Duality Theory and Propagation Rules for Generalized Nets

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Abstract

Generalized nets and sequences are used in quasi-Monte Carlo rules for the approximation of high dimensional integrals over the unit cube. Hence one wants to have generalized nets and sequences of high quality.

In this paper we introduce a duality theory for generalized nets whose construction is not necessarily based on linear algebra over finite fields. We use this duality theory to prove propagation rules for such nets. This way we can obtain new generalized nets (sometimes with improved quality) from existing ones. We also extend our approach to the construction of generalized sequences.

1 Introduction

Generalizing the concept of (t, m, s)-nets and (t, s)-sequences in base b due to Niederreiter [12] the notion of $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences in base b was introduced in the recent paper [8]. Such nets and sequences are point sets in the unit cube $[0, 1)^s$ which can be used in a quasi-Monte Carlo rule $N^{-1} \sum_{h=0}^{N-1} f(\boldsymbol{x}_h)$ to approximate the integral $\int_{[0,1]^s} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$, i.e., the quadrature points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1} \in [0, 1)^s$ can be taken from a (t, m, s)-net (or one uses the first N points of a sequence). Throughout the paper a point set is always understood as a multiset, i.e., points may occur repeatedly.

The advantage of the more general concept due to [8] is that $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences in base b can exploit the smoothness α of a function f (which is not the case for the classical concepts of (t, m, s)-nets and (t, s)-sequences). For functions f having square integrable mixed partial derivatives of order α in each variable, the integration errors using $(t, \alpha, \beta, n, m, s)$-nets (having cardinality b^m) with $\alpha m = \beta n$ converge at least at a rate of $b^{t-(\alpha-1+\delta)m}$ for any $\delta > 0$ (see [1]). In special cases this can be improved to $b^{t-(\alpha+\delta)m}$ for any $\delta > 0$.

Special constructions of such point sets are based on the digital construction scheme introduced by Niederreiter [12] and generalized in [5, 6]; the resulting point sets are referred to as digital nets and sequences. Nowadays many propagation rules for nets

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and sequences, digital or not, and also for generalized digital nets (see [9]) are known. Roughly speaking, propagation rules are methods by which one can construct new nets and sequences from existing ones (sometimes the net or sequence does not change, only the parameters change and the net or sequence with such parameters might not have been known before).

In this paper we introduce generalizations of the following propagation rules, which all appeared in [9] for the special case of digital nets: the direct product of two generalized digital nets, the $(u, u+v)$-construction, the matrix product construction, the double m-construction, base change propagation rules and the higher order to higher order construction. In some cases the proofs are based on a new duality theory for not necessarily digital nets (in the digital case, duality theory is already well known to be an important tool for the analysis and construction of digital nets).

The paper is organized as follows: In Section 2, the definitions of $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences and some of their properties are recalled. Duality theory for not necessarily digital nets is presented in Section 3 and propagation rules for $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences are presented in Sections 4 and 5 respectively.

Throughout the paper $\mathbb{N}_0$ is used to denote non-negative integers and $\mathbb{N}$ is used to denote natural numbers.

2 The definitions of $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences

In this section, we give the definitions and some properties of $(t, \alpha, \beta, n, m, s)$-nets and $(t, \alpha, \beta, \sigma, s)$-sequences in base b. To this end some notation has to be fixed which is used throughout the paper.

Let $n, s \geq 1$, $b \geq 2$ be integers. For $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_s) \in \{0, \ldots, n\}^s$ let $|\boldsymbol{\nu}|_1 = \sum_{j=1}^s \nu_j$ and define $\boldsymbol{i}_{\boldsymbol{\nu}} = (i_{1,1}, \ldots, i_{1,\nu_1}, \ldots, i_{s,1}, \ldots, i_{s,\nu_s})$ with integers $1 \leq i_{j,\nu_j} < \ldots < i_{j,1} \leq n$ in case $\nu_j > 0$ and $\left\{i_{j,1}, \ldots, i_{j,\nu_j}\right\} = \emptyset$ in case $\nu_j = 0$, for $j = 1, \ldots, s$. For given $\boldsymbol{\nu}$ and $\boldsymbol{i}_{\boldsymbol{\nu}}$ let $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$, which we write as $\boldsymbol{a}_{\boldsymbol{\nu}} = (a_{1,i_{1,1}}, \ldots, a_{1,i_{1,\nu_1}}, \ldots, a_{s,i_{s,1}}, \ldots, a_{s,i_{s,\nu_s}})$.

A *generalized elementary interval in base* b is a subset of $[0,1)^s$ of the form

$$J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \{i_{j,1},\ldots,i_{j,\nu_j}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right),$$

where $\left\{i_{j,1}, \ldots, i_{j,\nu_j}\right\} = \emptyset$ in case $\nu_j = 0$ for $1 \leq j \leq s$.

From [8, Lemmas 3.1 and 3.2] it is known that for $\boldsymbol{\nu} \in \{0, \ldots, n\}^s$ and $\boldsymbol{i}_{\boldsymbol{\nu}}$ defined as above and fixed, the generalized elementary intervals $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ for $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0, \ldots, b-1\}^{|\boldsymbol{\nu}|_1}$ form a partition of $[0,1)^s$ and the volume of $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ is $b^{-|\boldsymbol{\nu}|_1}$.

We can now give the definition of $(t, \alpha, \beta, n, m, s)$-nets based on [8, Definition 4].

Definition 2.1 Let n, m, s, $\alpha \geq 1$ be natural numbers, let $0 < \beta \leq 1$ be a real number, and let $0 \leq t \leq \beta n$ be an integer. Let $b \geq 2$ be an integer and $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a multiset in $[0,1)^s$. We say that $\mathcal{P}$ is a (t,α,β,n,m,s)-*net in base* b, if for all integers $1 \le i_{j,\nu_j} < \cdots < i_{j,1}$, where $\nu_j \ge 0$, with

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \le \beta n - t,$$

where for $\nu_j = 0$ we set the empty sum $\sum_{l=1}^{0} i_{j,l} = 0$, the generalized elementary interval $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains exactly $b^{m-|\boldsymbol{\nu}|_1}$ points of $\mathcal{P}$ for each $\boldsymbol{a}_{\boldsymbol{\nu}} \in \{0,\dots,b-1\}^{|\boldsymbol{\nu}|_1}$.

A (t,α,β,n,m,s)-net in base b is called a *strict* (t,α,β,n,m,s)-*net in base* b, if it is not a (u,α,β,n,m,s)-net in base b with $u < t$.

Note that in the definition above the specific order of elements of a multiset is not important. The parameter t is often referred to as the *quality-parameter* of the net. By the *strength* of the net one means the quantity $\beta n - t$.

Remark 2.1 We obtain the definition of a classical (t,m,s)-net in base b due to Niederreiter [12, Definition 4.1] from Definition 2.1 by setting $\alpha = \beta = 1$, $n = m$, and considering all $\nu_1,\dots,\nu_s \ge 0$ so that $\sum_{j=1}^{s} \nu_j \le m-t$, where we set $i_{j,k} = \nu_j - k + 1$ for $k = 1,\dots,\nu_j$. In this case the definition can be simplified to the following: a multiset $\mathcal{P} = \{\boldsymbol{x}_0,\dots,\boldsymbol{x}_{b^m-1}\}$ whose elements belong to $[0,1)^s$ is a (t,m,s)-net in base b if for all integers $d_1,\dots,d_s \ge 0$ with $d_1 + \cdots + d_s = m - t$ each elementary interval $J = \prod_{j=1}^{s} \left[\frac{a_j}{b^{d_j}}, \frac{a_j+1}{b^{d_j}}\right)$ with integers $0 \le a_j < b^{d_j}$ for $1 \le j \le s$ and of volume b^{t-m} contains exactly b^t elements of $\mathcal{P}$. Hence a $(t,1,1,m,m,s)$-net is a (t,m,s)-net.

Remark 2.2 Let n, m, s, $\alpha \ge 1$ be natural numbers and let $0 < \beta \le 1$ be a real number. It follows from Definition 2.1 that any multiset consisting of b^m points in $[0,1)^s$ is a $(\lfloor \beta n \rfloor, \alpha, \beta, n, m, s)$-net in base b.

Remark 2.3 Note that $b^{m-|\boldsymbol{\nu}|_1} = b^m \mathrm{Vol}(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}}))$. Hence Definition 2.1 says that the proportion of points of $\mathcal{P}$ in $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ equals the volume of $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$, i.e.,

$$\frac{|\{0 \le h < b^m \,:\, \boldsymbol{x}_h \in J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})\}|}{b^m} = \mathrm{Vol}(J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})).$$

We also give the definition of $(t,\alpha,\beta,\sigma,s)$-sequences from [8].

Definition 2.2 Let σ, s, $\alpha \ge 1$ be natural numbers, let $0 < \beta \le 1$ be a real number, and let $t \ge 0$ be an integer. Let $b \ge 2$ be an integer and $\omega = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots)$ be an infinite sequence in $[0,1)^s$. We say that ω is a $(t,\alpha,\beta,\sigma,s)$-*sequence in base* b, if for all integers $k \ge 0$ and $m > t/(\beta\sigma)$ we have that the finite subsequence $\{\boldsymbol{x}_{kb^m}, \boldsymbol{x}_{kb^m+1}, \dots, \boldsymbol{x}_{(k+1)b^m-1}\}$ is a $(t,\alpha,\beta,\sigma m,m,s)$-net in base b.

A $(t,\alpha,\beta,\sigma,s)$-sequence in base b is called a *strict* $(t,\alpha,\beta,\sigma,s)$-*sequence in base* b if it is not a $(u,\alpha,\beta,\sigma,s)$-sequence in base b with $u < t$.

Note that in the definition above the specific order of elements of an infinite sequence is of importance.

Remark 2.4 We obtain the definition of a classical (t, s)-sequence in base b due to Niederreiter [12, Definition 4.2] from Definition 2.2 and Remark 2.1 by setting $\alpha = \beta = \sigma = 1$. Hence a $(t, 1, 1, 1, s)$-sequence in base b is a (t, s)-sequence in base b.

Explicit constructions of $(t, \alpha, \beta, n, m, s)$-nets, respectively $(t, \alpha, \beta, \sigma, s)$-sequences, in prime-power bases b are known using the digital construction scheme. Nets (sequences) constructed in this manner are referred to as digital $(t, \alpha, \beta, n{\times}m, s)$-nets (digital $(t, \alpha, \beta, \sigma, s)$-sequences) over a finite field $\mathbb{F}_b$. For more information we refer to [6, Subsection 4.4] and [9]. The proof that digital $(t, \alpha, \beta, n \times m, s)$-nets, respectively digital $(t, \alpha, \beta, \sigma, s)$-sequences, over $\mathbb{F}_b$ are in fact special cases of $(t, \alpha, \beta, n, m, s)$-nets, respectively $(t, \alpha, \beta, \sigma, s)$-sequences, in base b is to be found in [8, Theorem 3.5].

Some simple propagation rules for $(t, \alpha, \beta, n, m, s)$-nets, respectively $(t, \alpha, \beta, \sigma, s)$-sequences, in base b were already listed in [8]. For completeness, we repeat them here. We also add some further trivial propagation rules in the following list:

Let $\mathcal{P}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b and let ω be a $(t, \alpha, \beta, \sigma, s)$-sequence in base b. Then we have the following:

(i) $\mathcal{P}$ is a $(t', \alpha, \beta', n, m, s)$-net in base b for all $0 < \beta' \leq \beta$ and all $t \leq t' \leq \beta' n$, and ω is a $(t', \alpha, \beta', \sigma, s)$-sequence in base b for all $0 < \beta' \leq \beta$ and all $t \leq t'$.

