for the reproducing kernel $K_{\mathrm{sob},s,\boldsymbol{w},\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y})$, where the digital shift is taken in prime-power base q and with respect to the bijection φ, is given by*

$$K_{\mathrm{ds},q,\boldsymbol{\gamma},\boldsymbol{w},\varphi}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{r}_q(\boldsymbol{w}, \boldsymbol{\gamma}, \boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{y})}.$$

Appendix B. Some other useful results.

Here we prove two results which are used in Section 3.

LEMMA B.1. *With $T_\kappa(u, v) := (v - u)\mathrm{wal}_\kappa((u \ominus_\varphi v)/q)$, for $l \in \{0, \ldots, q-1\}$ we have*

$$\sum_{\kappa=1}^{q-1} \sum_{\substack{u,v=0 \\ v \geq u}}^{q-1} \mathrm{wal}_\kappa \left(\frac{l}{q}\right) T_\kappa(u, v) = q\Big(\frac{1-q^2}{6} + \sum_{\substack{u=0, \\ u < u \oplus_\varphi l}}^{q-1} (u \oplus_\varphi l) - u\Big)$$

and

$$\sum_{\kappa=1}^{q-1} \sum_{\substack{u,v=0 \\ v \geq u}}^{q-1} \mathrm{wal}_\kappa \left(\frac{l}{q}\right) \overline{T_\kappa(u, v)} = q\Big(\frac{1-q^2}{6} + \sum_{\substack{u=0, \\ u < u \ominus_\varphi l}}^{q-1} (u \ominus_\varphi l) - u\Big).$$

Proof. For the first sum we have

$$\sum_{\kappa=1}^{q-1} \sum_{\substack{u,v=0 \\ v \geq u}}^{q-1} (v-u)\mathrm{wal}_\kappa \left(\frac{l \oplus_\varphi u \ominus_\varphi v}{q}\right) = \sum_{\substack{u,v=0 \\ v \geq u}}^{q-1} (v-u) \sum_{\kappa=1}^{q-1} \mathrm{wal}_\kappa \left(\frac{l \oplus_\varphi u \ominus_\varphi v}{q}\right),$$

where the right sum can be simplified such that the last line is equal to

$$\begin{aligned}
&\sum_{\substack{u,v=0 \\ v \geq u}}^{q-1} (v-u) \times \begin{cases} q-1 & \text{if } v = u \oplus_\varphi l, \\ -1 & \text{else,} \end{cases} \\
=&\sum_{u=0}^{q-1} \sum_{v=u}^{q-1} -(v-u) + q \sum_{\substack{u,v=0 \\ v \geq u, v = u \oplus_\varphi l}}^{q-1} (v-u) = \sum_{d=0}^{q-1} -(q-d)d + q \sum_{\substack{u=0 \\ u < u \oplus_\varphi l}}^{q-1} ((u \oplus_\varphi l) - u) \\
=&q\Big(\frac{-(q^2-1)}{6} + \sum_{\substack{u=0 \\ u < u \oplus_\varphi l}}^{q-1} ((u \oplus_\varphi l) - u)\Big).
\end{aligned}$$

The second part follows from the first by

$$\sum_{\kappa=1}^{q-1} \sum_{\substack{u,v=0 \\ v \geq u}}^{q-1} \mathrm{wal}_\kappa \Big(\frac{l}{q}\Big) \overline{T_\kappa\Big(u, v\Big)}$$

$$=\overline{\sum_{\kappa=1}^{q-1}\sum_{\substack{u,v=0\\v\geq u}}^{q-1}\overline{\mathrm{wal}_\kappa\Big(\frac{l}{q}\Big)}T_\kappa\Big(u,v\Big)}=\overline{\sum_{\kappa=1}^{q-1}\sum_{\substack{u,v=0\\v\geq u}}^{q-1}\mathrm{wal}_\kappa\Big(\frac{\ominus_\varphi l}{q}\Big)T_\kappa\Big(u,v\Big)},$$

which finishes the proof. □

The next lemma was used in Section 3 to give an explicit computable representation of the shift invariant kernel $K_{\mathrm{ds},q,\boldsymbol{\gamma},\boldsymbol{w},\varphi}$.

LEMMA B.2. *If $x \neq y$,*

$$\sum_{k=1}^{\infty}\frac{-\tau(k)}{2}\mathrm{wal}_k(x)\overline{\mathrm{wal}_k(y)}=\frac{1}{6}-\frac{1}{2q^{i_0+1}}\times$$

$$\times\Big(\sum_{\substack{u=0,\\u<u\oplus_\varphi x_{i_0}\ominus_\varphi y_{i_0}}}^{q-1}(u\oplus_\varphi x_{i_0}\ominus_\varphi y_{i_0}-u)+\sum_{\substack{u=0,\\u<u\oplus_\varphi y_{i_0}\ominus_\varphi x_{i_0}}}^{q-1}(u\oplus_\varphi y_{i_0}\ominus_\varphi x_{i_0}-u)\Big),$$

with x_{i_0}, y_{i_0} denoting the i_0-th fractional q-adic digits of x and y, where i_0 is the smallest index such that the digits differ. For $x = y$ the sum is equal to $1/6$. (Note that for $k > 0$ we have $\widehat{r}_q(w,\gamma,k) = -\frac{\gamma}{2}\tau(k)$.)