(ii) $\mathcal{P}$ is a $(t', \alpha', \beta', n, m, s)$-net in base b for all $\alpha' \geq 1$ where $\beta' = \beta \min(\alpha, \alpha')/\alpha$ and $t' = \lceil t \min(\alpha, \alpha')/\alpha \rceil$, and ω is a $(t', \alpha', \beta', \sigma, s)$-sequence in base b for all $\alpha' \geq 1$ where $\beta' = \beta \min(\alpha, \alpha')/\alpha$ and where $t' = \lceil t \min(\alpha, \alpha')/\alpha \rceil$.

(iii) Consider the point set $\mathcal{P}'$ obtained by truncating each coordinate of each element of $\mathcal{P}$ in base b representation after n' digits, $1 \leq n' \leq n$. The resulting point set is a $(t', \alpha, \beta, n', m, s)$-net in base b, where $t' = \max(t - \beta(n - n'), 0)$.

(iv) Consider the point set $\mathcal{P}'$ obtained by truncating each coordinate of each element of $\mathcal{P}'$ in base b representation after n digits and adding $n' - n$ extra digits to every element, all of which are zero, $n' \geq n$. The resulting point set is a $(t, \alpha, \beta', n', m, s)$-net, where $\beta' = \beta n/n'$.

(v) The point set obtained by projecting $\mathcal{P}$ onto the coordinates in $\mathfrak{u}$, where $\mathfrak{u} \subseteq \{1, \ldots, s\}$, is a $(t_{\mathfrak{u}}, \alpha, \beta, n, m, |\mathfrak{u}|)$-net in base b, where $t_{\mathfrak{u}} \leq t$.

(vi) Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{b^r}$ be $(t, \alpha, \beta, n, m, s)$-nets in base b. Then the multiset obtained from the union of the elements of $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{b^r}$ is a $(t, \alpha, \beta, n, m + r, s)$-net in base b.

We remark that these propagation rules are analogous to Propagation rules I-VI in [9] for digital nets.

3 Duality theory

Duality theory, as introduced by Niederreiter and Pirsic [15] (see also [9]), is a helpful tool in the analysis and construction of digital nets. Here we introduce a duality theory for not necessarily digital constructions. The basic tool are Walsh functions in integer base $b \geq 2$ whose definition and basic properties are recalled in the following.

Definition 3.1 Let $b \geq 2$ be an integer and represent $k \in \mathbb{N}_0$ in base b, $k = \kappa_{a-1} b^{a-1} + \cdots + \kappa_0$, with $\kappa_i \in \{0, \ldots, b-1\}$. Further let $\omega_b = \mathrm{e}^{2\pi \mathtt{i}/b}$ be the bth root of unity. Then the *kth b-adic Walsh function* ${}_b\mathrm{wal}_k(x) : [0,1) \to \left\{1, \omega_b, \ldots, \omega_b^{b-1}\right\}$ is given by

$$ {}_b\mathrm{wal}_k(x) = \omega_b^{\xi_1 \kappa_0 + \cdots + \xi_a \kappa_{a-1}}, $$

for $x \in [0,1)$ with base b representation $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \ldots$ (unique in the sense that infinitely many of the ξ_i are different from $b-1$).

For dimension $s \geq 2$, $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1)^s$, and $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, we define ${}_b\mathrm{wal}_{\boldsymbol{k}} : [0,1)^s \to \left\{1, \omega_b, \ldots, \omega_b^{b-1}\right\}$ by

$$ {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{s} {}_b\mathrm{wal}_{k_j}(x_j). $$

Dealing with Walsh functions in conjunction with a $(t, \alpha, \beta, n, m, s)$-net in base b or a $(t, \alpha, \beta, \sigma, s)$-sequence in base b, we will always assume that b-adic Walsh functions are in the same base b. If we deal with an arbitrary point set, which is not necessarily a $(t, \alpha, \beta, n, m, s)$-net or a $(t, \alpha, \beta, \sigma, s)$-sequence in base b, we will deal with an arbitrary but fixed integer base $b \geq 2$. Consequently, we will in the following often write wal instead of ${}_b\mathrm{wal}$.

The following notation will be used throughout the paper: By $\oplus$ we denote the digit-wise addition modulo b, i.e., for $x, y \in [0,1)$ with base b expansions $x = \sum_{l=1}^{\infty} \xi_l b^{-l}$ and $y = \sum_{l=1}^{\infty} \eta_l b^{-l}$, we define

$$ x \oplus y = \sum_{l=1}^{\infty} \zeta_l b^{-l}, $$

where $\zeta_l \in \{0, \ldots, b-1\}$ is given by $\zeta_l \equiv \xi_l + \eta_l \pmod{b}$, and let $\ominus$ denote the digitwise subtraction modulo b (for short we use $\ominus x := 0 \ominus x$). In the same fashion we also define the digitwise addition and digitwise subtraction for nonnegative integers based on the b-adic expansion. For vectors in $[0,1)^s$ or $\mathbb{N}_0^s$, the operations $\oplus$ and $\ominus$ are carried out componentwise. Throughout the paper, we always use the same base b for the operations $\oplus$ and $\ominus$ as is used for the Walsh functions. Further we call $x \in [0,1)$ a b-adic rational if it can be written in a finite base b expansion. The following simple properties of Walsh functions are often used in the sequel:

For all $k, l \in \mathbb{N}_0$ and all $x, y \in [0,1)$, with the restriction that if x, y are not b-adic rationals, then $x \oplus y$ is not allowed to be a b-adic rational, we have $\mathrm{wal}_k(x) \cdot \mathrm{wal}_l(x) = \mathrm{wal}_{k \oplus l}(x)$ and $\mathrm{wal}_k(x) \cdot \mathrm{wal}_k(y) = \mathrm{wal}_k(x \oplus y)$. Furthermore, $\overline{\mathrm{wal}_k(x)} = \mathrm{wal}_{\ominus k}(x)$.

Now we turn to duality theory for not necessarily digital nets. Let $\mathcal{K}_{n,b}^s = \{0, \ldots, b^n - 1\}^s$. We also assume there is an ordering of the elements in $\mathcal{K}_{n,b}^s$ which can be arbitrary but needs to be the same in each instance, i.e., let $\mathcal{K}_{n,b}^s = \{\boldsymbol{k}_0, \ldots, \boldsymbol{k}_{b^{sn}-1}\}$. (Note that $|\mathcal{K}_{n,b}^s| = b^{sn}$.) By this we mean that in expressions like $\sum_{\boldsymbol{k} \in \mathcal{K}_{n,b}^s}$, $(a_{\boldsymbol{k},\boldsymbol{l}})_{\boldsymbol{k},\boldsymbol{l} \in \mathcal{K}_{n,b}^s}$, and $(c_{\boldsymbol{k}})_{\boldsymbol{k} \in \mathcal{K}_{n,b}^s}$ the elements $\boldsymbol{k}$ and $\boldsymbol{l}$ run through the set $\mathcal{K}_{n,b}^s$ always in the same order.

The following $b^{sn} \times b^{sn}$ matrix plays a central role in the duality theory for generalized nets

$$ \mathbf{W}_n := \left(\mathrm{wal}_{\boldsymbol{k}} \left(b^{-n} \boldsymbol{l} \right) \right)_{\boldsymbol{k},\boldsymbol{l} \in \mathcal{K}_{n,b}^s}. $$

We call $\mathbf{W}_n$ a *Walsh matrix.*

In the following we denote by A^* the conjugate transpose of a matrix A over the complex numbers $\mathbb{C}$, i.e., $A^* = \overline{A}^{\top}$.

Lemma 3.1 *The Walsh matrix* $\mathbf{W}_n$ *is invertible and its inverse is given by* $\mathbf{W}_n^{-1} = \frac{1}{b^{sn}}\mathbf{W}_n^*$.

Proof. Let $\mathbf{A} = (a_{\boldsymbol{k},\boldsymbol{l}})_{\boldsymbol{k},\boldsymbol{l}\in\mathcal{K}_{n,b}^s} = \mathbf{W}_n \frac{1}{b^{sn}}\mathbf{W}_n^*$. Then, using the orthogonality of the Walsh functions, we obtain

$$\begin{aligned} a_{\boldsymbol{k},\boldsymbol{l}} &= \frac{1}{b^{sn}} \sum_{\boldsymbol{h}\in\mathcal{K}_{n,b}^s} \mathrm{wal}_{\boldsymbol{k}}\left(b^{-n}\boldsymbol{h}\right) \overline{\mathrm{wal}_{\boldsymbol{l}}\left(b^{-n}\boldsymbol{h}\right)} = \frac{1}{b^{sn}} \prod_{j=1}^{s} \sum_{h=0}^{b^n-1} \mathrm{wal}_{k_j\ominus l_j}(h/b^n) \\ &= \begin{cases} 1 & \text{if } \boldsymbol{k} = \boldsymbol{l}, \\ 0 & \text{if } \boldsymbol{k} \neq \boldsymbol{l}. \end{cases} \end{aligned}$$

where $\boldsymbol{k} = (k_1, \ldots, k_s)$ and $\boldsymbol{l} = (l_1, \ldots, l_s)$ are in $\mathcal{K}_{n,b}^s$. □

For a multiset $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in $[0,1)^s$ and $\boldsymbol{k} \in \mathcal{K}_{n,b}^s$ we define

$$c_{\boldsymbol{k}} = c_{\boldsymbol{k}}(\mathcal{P}) := \sum_{h=0}^{N-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h)$$

(note that $|c_{\boldsymbol{k}}| \leq N$) and the vector

$$\vec{C} = \vec{C}(\mathcal{P}) := (c_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathcal{K}_{n,b}^s}. \tag{1}$$

For $\boldsymbol{a} = (a_1, \ldots, a_s) \in \mathcal{K}_{n,b}^s$ define the elementary b-adic interval

$$E_{\boldsymbol{a}} := \prod_{j=1}^{s} \left[\frac{a_j}{b^n}, \frac{a_j+1}{b^n}\right).$$

Lemma 3.2 *We have*

$$\sum_{\boldsymbol{k}\in\mathcal{K}_{n,b}^s} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}\ominus\boldsymbol{y}) = \begin{cases} |\mathcal{K}_{n,b}^s| & \text{if } \boldsymbol{x},\boldsymbol{y}\in E_{\boldsymbol{a}} \text{ for some } \boldsymbol{a}\in\mathcal{K}_{n,b}^s, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have $x, y \in [ab^{-n}, (a+1)b^{-n})$ for some $0 \leq a < b^n$ if and only if the b-adic digit expansions of x and y coincide at the first n digits. From this the result follows. □

Let $\boldsymbol{x} \in E_{\boldsymbol{a}}$ for some $\boldsymbol{a} \in \mathcal{K}_{n,b}^s$. Then, using Lemma 3.2, we have

$$\begin{aligned} \frac{1}{|\mathcal{K}_{n,b}^s|} \sum_{\boldsymbol{k}\in\mathcal{K}_{n,b}^s} c_{\boldsymbol{k}} \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})} &= \frac{1}{|\mathcal{K}_{n,b}^s|} \sum_{\boldsymbol{k}\in\mathcal{K}_{n,b}^s} \sum_{h=0}^{N-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h\ominus\boldsymbol{x}) \\ &= \sum_{h=0}^{N-1} \frac{1}{|\mathcal{K}_{n,b}^s|} \sum_{\boldsymbol{k}\in\mathcal{K}_{n,b}^s} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h\ominus\boldsymbol{x}) \\ &= |\{h : \boldsymbol{x}_h \in E_{\boldsymbol{a}}\}| =: m_{\boldsymbol{a}}. \end{aligned}$$

Definition 3.2 Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ be a multiset in $[0,1)^s$, let n be such that $N \leq b^n$, and let $\mathcal{K}_{n,b}^s = \{0, \ldots, b^n - 1\}^s$.

1. For $\boldsymbol{a} \in \mathcal{K}_{n,b}^s$ let
$$m_{\boldsymbol{a}} = m_{\boldsymbol{a}}(\mathcal{P}) := |\{h \,:\, \boldsymbol{x}_h \in E_{\boldsymbol{a}}\}|$$
and
$$\vec{M} = \vec{M}(\mathcal{P}) := (m_{\boldsymbol{a}})_{\boldsymbol{a} \in \mathcal{K}_{n,b}^s}.$$
Then we call the vector $\vec{M}$ the *point set vector.*

2. The vector $\vec{C} = \vec{C}(\mathcal{P})$ from (1) is called the *dual vector* (with respect to the Walsh matrix $\mathbf{W}_n$).

3. The set
$$\mathcal{D}_n = \mathcal{D}_n(\mathcal{P}) := \{\boldsymbol{k} \in \mathcal{K}_{n,b}^s : c_{\boldsymbol{k}} \neq 0\}$$
is called the *dual set* (with respect to the Walsh matrix $\mathbf{W}_n$).

The relationship between a point set vector and its dual vector is stated in the following theorem.

Theorem 3.1 *Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ be a multiset in $[0,1)^s$ and let n be such that $N \leq b^n$. Let $\vec{M}$ and $\vec{C}$ be defined as above. Then*
$$\frac{1}{|\mathcal{K}_{n,b}^s|}\mathbf{W}_n\vec{C} = \vec{M} \quad \textit{and} \quad \vec{C} = \mathbf{W}_n^*\vec{M}. \tag{2}$$

Proof. The first result follows from Lemma 3.2 and the second result follows from Lemma 3.1 and the identity $\vec{C} = |\mathcal{K}_{n,b}^s|\mathbf{W}_n^{-1}\vec{M} = \mathbf{W}_n^*\vec{M}$. □

The vector $\vec{C}$ carries sufficient information to construct a point set in the following way: Given $\vec{C}$, we can use Theorem 3.1 to determine how many points are to be placed in the interval $E_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{K}_{n,b}^s$. Note that for the $(t, \alpha, \beta, n, m, s)$-net property it is of no importance where exactly within an interval $E_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{K}_{n,b}^s$, the points are placed.

In analogy, the dual space of a digital net also allows us to reconstruct the original point set, see [15]. Although $\vec{C}$ is different from the dual space for digital nets, it contains the same information and can be used in a manner similar to the dual space. This will be shown below by example of the $(u, u+v)$-construction, the matrix-product construction and the double m construction for generalized nets. In case $\mathcal{P}$ is a digital $(t, \alpha, \beta, n \times m, s)$-net, the dual set $\mathcal{D}_n$ defined in Definition 3.2 coincides with the dual space defined in [9] intersected with $\mathcal{K}_{n,b}^s$, and if $\mathcal{P}$ is a digital (t, m, s)-net, it coincides with the dual space in [15] intersected with $\mathcal{K}_{m,b}^s$.

Although the above results hold for arbitrary point sets, in the following we consider point sets which are nets and show how to relate the quality of a $(t, \alpha, \beta, n, m, s)$-net to its dual set. To this end we need to introduce a function which was first introduced in [6] in the context of applying digital nets to quasi-Monte Carlo integration of smooth functions and which is related to the quality of suitable digital nets. For $k \in \mathbb{N}_0$ and $\alpha \geq 1$ let
$$\mu_\alpha(k) = \begin{cases} a_1 + \cdots + a_{\min(\nu,\alpha)} & \text{for } k > 0, \\ 0 & \text{for } k = 0, \end{cases}$$
where for $k > 0$ we assume that $k = \kappa_1 b^{a_1 - 1} + \cdots + \kappa_\nu b^{a_\nu - 1}$ with $0 < \kappa_1, \ldots, \kappa_\nu < b$ and $1 \leq a_\nu < \cdots < a_1$.

For a vector $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ we define $\mu_\alpha(\boldsymbol{k}) = \mu_\alpha(k_1) + \cdots + \mu_\alpha(k_s)$ and for a subset $\mathcal{Q}$ of $\mathcal{K}_{n,b}^s$ and $\alpha \geq 1$ define

$$\rho_\alpha(\mathcal{Q}) := \min_{\boldsymbol{k} \in \mathcal{Q} \setminus \{\boldsymbol{0}\}} \mu_\alpha(\boldsymbol{k}).$$

The following theorem establishes a relationship between $\rho_\alpha(\mathcal{Q})$ and the quality of a $(t, \alpha, \beta, n, m, s)$-net.

Theorem 3.2 *A multiset $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ in $[0,1)^s$ with dual set $\mathcal{D}_n$ is a $(t, \alpha, \beta, n, m, s)$-net in base b if and only if $\rho_\alpha(\mathcal{D}_n) \geq \lfloor \beta n \rfloor - t + 1$. If $\mathcal{P}$ is a strict $(t, \alpha, \beta, n, m, s)$-net in base b, then $\rho_\alpha(\mathcal{D}_n) = \lfloor \beta n \rfloor - t + 1$.*

Proof. It was shown in [1, Theorem 1] that $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b if and only if for all $\boldsymbol{k} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \beta n - t$ we have $\sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = 0$ and this is obviously equivalent to $\rho_\alpha(\mathcal{D}_n) \geq \lfloor \beta n \rfloor - t + 1$. For the second assertion, we assume that $\mathcal{P}$ is a strict $(t, \alpha, \beta, n, m, s)$-net in base b, which implies that $\rho_\alpha(\mathcal{D}_n) \geq \lfloor \beta n \rfloor - t + 1$. We now assume $\rho_\alpha(\mathcal{D}_n) \geq \lfloor \beta n \rfloor - t + 2 = \lfloor \beta n \rfloor - (t-1) + 1$. Arguing in the same fashion as for the first part of the proof this is equivalent to $\sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_h) = 0$ for all $\boldsymbol{k}$ satisfying $0 < \mu_\alpha(\boldsymbol{k}) \leq \lfloor \beta n \rfloor - (t-1)$. Using [1, Theorem 1] again this implies that $\mathcal{P}$ is a $(t-1, \alpha, \beta, n, m, s)$-net in base b, which contradicts the assumption that $\mathcal{P}$ is a strict $(t, \alpha, \beta, n, m, s)$-net in base b. $\square$

4 Propagation rules for $(t, \alpha, \beta, n, m, s)$-nets

In this section, we introduce several propagation rules for $(t, \alpha, \beta, n, m, s)$-nets which generalize the analogous results for the digital case given in [9].

4.1 Direct product of two $(t, \alpha, \beta, n, m, s)$-nets

Let $\mathcal{P}_1 = (\boldsymbol{x}_h)_{h=0}^{b^{m_1}-1}$ be a $(t_1, \alpha_1, \beta_1, n_1, m_1, s_1)$-net in base b and $\mathcal{P}_2 = (\boldsymbol{y}_i)_{i=0}^{b^{m_2}-1}$ be a $(t_2, \alpha_2, \beta_2, n_2, m_2, s_2)$-net in base b. Based on $\mathcal{P}_1$ and $\mathcal{P}_2$ a new $(t, \alpha, \beta, n, m, s)$-net in base b is formed, where $n = n_1 + n_2$, $m = m_1 + m_2$, and $s = s_1 + s_2$. The points of $\mathcal{P}$ are defined to be the direct product of the points from $\mathcal{P}_1$ and $\mathcal{P}_2$, i.e., $\mathcal{P}$ is the multiset of b^m points

$$(\boldsymbol{x}_h, \boldsymbol{y}_i), \quad \text{for} \quad 0 \leq h \leq b^{m_1} - 1 \text{ and } 0 \leq i \leq b^{m_2} - 1$$

in some order. The following theorem gives information on the t-value of the resulting $(t, \alpha, \beta, n, m, s)$-net.

Theorem 4.1 *Let $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}$ be defined as above. Then $\mathcal{P}$ is a $(t, \alpha, \beta, n, m, s)$-net in base b, where $\alpha = \max(\alpha_1, \alpha_2)$, $\beta = \min(\beta_1, \beta_2)$ and*

$$t = \max(\beta_1 n_1 + t_2, \beta_2 n_2 + t_1).$$

Proof. Note first that

$$\begin{aligned} \beta n - t &= \beta n_1 + \beta n_2 - \max(\beta_1 n_1 + t_2, \beta_2 n_2 + t_1) \\ &= \min(\beta n_1 + \beta n_2 - \beta_1 n_1 - t_2, \beta n_1 + \beta n_2 - \beta_2 n_2 - t_1) \end{aligned}$$

$$\leq \quad \min(\beta_1 n_1 - t_1, \beta_2 n_2 - t_2). \tag{3}$$

Let $\nu_1, \ldots, \nu_s \geq 0$ and, for $1 \leq j \leq s$, let $1 \leq i_{j,\nu_j} < \cdots < i_{j,1}$, with

$$i_{1,1} + \cdots + i_{1,\min(\nu_1,\alpha)} + \cdots + i_{s,1} + \cdots + i_{s,\min(\nu_s,\alpha)} \leq \beta n - t. \tag{4}$$

We need to check that the generalized elementary interval

$$\begin{aligned} J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu}) &= \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \{i_{j,\nu_j},\ldots,i_{j,1}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right) \\ &= \left[\prod_{j=1}^{s_1} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \{i_{j,\nu_j},\ldots,i_{j,1}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right)\right] \times \\ &\quad \left[\prod_{j=s_1+1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \{i_{j,\nu_j},\ldots,i_{j,1}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right)\right] \end{aligned}$$

contains $b^{m-|\boldsymbol{\nu}|_1}$ points.