Proof. Let $D_{a,\kappa}$ denote $D_{a,\kappa}=\sum_{k=\kappa q^{a-1}+\cdots}\mathrm{wal}_k$, where the sum ranges over all k with the leading term κq^{a-1} in the q-adic expansion. By the character properties of the Walsh function set we have

$$D_{a,\kappa}=\mathrm{wal}_{\kappa q^{a-1}}\sum_{0\leq k<q^{a-1}}\mathrm{wal}_k=\mathrm{wal}_{\kappa q^{a-1}}\cdot q^{a-1}\cdot 1_{[0,q^{-(a-1)})}.$$

First, let $x \neq y$, $i_0 = \max(\{i : x_j = y_j, \forall j = 0,\ldots,i \geq 0\}) + 1$, where x_i, y_i are the q-adic digits of x and y. Then, with $T_\kappa(u,v) = (v-u)\mathrm{wal}_\kappa((u\ominus_\varphi v)/q)$ as above, and since $\tau(k)$ depends only on the q-adic length a and most significant digit κ,

$$\sum_{k=1}^{\infty}\frac{-\tau(k)}{2}\mathrm{wal}_k(x)\overline{\mathrm{wal}_k(y)}=\sum_{a=1}^{\infty}\sum_{\kappa=1}^{q-1}\frac{-\tau(\kappa q^{a-1})}{2}D_{a,\kappa}(x\ominus_\varphi y)$$

$$=\sum_{a=1}^{\infty}\sum_{\kappa=1}^{q-1}\frac{1}{q^{2a}}\Re\Big(\frac{-1}{q}\sum_{\substack{u,v=0,\\v\geq u}}^{q-1}T_\kappa(u,v)-\frac{1}{6}\Big)D_{a,\kappa}(x\ominus_\varphi y)$$

$$=\sum_{a=1}^{i_0-1}\sum_{\kappa=1}^{q-1}\frac{1}{q^{2a}}\Re\Big(\frac{-1}{q}\sum_{\substack{u,v=0,\\v\geq u}}^{q-1}T_\kappa(u,v)-\frac{1}{6}\Big)D_{a,\kappa}(x\ominus_\varphi y)$$

$$+\sum_{\kappa=1}^{q-1}\frac{1}{q^{2i_0}}\Re\Big(\frac{-1}{q}\sum_{\substack{u,v=0,\\v\geq u}}^{q-1}T_\kappa(u,v)-\frac{1}{6}\Big)D_{i_0,\kappa}(x\ominus_\varphi y)$$

$$=\sum_{a=1}^{i_0-1}\sum_{\kappa=1}^{q-1}\frac{1}{q^{a+1}}\Re\Big(\frac{-1}{q}\sum_{\substack{u,v=0,\\v\geq u}}^{q-1}T_\kappa(u,v)-\frac{1}{6}\Big)$$

$$+\sum_{\kappa=1}^{q-1}\frac{1}{2q^{i_0+1}}\Big(\frac{-1}{q}\sum_{\substack{u,v=0,\\v\geq u}}^{q-1}\mathrm{wal}_{\kappa q^{i_0-1}}(x\ominus_\varphi y)T_\kappa(u,v)-\frac{\mathrm{wal}_{\kappa q^{i_0-1}}(x\ominus_\varphi y)}{6}\Big)$$

$$+\sum_{\kappa=1}^{q-1}\frac{1}{2q^{i_0+1}}\Big(\frac{-1}{q}\sum_{\substack{u,v=0,\\v\geq u}}^{q-1}\mathrm{wal}_{\kappa q^{i_0-1}}(x\ominus_\varphi y)\overline{T_\kappa(u,v)}-\frac{\mathrm{wal}_{\kappa q^{i_0-1}}(x\ominus_\varphi y)}{6}\Big)$$

$$= \sum_{a=1}^{i_0-1} \frac{1}{q^{a+1}} \Big(\frac{q^2-1}{6} - \frac{q-1}{6} \Big) + \frac{1}{2q^{i_0+1}} \Big(2 \cdot \frac{q^2-1}{6} -$$

$$- \Big(\sum_{\substack{u=0, \\ u < u \oplus_\varphi x_{i_0} \ominus_\varphi y_{i_0}}}^{q-1} (u \oplus_\varphi x_{i_0} \ominus_\varphi y_{i_0} - u) + \sum_{\substack{u=0, \\ u < u \oplus_\varphi y_{i_0} \ominus_\varphi x_{i_0}}}^{q-1} (u \oplus_\varphi y_{i_0} \ominus_\varphi x_{i_0} - u) \Big) - 2 \cdot \frac{-1}{6} \Big)$$

$$= \frac{1}{6} - \frac{1}{2q^{i_0+1}} \Big(\sum_{\substack{u=0, \\ u < u \oplus_\varphi x_{i_0} \ominus_\varphi y_{i_0}}}^{q-1} (u \oplus_\varphi x_{i_0} \ominus_\varphi y_{i_0} - u) + \sum_{\substack{u=0, \\ u < u \oplus_\varphi y_{i_0} \ominus_\varphi x_{i_0}}}^{q-1} (u \oplus_\varphi y_{i_0} \ominus_\varphi x_{i_0} - u) \Big),$$

by Lemma B.1.

If $x = y$, it is easy to see (e.g. by letting $i_0 \to \infty$ in the above term) that the second term vanishes. □

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