Since $\beta_1, \beta_2 \leq 1$ we find from (3) and (4) that $i_{j,1} \leq n_1$ for $1 \leq j \leq s_1$ and $i_{j,1} \leq n_2$ for $j = s_1 + 1, \ldots, s_1 + s_2$. Hence the previous expression becomes

$$\begin{aligned} J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu}) &= \left[\prod_{j=1}^{s_1} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n_1\} \setminus \{i_{j,\nu_j},\ldots,i_{j,1}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n_1}}{b^{n_1}}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n_1}}{b^{n_1}} + \frac{1}{b^{n_1}}\right)\right] \times \\ &\quad \left[\prod_{j=s_1+1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n_2\} \setminus \{i_{j,\nu_j},\ldots,i_{j,1}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n_2}}{b^{n_2}}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n_2}}{b^{n_2}} + \frac{1}{b^{n_2}}\right)\right]. \end{aligned}$$

Again from (3) and (4) we deduce that

$$i_{1,1} + \cdots + i_{1,\min(\nu_1,\alpha)} + \cdots + i_{s_1,1} + \cdots + i_{s_1,\min(\nu_{s_1},\alpha)} \leq \beta_1 n_1 - t_1$$

and

$$i_{s_1+1,1} + \cdots + i_{s_1+1,\min(\nu_{s_1+1},\alpha)} + \cdots + i_{s,1} + \cdots + i_{s,\min(\nu_s,\alpha)} \leq \beta_2 n_2 - t_2.$$

As $\mathcal{P}_1$ is a $(t_1, \alpha_1, \beta_1, n_1, m_1, s_1)$-net in base b and $\mathcal{P}_2$ a $(t_2, \alpha_2, \beta_2, n_2, m_2, s_2)$-net in base b, it follows that

$$\prod_{j=1}^{s_1} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n_1\} \setminus \{i_{j,\nu_j},\ldots,i_{j,1}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n_1}}{b^{n_1}}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n_1}}{b^{n_1}} + \frac{1}{b^{n_1}}\right)$$

contains $b^{m_1-\sum_{j=1}^{s_1}\nu_j}$ points of $\mathcal{P}_1$ and

$$\prod_{j=s_1+1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l\in\{1,\ldots,n_2\}\setminus\{i_{j,\nu_j},\ldots,i_{j,1}\}}}^{b-1} \left[\frac{a_{j,1}}{b}+\cdots+\frac{a_{j,n_2}}{b^{n_2}}, \frac{a_{j,1}}{b}+\cdots+\frac{a_{j,n_2}}{b^{n_2}}+\frac{1}{b^{n_2}}\right)$$

contains $b^{m_2-\sum_{j=s_1+1}^{s}\nu_j}$ points of $\mathcal{P}_2$. By the construction method, it follows that $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains $b^{m_1+m_2-\sum_{j=1}^{s}\nu_j} = b^{m-|\boldsymbol{\nu}|_1}$ points of $\mathcal{P}$, concluding the proof. □

4.2 The $(u, u+v)$-construction

The $(u, u+v)$-construction in the context of $(t, \alpha, \beta, n, m, s)$-nets in base b has already been discussed in the recent paper [1, Section 5]. Hence we simply recall the construction and state the result. Assume we are given a $(t_1, \alpha, \beta_1, n_1, m_1, s_1)$-net $\mathcal{P}_1$ denoted by $\{\boldsymbol{x}_h\}_{h=0}^{b^{m_1}-1}$ and a $(t_2, \alpha, \beta_2, n_2, m_2, s_2)$-net $\mathcal{P}_2$ denoted by $\{\boldsymbol{y}_i\}_{i=0}^{b^{m_2}-1}$, where we assume $s_1 \le s_2$. W.l.o.g. we may assume that $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s_1})$ with $x_{h,j} = \xi_{h,j,1}/b + \cdots + \xi_{h,j,n_1}/b^{n_1}$ and $\boldsymbol{y}_i = (y_{i,1}, \ldots, y_{i,s_2})$ with $y_{i,j} = \eta_{i,j,1}/b + \cdots + \eta_{i,j,n_2}/b^{n_2}$ (if there are digits $\xi_{h,j,r} \neq 0$ for $r > n_1$ or $\eta_{i,j,r} \neq 0$ for $r > n_2$ we can slightly modify $\mathcal{P}_1$, $\mathcal{P}_2$ by setting $\xi_{h,j,r} = 0$ for $r > n_1$ and $\eta_{i,j,r} = 0$ for $r > n_2$, without changing the $(t_w, \alpha, \beta_w, n_w, m_w, s_w)$-net property of $\mathcal{P}_w$, $w = 1, 2$). Set further $\ell := \min(2\beta_1 n_1 - 2t_1 + 1, \beta_2 n_2 - t_2)$.

We define a new point set $\mathcal{P} = \{\boldsymbol{z}_h\}_{h=0}^{b^{m_1+m_2}-1}$, $\boldsymbol{z}_h = (z_{h,1}, \ldots, z_{h,s_1+s_2})$, consisting of $b^{m_1+m_2}$ points in $[0,1)^{s_1+s_2}$ as follows:

- For $j = 1, \ldots, s_1$, $i = 0, \ldots, b^{m_2} - 1$ and $h = 0, \ldots, b^{m_1} - 1$ we set

$$\begin{aligned} z_{ib^{m_1}+h,j} &= \frac{\xi_{h,j,1} \ominus \eta_{i,j,1}}{b} + \cdots + \frac{\xi_{h,j,\min(\ell,n_1)} \ominus \eta_{i,j,\min(\ell,n_1)}}{b^{\min(\ell,n_1)}} \\ &\quad + \left(\frac{\xi_{h,j,\ell+1}}{b^{\ell+1}} + \cdots + \frac{\xi_{h,j,n_1}}{b^{n_1}}\right) \mathbf{1}_{n_1 \ge \ell} \\ &\quad + \left(\frac{\ominus \eta_{i,j,n_1+1}}{b^{n_1+1}} + \cdots + \frac{\ominus \eta_{i,j,\ell}}{b^{\ell}}\right) \mathbf{1}_{n_1 < \ell}. \end{aligned}$$

- For $j = s_1 + 1, \ldots, s_1 + s_2$, $i = 0, \ldots, b^{m_2} - 1$ and $h = 0, \ldots, b^{m_1} - 1$ we set

$$z_{ib^{m_1}+h,j} = y_{i,j-s_1}.$$

Then we have the following result, which was first shown in [1, Theorem 3]:

Theorem 4.2 *Let $b \ge 2$ be an integer, let $\mathcal{P}_1$ be a $(t_1, \alpha, \beta_1, n_1, m_1, s_1)$-net in base b, and $\mathcal{P}_2$ be a $(t_2, \alpha, \beta_2, n_2, m_2, s_2)$-net in base b. Then $\mathcal{P}$ defined as above is a $(t, \alpha, \beta, n, m, s)$-net in base b, where $n = n_1 + n_2$, $m = m_1 + m_2$, $s = s_1 + s_2$,*

$$\beta = \min(\beta_1, \beta_2), \text{ and } t = \beta n - \ell.$$

Remark 4.1 *Note that we defined the $(u, u+v)$-construction in such a way that it yields the same point set as the $(u, u+v)$-construction for digital nets as considered in [9].*

4.3 The matrix-product construction

In this subsection, we will assume that b is prime. We firstly introduce matrices which are non-singular by column (NSC), see [3]. Let A be an $M \times M$ matrix over $\mathbb{Z}_b$. For $1 \le l \le M$, let A_l denote the $l \times M$ matrix consisting of the first l rows of A. For $1 \le k_1 < \cdots < k_l \le M$, let $A(k_1, \ldots, k_l)$ denote the $l \times l$ matrix consisting of the columns $k_1, \ldots, k_l$ of A_l.

Definition 4.1 An $M \times M$ matrix A defined over $\mathbb{Z}_b$ is called *non-singular by column (NSC)* if $A(k_1, \ldots, k_l)$ is non-singular for each $1 \le l \le M$ and $1 \le k_1 < \cdots < k_l \le M$.

It is known that an $M \times M$ NSC matrix over $\mathbb{Z}_b$ exists if and only if $1 \le M \le b$, see [3, Section 3]. For any integer $1 \le M \le b$, an explicit $M \times M$ upper triangular NSC matrix over $\mathbb{Z}_b$ is given in [3, Section 5.2].

For the remainder of this section, we will assume that $A = (A_{k,l})$ is an $M \times M$ upper triangular NSC matrix over $\mathbb{Z}_b$ (upper triangular means that $A_{k,l} = 0$ for all $1 \le l < k \le M$).

We now describe how to construct the point set, based on the so-called matrix-product construction:

Let $1 \le s_1 \le \cdots \le s_M$ be integers and define $\sigma_0 := 0$ and $\sigma_k := s_1 + \cdots + s_k$ for $1 \le k \le M$. Let $s := \sigma_M$. For $1 \le k \le M$ let $\mathcal{P}_k = \{\boldsymbol{x}_h^{(k)}\}_{h=0}^{b^{m_k}-1}$, where $\boldsymbol{x}_h^{(k)} = (x_{h,\sigma_{k-1}+1}^{(k)}, \ldots, x_{h,\sigma_k}^{(k)})$ for $0 \le h < b^{m_k}$, be $(t_k, \alpha, \beta_k, n_k, m_k, s_k)$-nets in base b. (As with the $(u, u+v)$-construction, one can without loss of generality assume that $x_{h,j}^{(k)} = \xi_{h,j,1}^{(k)}/b + \xi_{h,j,2}^{(k)}/b^2 + \cdots$ with $\xi_{h,j,c}^{(k)} = 0$ for $c > n_k$, as setting the remaining digits to zero does not affect the quality of the net $\mathcal{P}_k$. However, this is not necessary as the results in this subsection also hold otherwise.)

We now define $V = (V_{k,l})_{k,l=1}^M := A^{-1} \in \mathbb{Z}_b^{M \times M}$ and note that V is upper triangular. For

$$h = h_1 + h_2 b^{m_1} + \cdots + h_M b^{m_1+m_2+\cdots+m_{M-1}},$$

with integers $0 \le h_k < b^{m_k}$ (hence $0 \le h < b^m$ where $m = m_1 + \cdots + m_M$) and for $\sigma_{k-1} < j \le \sigma_k$, $k = 1, \ldots, M$, define

$$z_{h,j} := V_{k,k} x_{h_k,j}^{(k)} \oplus \cdots \oplus V_{k,M} x_{h_M,j}^{(M)}, \tag{5}$$

where $\oplus$ and also the multiplication are carried out digit-wise modulo b, i.e., $z_{h,j} = \zeta_{h,j,1}/b + \zeta_{h,j,2}/b^2 + \cdots$ where

$$\zeta_{h,j,c} = V_{k,k}\xi_{h_k,j,c}^{(k)} + \cdots + V_{k,M}\xi_{h_M,j,c}^{(M)} \in \mathbb{Z}_b \quad \text{for all } c \ge 1,$$

with $x_{h_l,j}^{(l)} = \xi_{h_l,j,1}^{(l)}/b + \xi_{h_l,j,2}^{(l)}/b^2 + \cdots$ for $k \le l \le M$, where addition and multiplication are carried out in $\mathbb{Z}_b$, and where we assume that for each h and j infinitely many of the digits $\zeta_{h,j,c}$, $c = 1, 2, \ldots$ are different from $b-1$ (if this is not the case, then, for example by modifying any of the digits $\zeta_{h,j,c}$, $c = \max_{1 \le k \le M} n_k + 1, \max_{1 \le k \le M} n_k + 2, \ldots,$ will solve this problem without affecting the quality of the point set; indeed, the forthcoming Theorem 4.3 will establish that the digits $\zeta_{h,j,c}$ with $c > \min_{1 \le k \le M}(M-k+1)(\beta_k n_k - t_k)$ can be modified arbitrarily since they do not influence the quality of the net; this way, for $M = 2$, the $(u, u+v)$-construction can be viewed as a special case of the matrix product construction).

Analogously to the notation used above, we write $\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)} = A_{1,k}\boldsymbol{u}_1^{(k)} \oplus \cdots \oplus A_{k,k}\boldsymbol{u}_k^{(k)}$, where the addition and multiplication are carried out digit-wise modulo b.

Now we define $\mathcal{P} = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^m-1}\}$ with $m = m_1 + \cdots + m_M$ through $\boldsymbol{z}_h := (z_{h,1}, \ldots, z_{h,s})$ for $0 \le h < b^m$.

Lemma 4.1 *Let $\boldsymbol{d} = (\boldsymbol{d}_1, \ldots, \boldsymbol{d}_M) \in \mathcal{K}_{n,b}^{s}$ with $\boldsymbol{d}_k \in \mathcal{K}_{n,b}^{s_k}$ and assume that $\boldsymbol{d}_k = \bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}$ where for $l \le k$, $\boldsymbol{u}_l^{(k)} = (\boldsymbol{u}_l, \boldsymbol{0}) \in \mathcal{K}_{n,b}^{s_k}$ for some $\boldsymbol{u}_l \in \mathcal{K}_{n,b}^{s_l}$. Then we have*

$$\frac{1}{b^{m_1+m_2+\cdots+m_M}} \sum_{h=0}^{b^{m_1+m_2+\cdots+m_M}-1} \mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) = \prod_{r=1}^{M} \left(\frac{1}{b^{m_r}} \sum_{h_r=0}^{b^{m_r}-1} \mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}) \right).$$

Proof. Let $\boldsymbol{z}_h = (\boldsymbol{z}_h^{(s_1)}, \ldots, \boldsymbol{z}_h^{(s_M)}) \in [0,1)^{s_1+\cdots+s_M}$ where $\boldsymbol{z}_h^{(s_k)} = (z_{h,\sigma_{k-1}+1}, \ldots, z_{h,\sigma_k}) \in [0,1)^{s_k}$ for $1 \le k \le M$. For $\boldsymbol{d} = (\boldsymbol{d}_1, \ldots, \boldsymbol{d}_M) \in \mathcal{K}_{n,b}^{s}$ with $\boldsymbol{d}_k \in \mathcal{K}_{n,b}^{s_k}$ we have

$$\sum_{h=0}^{b^{m_1+\cdots+m_M}-1} \mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) = \sum_{h=0}^{b^{m_1+\cdots+m_M}-1} \prod_{k=1}^{M} \mathrm{wal}_{\boldsymbol{d}_k}(\boldsymbol{z}_h^{(s_k)}).$$

By assumption we have $\boldsymbol{d}_k = \bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}$, where for $l \le k$, $\boldsymbol{u}_l^{(k)} = (\boldsymbol{u}_l, \boldsymbol{0}) \in \mathcal{K}_{n,b}^{s_k}$ for some $\boldsymbol{u}_l \in \mathcal{K}_{n,b}^{s_l}$. Let furthermore $\overline{\boldsymbol{u}}_l = (\boldsymbol{u}_l, \boldsymbol{0}) \in \mathcal{K}_{n,b}^{s}$. Then for each of the above summands we have

$$\begin{aligned}
\prod_{k=1}^{M} \mathrm{wal}_{\boldsymbol{d}_k}(\boldsymbol{z}_h^{(s_k)}) &= \prod_{k=1}^{M} \mathrm{wal}_{\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}}(\boldsymbol{z}_h^{(s_k)}) \\
&= \prod_{k=1}^{M} \mathrm{wal}_{\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}}(z_{h,\sigma_{k-1}+1}, \ldots, z_{h,\sigma_k}) \\
&= \prod_{k=1}^{M} \prod_{r=k}^{M} \mathrm{wal}_{\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}}(V_{k,r}x_{h_r,\sigma_{k-1}+1}^{(r)}, \ldots, V_{k,r}x_{h_r,\sigma_k}^{(r)}) \\
&= \prod_{k=1}^{M} \prod_{r=k}^{M} \mathrm{wal}_{V_{k,r}\left(\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}\right)}(x_{h_r,\sigma_{k-1}+1}^{(r)}, \ldots, x_{h_r,\sigma_k}^{(r)}) \\
&= \prod_{k=1}^{M} \prod_{r=k}^{M} \mathrm{wal}_{V_{k,r}\left(\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}\right)}(\boldsymbol{x}_{h_r}^{(r)}) \\
&= \prod_{r=1}^{M} \prod_{k=1}^{r} \mathrm{wal}_{V_{k,r}\left(\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}\right)}(\boldsymbol{x}_{h_r}^{(r)}) \\
&= \prod_{r=1}^{M} \mathrm{wal}_{\bigoplus_{k=1}^{r} V_{k,r}\left(\bigoplus_{l=1}^{k} A_{l,k}\boldsymbol{u}_l^{(k)}\right)}(\boldsymbol{x}_{h_r}^{(r)}) \\
&= \prod_{r=1}^{M} \mathrm{wal}_{\bigoplus_{k=1}^{r} V_{k,r}\left(\bigoplus_{l=1}^{k} A_{l,k}\overline{\boldsymbol{u}}_l\right)}((\boldsymbol{x}_{h_r}^{(r)}, \boldsymbol{0})),
\end{aligned}$$

where $(\boldsymbol{x}_{h_r}^{(r)}, \boldsymbol{0}) \in [0,1)^s$ is just the concatenation of $\boldsymbol{x}_{h_r}^{(r)} \in [0,1)^{s_r}$ and the $s - s_r$ dimensional zero vector $\boldsymbol{0}$. Since $V = A^{-1}$ we now have

$$\bigoplus_{k=l}^{r} V_{k,r}A_{l,k} = \begin{cases} 1 & \text{if } r = l \\ 0 & \text{if } r \neq l. \end{cases}$$

Hence we obtain $\bigoplus_{k=1}^r V_{k,r} \bigoplus_{l=1}^k A_{l,k}\overline{\boldsymbol{u}}_l = \bigoplus_{l=1}^r \overline{\boldsymbol{u}}_l \bigoplus_{k=l}^r V_{k,r}A_{l,k} = \overline{\boldsymbol{u}}_r$ and hence

$$\prod_{r=1}^M \mathrm{wal}_{\bigoplus_{k=1}^r V_{k,r}\left(\bigoplus_{l=1}^k A_{l,k}\overline{\boldsymbol{u}}_l\right)}((\boldsymbol{x}_{h_r}^{(r)}, \boldsymbol{0})) = \prod_{r=1}^M \mathrm{wal}_{\overline{\boldsymbol{u}}_r}((\boldsymbol{x}_{h_r}^{(r)}, \boldsymbol{0})) = \prod_{r=1}^M \mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}).$$

Hence

$$\begin{aligned}\frac{1}{b^{m_1+\cdots+m_M}} \sum_{h=0}^{b^{m_1+\cdots+m_M}-1} \mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) &= \frac{1}{b^{m_1+\cdots+m_M}} \sum_{h=0}^{b^{m_1+\cdots+m_M}-1} \prod_{r=1}^M \mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}) \\ &= \prod_{r=1}^M \left(\frac{1}{b^{m_r}} \sum_{h_r=0}^{b^{m_r}-1} \mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}) \right).\end{aligned}$$

□

For the rest of the subsection, we make the convention that

$$\mu_\alpha(\boldsymbol{d}) = \sum_{k=1}^M \mu_\alpha(\boldsymbol{d}_k).$$

If $\mu_\alpha(\boldsymbol{d}) > 0$, then there exists at least one integer l, so that $\boldsymbol{u}_l \neq \boldsymbol{0}$; the largest integer l so that $\boldsymbol{u}_l \neq \boldsymbol{0}$ is denoted by l^*. We need the following lemma.

Lemma 4.2 *Let $\boldsymbol{d}$ be as in Lemma 4.1 with $\mu_\alpha(\boldsymbol{d}) > 0$ and let l^* denote the largest integer l so that $\boldsymbol{u}_l \neq \boldsymbol{0}$. Then we have $\mu_\alpha(\boldsymbol{d}) \geq (M - l^* + 1)\mu_\alpha(\boldsymbol{u}_{l^*})$.*

Proof. The proof follows along the same lines as the proofs of [9, Lemmas 2 and 3]. □

We can now show the main result of this subsection.

Theorem 4.3 *The multiset $\mathcal{P} = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^m-1}\}$ where $\boldsymbol{z}_h := (z_{h,1}, \ldots, z_{h,s})$ and where the $z_{h,j}$ are given by (5) forms a $(t, \alpha, \beta, n, m, s)$-net, where $s = s_1 + \cdots + s_M$, $n = \max_{1\leq k\leq M} n_k$, $m = m_1 + \cdots + m_M$, $\beta = \min(1, \frac{\alpha m}{n})$ and*

$$t \leq \beta n - \min_{1\leq l\leq M} (M - l + 1)(\beta_l n_l - t_l).$$

Proof. According to [1, Theorem 1] it is enough to show that

$$\frac{1}{b^{m_1+m_2+\cdots+m_M}} \sum_{h=0}^{b^{m_1+m_2+\cdots+m_M}-1} \mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) = 0$$

for all $\boldsymbol{d} \in \mathbb{N}_0^s$ satisfying $0 < \mu_\alpha(\boldsymbol{d}) \leq \beta n - t$. As $\boldsymbol{d}$ must satisfy $\mu_\alpha(\boldsymbol{d}) \leq \beta n - t$ we may restrict ourselves to $\boldsymbol{d} \in \mathcal{K}_{n,b}^s$ satisfying $0 < \mu_\alpha(\boldsymbol{d}) \leq \beta n - t$. From Lemma 4.1, we know that

$$\frac{1}{b^{m_1+m_2+\cdots+m_M}} \sum_{h=0}^{b^{m_1+m_2+\cdots+m_M}-1} \mathrm{wal}_{\boldsymbol{d}}(\boldsymbol{z}_h) = \prod_{r=1}^M \left(\frac{1}{b^{m_r}} \sum_{h_r=0}^{b^{m_r}-1} \mathrm{wal}_{\boldsymbol{u}_r}(\boldsymbol{x}_{h_r}^{(r)}) \right). \tag{6}$$

Assume now that $\boldsymbol{d} \in \mathcal{K}_{n,b}^s$ is such that $0 < \mu_\alpha(\boldsymbol{d}) \le \beta n - t$, then there exists an integer l so that $\mu_\alpha(\boldsymbol{u}_l) > 0$ and as before, we denote the largest integer l so that $\mu_\alpha(\boldsymbol{u}_l) > 0$ by l^*. We now use Lemma 4.2 to conclude that

$$\begin{aligned}(M - l^* + 1)(\beta_{l^*} n_{l^*} - t_{l^*}) \ \ &\ge \ \ \min_{1 \le l \le M} (M - l + 1)(\beta_l n_l - t_l) \\ &= \beta n - t \ge \mu_\alpha(\boldsymbol{d}) \ge (M - l^* + 1)\mu_\alpha(\boldsymbol{u}_{l^*}).\end{aligned}$$

Hence we have shown that $0 < \mu_\alpha(\boldsymbol{u}_{l^*}) \le \beta_{l^*} n_{l^*} - t_{l^*}$ and therefore

$$\frac{1}{b^{m_{l^*}}} \sum_{h_{l^*}=0}^{b^{m_{l^*}}-1} \mathrm{wal}_{\boldsymbol{u}_{l^*}}(\boldsymbol{x}_{h_{l^*}}^{(l^*)}) = 0,$$

i.e., the l^*th factor in equation (6) is zero. □

4.4 A double m-construction

In this section, we aim to generalize a propagation rule referred to as "double m-construction" in [9, Section 3.4], which again generalizes a propagation rule from [15] for digital (t, m, s)-nets. We remark that this is the first time that this propagation rule appears in the context of not necessarily digital nets.

Assume we are given a $(t_1, \alpha_1, \beta_1, n, m, s)$-net in base b, denoted by $\mathcal{P}_1 = \{\boldsymbol{x}_h\}_{h=0}^{b^m-1}$, and a $(t_2, \alpha_2, \beta_2, n, m, s)$-net in base b, denoted by $\mathcal{P}_2 = \{\boldsymbol{y}_i\}_{i=0}^{b^m-1}$. For $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$, we write

$$x_{h,j} = \frac{\xi_{h,j,1}}{b} + \cdots + \frac{\xi_{h,j,n}}{b^n}$$

and for $\boldsymbol{y}_i = (y_{i,1}, \ldots, y_{i,s})$, we set

$$y_{i,j} = \frac{\eta_{i,j,1}}{b} + \cdots + \frac{\eta_{i,j,n}}{b^n}.$$

Furthermore, the dual set associated with $\mathcal{P}_1$ is denoted by $\mathcal{D}_n^{(1)}$, the dual set associated with $\mathcal{P}_2$ by $\mathcal{D}_n^{(2)}$. We are now in a position to define a multiset $\mathcal{P} := \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^{2m}-1}\}$ as follows: For $h' = hb^m + i$, $0 \le h \le b^m - 1$, $0 \le i \le b^m - 1$, we set

$$z_{h',j} = \frac{\xi_{h,j,1} \oplus \eta_{i,j,1}}{b} + \cdots + \frac{\xi_{h,j,n} \oplus \eta_{i,j,n}}{b^n} + \frac{0 \ominus \eta_{i,j,1}}{b^{n+1}} + \cdots + \frac{0 \ominus \eta_{i,j,n}}{b^{2n}}, \tag{7}$$

$h' = 0, \ldots, b^{2m} - 1$, $j = 1, \ldots, s$. We now define a set $\mathcal{N}$, which in the forthcoming Lemma 4.3 will be shown to be the dual set of $\mathcal{P}$. Let $\boldsymbol{a}_r = (a_{r,1}, \ldots, a_{r,s}) \in \mathcal{D}_n^{(r)}$, $r = 1, 2$ and define $\boldsymbol{k} = \boldsymbol{k}(\boldsymbol{a}_1, \boldsymbol{a}_2) := (k_1, \ldots, k_s)$, where

$$k_j = a_{1,j} + b^n(a_{1,j} \oplus a_{2,j}), \quad j = 1, \ldots, s,$$

then we set $\mathcal{N} = \left\{\boldsymbol{k}(\boldsymbol{a}_1, \boldsymbol{a}_2) \in \mathcal{K}_{2n,b}^s : \boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}, \boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}\right\}$.

Lemma 4.3 *The set* $\mathcal{N} = \left\{\boldsymbol{k} \in \mathcal{K}_{2n,b}^s : \boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}, \boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}\right\}$ *is the dual set of* $\mathcal{P} = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^{2m}-1}\}$ *where* $\boldsymbol{z}_h := (z_{h,1}, \ldots, z_{h,s})$ *and where the* $z_{h,j}$ *are given by Equation (7).*

Proof. Let $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathcal{K}_{2n,b}^s$, where $k_j = a_{1,j} + b^n(a_{1,j} \oplus a_{2,j})$, $j = 1, \ldots, s$, and where $\boldsymbol{a}_r = (a_{r,1}, \ldots, a_{r,s}) \in \mathcal{K}_{n,b}^s$, $r = 1, 2$. Clearly,

$$c_{\boldsymbol{k}} = \sum_{h'=0}^{b^{2m}-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_{h'}) = \sum_{h=0}^{b^m-1} \sum_{i=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{z}_{hb^m+i}) = \sum_{h=0}^{b^m-1} \sum_{i=0}^{b^m-1} \prod_{j=1}^{s} \mathrm{wal}_{k_j}(z_{hb^m+i,j}).$$

For brevity, we set $k_j = k_j^{(1)} + b^n k_j^{(2)}$, where $k_j^{(1)}$ and $k_j^{(2)}$ have the b-adic expansions $k_j^{(1)} = \sum_{l=1}^n k_{j,l}^{(1)} b^{l-1}$ and $k_j^{(2)} = \sum_{l=1}^n k_{j,l}^{(2)} b^{l-1}$. Hence

$$\begin{aligned}
\mathrm{wal}_{k_j}(z_{hb^m+i,j}) &= \exp\left[\frac{2\pi\mathtt{i}}{b}\left(\sum_{l=1}^{n} k_{j,l}^{(1)}(\xi_{h,j,l} \oplus \eta_{i,j,l}) + \sum_{l=n+1}^{2n} k_{j,l-n}^{(2)}(0 \ominus \eta_{i,j,l-n})\right)\right] \\
&= \exp\left[\frac{2\pi\mathtt{i}}{b}\sum_{l=1}^{n} k_{j,l}^{(1)}(\xi_{h,j,l} \oplus \eta_{i,j,l})\right] \exp\left[\frac{2\pi\mathtt{i}}{b}\sum_{l=1}^{n} k_{j,l}^{(2)}(0 \ominus \eta_{i,j,l})\right] \\
&= \mathrm{wal}_{k_j^{(1)}}(x_{h,j} \oplus y_{i,j})\mathrm{wal}_{k_j^{(2)}}(0 \ominus y_{i,j}) \\
&= \mathrm{wal}_{a_{1,j}}(x_{h,j})\mathrm{wal}_{a_{1,j}}(y_{i,j})\mathrm{wal}_{a_{1,j}}(0 \ominus y_{i,j})\mathrm{wal}_{a_{2,j}}(0 \ominus y_{i,j}) \\
&= \mathrm{wal}_{a_{1,j}}(x_{h,j})\overline{\mathrm{wal}_{a_{2,j}}(y_{i,j})},
\end{aligned}$$

and further

$$\begin{aligned}
c_{\boldsymbol{k}} &= \sum_{h=0}^{b^m-1} \sum_{i=0}^{b^m-1} \prod_{j=1}^{s} \mathrm{wal}_{a_{1,j}}(x_{h,j})\overline{\mathrm{wal}_{a_{2,j}}(y_{i,j})} \\
&= \sum_{h=0}^{b^m-1} \sum_{i=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{a}_1}(\boldsymbol{x}_h)\overline{\mathrm{wal}_{\boldsymbol{a}_2}(\boldsymbol{y}_i)} \\
&= \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{a}_1}(\boldsymbol{x}_h) \sum_{i=0}^{b^m-1} \overline{\mathrm{wal}_{\boldsymbol{a}_2}(\boldsymbol{y}_i)} \\
&= \sum_{h=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{a}_1}(\boldsymbol{x}_h) \overline{\sum_{i=0}^{b^m-1} \mathrm{wal}_{\boldsymbol{a}_2}(\boldsymbol{y}_i)}.
\end{aligned}$$

If $\boldsymbol{k} \in \mathcal{N}$, then $\boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}$ and $\boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}$, so we have $c_{\boldsymbol{k}} \neq 0$ and hence $\boldsymbol{k}$ is in the dual set of $\mathcal{P}$. If on the other hand $\boldsymbol{k}$ is in the dual set of $\mathcal{P}$, then $c_{\boldsymbol{k}} \neq 0$ and hence $\boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}$ and $\boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}$, so $\boldsymbol{k} \in \mathcal{N}$. □

In order to bound the quality parameter of $\mathcal{P} = \{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_{b^{2m}-1}\}$, we define

$$d = d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}) := \max_{1 \le j \le s} \max_{R_j} \max(0, \mu_\alpha(a_{1,j}) - \mu_\alpha(a_{1,j} \oplus a_{2,j})),$$

where R_j is the set of all ordered pairs $(\boldsymbol{a}_1, \boldsymbol{a}_2)$, with $\boldsymbol{a}_r = (a_{r,1}, \ldots, a_{r,s}) \in \mathcal{D}_n^{(r)} \setminus \{\boldsymbol{0}\}$, $a_{1,i} \oplus a_{2,i} = 0$ for $i \neq j$ and $a_{1,j} \oplus a_{2,j} \neq 0$. We define the max over R_j to be zero if R_j is empty. We can now prove the main result of this subsection.

Theorem 4.4 *Let $\mathcal{P}_1$ be a $(t_1, \alpha_1, \beta_1, n, m, s)$-net in base b with dual set $\mathcal{D}_n^{(1)}$ and $\mathcal{P}_2$ be a $(t_2, \alpha_2, \beta_2, n, m, s)$-net in base b with dual set $\mathcal{D}_n^{(2)}$. Let $d = d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)})$. Then the* *point set given by Equation* (7) *is a* $(t, \alpha, \beta, 2n, 2m, s)$*-net in base* b *with* $\alpha = \max(\alpha_1, \alpha_2)$, $\beta = \min(\beta_1, \beta_2)$ *and*

$$t \le \max(\lfloor 2\beta n \rfloor - n - \lfloor \beta_1 n \rfloor + t_1 + d, \lfloor 2\beta n \rfloor - n - \lfloor \beta_2 n \rfloor + t_2, 0),$$

if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} = \{\boldsymbol{0}\}$, *and*

$$t \le \max(\lfloor 2\beta n \rfloor - n - \lfloor \beta_1 n \rfloor + t_1 + d, \lfloor 2\beta n \rfloor - n - \lfloor \beta_2 n \rfloor + t_2, \lfloor 2\beta n \rfloor + 1 - \rho_\alpha(\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)}), 0),$$

if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} \neq \{\boldsymbol{0}\}$.

Proof. Clearly, $0 < \beta \le 1$, $\alpha \ge 1$. We show a lower bound for $\mu_\alpha(\boldsymbol{k})$ for all non-zero vectors $\boldsymbol{k} \in \mathcal{N}$, which by Lemma 4.3 is the dual set of the point set given by Equation (7). To this end we use the property that $\rho_\alpha(\mathcal{D}_n^{(r)}) \ge \rho_{\alpha_r}(\mathcal{D}_n^{(r)}) \ge \lfloor \beta_r n \rfloor - t_r + 1$, as $\alpha \ge \alpha_r$, $r = 1, 2$. For $\boldsymbol{k} \in \mathcal{N}$, $\boldsymbol{k} \neq \boldsymbol{0}$, we have $\boldsymbol{k} = \boldsymbol{a}_1 + b^n(\boldsymbol{a}_1 \oplus \boldsymbol{a}_2)$ with $\boldsymbol{a}_1 \in \mathcal{D}_n^{(1)}$ and $\boldsymbol{a}_2 \in \mathcal{D}_n^{(2)}$ (not both of them are zero) and hence

$$\mu_\alpha(\boldsymbol{k}) = \mu_\alpha(\boldsymbol{a}_1 + b^n(\boldsymbol{a}_1 \oplus \boldsymbol{a}_2)).$$

We consider four different cases:

1. If $\boldsymbol{a}_1 = \boldsymbol{0}$, then $\boldsymbol{a}_2 \neq \boldsymbol{0}$, and hence

$$\mu_\alpha(\boldsymbol{k}) = \mu_\alpha(b^n \boldsymbol{a}_2) \ge n + \mu_\alpha(\boldsymbol{a}_2) \ge n + \rho_\alpha(\mathcal{D}_n^{(2)}) \ge n + \lfloor \beta_2 n \rfloor - t_2 + 1.$$

2. If $\boldsymbol{a}_2 = \boldsymbol{0}$, then $\boldsymbol{a}_1 \neq \boldsymbol{0}$, and we obtain in a similar manner that

$$\mu_\alpha(\boldsymbol{k}) \ge \mu_\alpha(b^n \boldsymbol{a}_1) \ge n + \rho_\alpha(\mathcal{D}_n^{(1)}) \ge n + \lfloor \beta_1 n \rfloor - t_1 + 1.$$

3. If $\boldsymbol{a}_1, \boldsymbol{a}_2 \neq \boldsymbol{0}$, but $\boldsymbol{a}_1 \oplus \boldsymbol{a}_2 = \boldsymbol{0}$, then $\boldsymbol{a}_1 \in \mathcal{D}_n^{(2)}$, so $\boldsymbol{a}_1 \in \mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)}$. Consequently, if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} = \{\boldsymbol{0}\}$, this is not possible. If $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} \neq \{\boldsymbol{0}\}$, then

$$\mu_\alpha(\boldsymbol{k}) = \mu_\alpha(\boldsymbol{a}_1) \ge \rho_\alpha(\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)}).$$

4. If $\boldsymbol{a}_1, \boldsymbol{a}_2 \neq \boldsymbol{0}$ and $\boldsymbol{a}_1 \oplus \boldsymbol{a}_2 \neq \boldsymbol{0}$, then we have

$$\begin{aligned} \mu_\alpha(\boldsymbol{k}) &= \sum_{j=1}^{s} \mu_\alpha(a_{1,j} + b^n(a_{1,j} \oplus a_{2,j})) \\ &= \sum_{\substack{j=1 \\ a_{j,1} \oplus a_{j,2} \neq 0}}^{s} \mu_\alpha(a_{1,j} + b^n(a_{1,j} \oplus a_{2,j})) + \sum_{\substack{j=1 \\ a_{j,1} \oplus a_{j,2} = 0}}^{s} \mu_\alpha(a_{1,j}) \\ &\ge \sum_{\substack{j=1 \\ a_{j,1} \oplus a_{j,2} \neq 0}}^{s} \mu_\alpha(b^n(a_{1,j} \oplus a_{2,j})) + \sum_{\substack{j=1 \\ a_{j,1} \oplus a_{j,2} = 0}}^{s} \mu_\alpha(a_{1,j}). \end{aligned} \tag{8}$$

We now distinguish between two sub-cases: firstly, assume that the first sum in equation (8) has at least two terms, then $\mu_\alpha(\boldsymbol{k}) \ge 2n + 2$. Otherwise, it has exactly one term, say for $j = j_0$, which gives a smaller value than $2n + 2$. In this sub-case we have

$$\begin{aligned}\mu_\alpha(\boldsymbol{k}) &= \mu_\alpha(b^n(a_{1,j_0} \oplus a_{2,j_0})) + \mu_\alpha(\boldsymbol{a}_1) - \mu_\alpha(a_{1,j_0}) \\ &\geq n + \mu_\alpha(\boldsymbol{a}_1) - (\mu_\alpha(a_{1,j_0}) - \mu_\alpha(a_{1,j_0} \oplus a_{2,j_0})) \\ &\geq n + \rho_\alpha(\mathcal{D}_n^{(1)}) - d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}) \\ &\geq n + \lfloor \beta_1 n \rfloor - t_1 + 1 - d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}).\end{aligned}$$

Hence combining the four cases we have

$$\rho_\alpha(\mathcal{N}) \geq \min(n + \lfloor \beta_1 n \rfloor - t_1 + 1 - d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}), n + \lfloor \beta_2 n \rfloor - t_2 + 1, \rho_\alpha(\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)})),$$

if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} \neq \{\mathbf{0}\}$, and

$$\rho_\alpha(\mathcal{N}) \geq \min(n + \lfloor \beta_1 n \rfloor - t_1 + 1 - d(\mathcal{D}_n^{(1)}, \mathcal{D}_n^{(2)}), n + \lfloor \beta_2 n \rfloor - t_2 + 1),$$

if $\mathcal{D}_n^{(1)} \cap \mathcal{D}_n^{(2)} = \{\mathbf{0}\}$. Now the result follows from Theorem 3.2. □

4.5 A base change propagation rule

In this subsection we show how one can obtain a net in base b from a net in base b^L. Thereby we generalize [13, Propagation Rule 7] (see also [9, Propagation Rule XI]) to $(t, \alpha, \beta, n, m, s)$-nets. The proof technique and the construction follows [13, Proposition 7] very closely.

Theorem 4.5 *If there exists a $(t, \alpha, \beta, n, m, s)$-net in base b^L with an integer $L \geq 1$, then there exists a $(t, \alpha, \beta, n, mL, sL)$-net in base b.*

Proof. Let $\mathcal{P} = \{\boldsymbol{x}_h\}_{h=0}^{(b^L)^m-1}$ be a $(t, \alpha, \beta, n, m, s)$-net in base b^L. Without loss of generality we may assume that $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,s})$ with

$$x_{h,j} = \sum_{l=1}^{n} \xi_{h,j,l} (b^L)^{-l} \quad \text{for } 0 \leq h \leq (b^L)^m - 1,$$

where all $\xi_{h,j,l} \in \mathbb{Z}_{b^L}$. Let the expansion of $\xi_{h,j,l}$ in base b be

$$\xi_{h,j,l} = \sum_{k=1}^{L} z_{h,l,k}^{(j)} b^{k-1} \quad \text{for } 0 \leq h \leq (b^L)^m - 1, 1 \leq j \leq s, 1 \leq l \leq n,$$

where all $z_{h,l,k}^{(j)} \in \mathbb{Z}_b$. Now we define a multiset $\mathcal{Q} = \{\boldsymbol{w}_0, \ldots, \boldsymbol{w}_{b^{mL}-1}\}$ whose elements are in $[0, 1)^{sL}$. The coordinate indices range from 1 to sL, and so we can denote them by $(j-1)L + k$ with $1 \leq j \leq s$ and $1 \leq k \leq L$. Let $w_{h,(j-1)L+k}$ denote the corresponding coordinates of the point $\boldsymbol{w}_h$. To complete the definition of $\mathcal{Q}$, we put

$$w_{h,(j-1)L+k} = \sum_{l=1}^{n} z_{h,l,k}^{(j)} b^{-l} \quad \text{for } 1 \leq j \leq s, 1 \leq k \leq L, 0 \leq h \leq b^{mL} - 1.$$

We will now show that $\mathcal{Q}$ is a (t,α,β,n,mL,sL)-net in base b. To this end we fix $\boldsymbol{\nu},\boldsymbol{a}_{\boldsymbol{\nu}},\boldsymbol{i}_{\boldsymbol{\nu}}$ so that $1 \le i_{(j-1)L+k,\nu_{(j-1)L+k}} < \cdots < i_{(j-1)L+k,1}$, for $1 \le k \le L$ and $1 \le j \le s$, so that $\sum_{j=1}^s \sum_{k=1}^L \sum_{l=1}^{\min(\nu_{(j-1)L+k},\alpha)} i_{(j-1)L+k,l} \le \beta n - t$.

For $\boldsymbol{w}_h$ to be in $J(\boldsymbol{a}_{\boldsymbol{\nu}},\boldsymbol{i}_{\boldsymbol{\nu}})$, we need

$$w_{h,(j-1)L+k,l} = a_{(j-1)L+k,l} \qquad \text{for all } l \in \left\{i_{(j-1)L+k,\nu_{(j-1)L+k}},\ldots,i_{(j-1)L+k,1}\right\},$$

which is satisfied if and only if

$$z_{h,l,k}^{(j)} = a_{(j-1)L+k,l} \qquad \text{for all } l \in \left\{i_{(j-1)L+k,\nu_{(j-1)L+k}},\ldots,i_{(j-1)L+k,1}\right\}.$$

For $1 \le j \le s$ we define $\bigcup_{k=1}^L \left\{i_{(j-1)L+k,\nu_{(j-1)L+k}},\ldots,i_{(j-1)L+k,1}\right\} = \left\{e_{j,\widetilde{\nu}_j},\ldots,e_{j,1}\right\}$. For $l \in \left\{e_{j,\widetilde{\nu}_j},\ldots,e_{j,1}\right\}$, we set $\widetilde{a}_{j,l} = \sum_{k=1}^L a_{(j-1)L+k,l} b^{k-1}$, where unspecified $a_{(j-1)L+k,l}$ are chosen arbitrarily. In fact, the number of $a_{(j-1)L+k,l}$ chosen arbitrarily is given by

$$\sum_{j=1}^s \sum_{k=1}^L (\widetilde{\nu}_j - \nu_{(j-1)L+k}) = L\sum_{j=1}^s \widetilde{\nu}_j - \sum_{j=1}^s \sum_{k=1}^L \nu_{(j-1)L+k}.$$

Hence there are $b^{L\sum_{j=1}^s \widetilde{\nu}_j - \sum_{j=1}^s \sum_{k=1}^L \nu_{(j-1)L+k}}$ generalized elementary intervals of format

$$J(\widetilde{\boldsymbol{a}},\boldsymbol{e}) = \prod_{j=1}^s \bigcup_{\substack{\widetilde{a}_{j,l}=0 \\ l\in\{1,\ldots,n\}\setminus\left\{e_{j,\widetilde{\nu}_j},\ldots,e_{j,1}\right\}}}^{b^L-1} \left[\frac{\widetilde{a}_{j,1}}{b^L} + \cdots + \frac{\widetilde{a}_{j,n}}{(b^L)^n}, \frac{\widetilde{a}_{j,1}}{b^L} + \cdots + \frac{\widetilde{a}_{j,n}}{(b^L)^n} + \frac{1}{(b^L)^n}\right)$$

of volume $(b^L)^{-\sum_{j=1}^s \widetilde{\nu}_j}$. However,

$$\sum_{j=1}^s \sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha)} e_{j,l} \le \sum_{j=1}^s \sum_{k=1}^L \sum_{l=1}^{\min(\nu_{(j-1)L+k},\alpha)} i_{(j-1)L+k,l} \le \beta n - t,$$

hence by the (t,α,β,n,m,s)-net property of $\mathcal{P}$, $J(\widetilde{\boldsymbol{a}},\boldsymbol{e})$ contains $(b^L)^{m-\sum_{j=1}^s \widetilde{\nu}_j}$ points and hence $J(\boldsymbol{i}_{\boldsymbol{\nu}},\boldsymbol{a}_{\boldsymbol{\nu}})$ contains

$$b^{L\sum_{j=1}^s \widetilde{\nu}_j - \sum_{j=1}^s \sum_{k=1}^L \nu_{(j-1)L+k}} (b^L)^{(m-\sum_{j=1}^s \widetilde{\nu}_j)} = b^{Lm - \sum_{j=1}^s \sum_{k=1}^L \nu_{(j-1)L+k}}$$

points of $\mathcal{Q}$ as required. $\square$

4.6 Pirsic's base change rule

In this subsection, we present a generalization of Pirsic's base change rule, see [18, Lemma 12], also [17]. This result shows how to interpret a (t,α,β,n,m,s)-net in base b^L as a $(t',\alpha',\beta',n',m',s)$-net in base $b^{L'}$. Furthermore, we state some special cases, in particular, we show how to interpret a (t,α,β,n,m,s)-net in base b as a $(t',\alpha',\beta',n',m',s)$-net in base $b^{L'}$ and how to interpret a (t,α,β,n,m,s)-net in base b^L as a $(t',\alpha',\beta',n',m',s)$-net in base b.

Theorem 4.6 *Let $n, n', m, m', s, \alpha, L$ and $L' \in \mathbb{N}$, where $\gcd(L, L') = 1$, $mL = m'L'$, $nL = n'L'$, let $0 < \beta \leq 1$ be a real number and let $0 \leq t \leq \beta n$ and βn be integers. Then a $(t, \alpha L', \beta, n, m, s)$-net in base b^L is a $(t', \alpha, \frac{\beta}{L'}, n', m', s)$-net in base $b^{L'}$, where*

$$t' = \min\left(\left\lceil \frac{tL + s\alpha(L-1)L' - \frac{(L'-1)L'}{2} + (-L' \pmod L)\beta n'}{L'(L' + (-L' \pmod L))} \right\rceil, \left\lceil \frac{tL + (s\alpha L' - 1)(L-1) - \frac{(L'-1)L'}{2}}{L'^2} \right\rceil \right).$$

Proof. The proof proceeds as follows: we start with a generalized elementary interval for the point set in base $b^{L'}$, then change this into a generalized elementary interval in base b and consequently rewrite the latter as a union of intervals in base b^L.

Assume we are given an arbitrary generalized elementary interval $J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ in base $b^{L'}$ for some given values of $\boldsymbol{\nu}$, $\boldsymbol{i_\nu}$, $\boldsymbol{a_\nu}$, such that $\nu_j \geq 0$, $1 \leq i_{j,\nu_j} < \cdots < i_{j,1}$, $j = 1, \ldots, s$, and such that for a non-negative integer t''

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} \leq \frac{\beta}{L'} n' - t''. \tag{9}$$

Without loss of generality, we assume that there exists at least one ν_j satisfying $\nu_j > 0$, then $J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ admits the following representation:

$$J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu}) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,n'\} \setminus \left\{i_{j,\nu_j},\ldots,i_{j,1}\right\}}}^{b^{L'}-1} \left[\frac{a_{j,1}}{b^{L'}} + \cdots + \frac{a_{j,n'}}{(b^{L'})^{n'}}, \frac{a_{j,1}}{b^{L'}} + \cdots + \frac{a_{j,n'}}{(b^{L'})^{n'}} + \frac{1}{(b^{L'})^{n'}}\right).$$

As $a_{j,l} \in \left\{0, \ldots, b^{L'} - 1\right\}$ it has a b-adic representation of the form $a_{j,l} = a_{j,l,1} + a_{j,l,2} b + \cdots + a_{j,l,L'} b^{L'-1}$, and hence

$$\frac{a_{j,l}}{(b^{L'})^l} = \frac{a_{j,l,L'}}{b^{(l-1)L'+1}} + \cdots + \frac{a_{j,l,2}}{b^{lL'-1}} + \frac{a_{j,l,1}}{b^{lL'}},$$

for $1 \leq l \leq n'$ where $a_{j,l,g} \in \{0, \ldots, b-1\}$. We now set

$$\frac{a_{j,l}}{(b^{L'})^l} = \sum_{k=(l-1)L'+1}^{lL'} \frac{\widetilde{a}_{j,k}}{b^k},$$

i.e. $\widetilde{a}_{j,lL'-g+1} = a_{j,l,g}$, $1 \leq l \leq n'$, $1 \leq g \leq L'$ and $1 \leq j \leq s$. We can now rewrite the above interval as a generalized elementary interval in base b,

$$J(\widetilde{\boldsymbol{i}}_{\boldsymbol{\nu}}, \widetilde{\boldsymbol{a}}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{\widetilde{a}_{j,l}=0 \\ l \in \{1,\ldots,n'L'\} \setminus \left\{\widetilde{i}_{j,\nu_j L'}, \widetilde{i}_{j,\nu_j L'-1},\ldots,\widetilde{i}_{j,1}\right\}}}^{b-1} \left[\frac{\widetilde{a}_{j,1}}{b} + \cdots + \frac{\widetilde{a}_{j,L'}}{b^{L'}} + \cdots + \frac{\widetilde{a}_{j,n'L'}}{b^{n'L'}}, \frac{\widetilde{a}_{j,1}}{b} + \cdots + \frac{\widetilde{a}_{j,L'}}{b^{L'}} + \cdots + \frac{\widetilde{a}_{j,n'L'}}{b^{n'L'}} + \frac{1}{b^{n'L'}}\right),$$

where

$$\widetilde{i}_{j,(k-1)L'+g} = i_{j,k}L' + 1 - g$$

for $1 \le g \le L'$ and $1 \le k \le \nu_j$. Clearly,

$$J(\widetilde{\boldsymbol{i}}_{\boldsymbol{\nu}}, \widetilde{\boldsymbol{a}}_{\boldsymbol{\nu}}) = \prod_{j=1}^{s} \bigcup_{\substack{\widetilde{a}_{j,l}=0 \\ l \in \{1,\dots,nL\} \setminus \left\{\widetilde{i}_{j,\nu_j L'},\dots,\widetilde{i}_{j,1}\right\}}}^{b-1} \left[\frac{\widetilde{a}_{j,1}}{b} + \dots + \frac{\widetilde{a}_{j,nL}}{b^{nL}}, \frac{\widetilde{a}_{j,1}}{b} + \dots + \frac{\widetilde{a}_{j,nL}}{b^{nL}} + \frac{1}{b^{nL}}\right).$$

Now for $1 \le j \le s$ and $1 \le k \le \nu_j L'$ we define integers $r_{j,k}$ and $e_{j,k}$ such that $0 \le r_{j,k} < L$ and

$$\widetilde{i}_{j,k} = e_{j,k}L - r_{j,k}.$$

Note that it is possible that $e_{j,k} = e_{j,k'}$ for $k \neq k'$. Let now $\left\{\widetilde{e}_{j,\widetilde{\nu}_j},\dots,\widetilde{e}_{j,1}\right\}$ be the set of distinct elements of $\left\{e_{j,\nu_j L'},\dots,e_{j,1}\right\}$. Then $\widetilde{\nu}_j \le \nu_j L'$ and $\left\{e_{j,\nu_j L'},\dots,e_{j,1}\right\} = \left\{\widetilde{e}_{j,\widetilde{\nu}_j},\dots,\widetilde{e}_{j,1}\right\}$.

Let $\widetilde{\boldsymbol{\nu}} = (\widetilde{\nu}_1,\dots,\widetilde{\nu}_s)$. For $1 \le j \le s$ for fixed $\widetilde{a}_{j,l}$ and $\widetilde{e}_{j,k}L - (L-1) \le l \le \widetilde{e}_{j,k}L$, where $1 \le k \le \widetilde{\nu}_j$, we set

$$\widetilde{\widetilde{a}}_{j,\widetilde{e}_{j,k}} = b^{L-1}\widetilde{a}_{j,\widetilde{e}_{j,k}L-(L-1)} + b^{L-2}\widetilde{a}_{j,\widetilde{e}_{j,k}L-(L-2)} + \dots + \widetilde{a}_{j,\widetilde{e}_{j,k}L}.$$

Furthermore, for fixed j, only $\nu_j L'$ of the $\widetilde{a}_{j,l}$, where $\widetilde{e}_{j,k}L - (L-1) \le l \le \widetilde{e}_{j,k}L$ and $1 \le k \le \widetilde{\nu}_j$, are specified in $\widetilde{\boldsymbol{a}}_{\boldsymbol{\nu}}$. Hence $J(\widetilde{\boldsymbol{i}}_{\boldsymbol{\nu}}, \widetilde{\boldsymbol{a}}_{\boldsymbol{\nu}})$, and therefore also $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$, is the union of $b^{L\sum_{j=1}^{s}\widetilde{\nu}_j - L'\sum_{j=1}^{s}\nu_j}$ disjoint intervals of the format

$$\prod_{j=1}^{s} \bigcup_{\substack{\widetilde{\widetilde{a}}_{j,l}=0 \\ l \in \{1,\dots,n\} \setminus \left\{\widetilde{e}_{j,\widetilde{\nu}_j},\dots,\widetilde{e}_{j,1}\right\}}}^{b^L-1} \left[\frac{\widetilde{\widetilde{a}}_{j,1}}{(b^L)} + \frac{\widetilde{\widetilde{a}}_{j,2}}{(b^L)^2} + \dots + \frac{\widetilde{\widetilde{a}}_{j,n}}{(b^L)^n}, \frac{\widetilde{\widetilde{a}}_{j,1}}{(b^L)} + \frac{\widetilde{\widetilde{a}}_{j,2}}{(b^L)^2} + \dots + \frac{\widetilde{\widetilde{a}}_{j,n}}{(b^L)^n} + \frac{1}{(b^L)^n}\right).$$

If we can show that

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha L')} \widetilde{e}_{j,l} \le \beta n - t, \tag{10}$$

then each interval contains $(b^L)^{m-|\widetilde{\boldsymbol{\nu}}|_1}$ points, and consequently $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains

$$(b^L)^{m-|\widetilde{\boldsymbol{\nu}}|_1} b^{|\widetilde{\boldsymbol{\nu}}|_1 L - |\boldsymbol{\nu}|_1 L'} = b^{mL - |\boldsymbol{\nu}|_1 L'} = b^{m'L' - |\boldsymbol{\nu}|_1 L'} = (b^{L'})^{m' - |\boldsymbol{\nu}|_1}$$

points and the proof is complete. Hence $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains the right number of points if Equation (10) is satisfied, or equivalently, if

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha L')} \widetilde{e}_{j,l} L \le L(\beta n - t).$$

So $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ still contains the right number of points if

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L',\alpha L')} \widetilde{i}_{j,l} + \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L',\alpha L')} r_{j,l} \le L(\beta n - t). \tag{11}$$

We now find a bound for $\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha L')}\widetilde{i}_{j,l}$:

$$\begin{aligned}
\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha L')}\widetilde{i}_{j,l} &= \sum_{j=1}^{s}\sum_{l=1}^{L'\min(\nu_j,\alpha)}\widetilde{i}_{j,l} \\
&= \sum_{j=1}^{s}\sum_{k=1}^{\min(\nu_j,\alpha)}\sum_{g=1}^{L'}\widetilde{i}_{j,(k-1)L'+g} = \sum_{j=1}^{s}\sum_{k=1}^{\min(\nu_j,\alpha)}\sum_{g=1}^{L'}(i_{j,k}L'+1-g) \\
&= \sum_{j=1}^{s}\sum_{k=1}^{\min(\nu_j,\alpha)}\left[\sum_{g=1}^{L'}i_{j,k}L' - \sum_{g=1}^{L'-1}g\right] \\
&= \sum_{j=1}^{s}\sum_{k=1}^{\min(\nu_j,\alpha)}\left[i_{j,k}L'^2 - \frac{(L'-1)L'}{2}\right] \\
&\le \sum_{j=1}^{s}\sum_{k=1}^{\min(\nu_j,\alpha)}\left[i_{j,k}L'^2\right] - \frac{(L'-1)L'}{2} \\
&\le \beta n'L' - t''L'^2 - \frac{(L'-1)L'}{2}, \qquad (12)
\end{aligned}$$

where we used Equation (9). Combining Equations (11) and (12) we find that $J(\boldsymbol{i_\nu},\boldsymbol{a_\nu})$ contains the right number of points if

$$t''L'^2 + \frac{(L'-1)L'}{2} - \sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha L')} r_{j,l} \ge tL.$$

That is, we can set

$$t' = \min\left\{t'' : t''L'^2 + \frac{(L'-1)L'}{2} - M(t'') \ge tL\right\}, \qquad (13)$$

where

$$M(t'') = \max\left\{\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha L')}(-\widetilde{i}_{j,l} \pmod L) \;:\; i_{j,l}\ge 0 \text{ and } \sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \le \frac{\beta}{L'}n' - t''\right\},$$

where we recall $\widetilde{i}_{j,l} = \widetilde{e}_{j,l}L - r_{j,l}$ for $1\le l\le \nu_j L'$ and $1\le j\le s$, and $\widetilde{i}_{j,(k-1)L'+g} = i_{j,k}L'+1-g$ for $1\le g\le L'$ and $1\le k\le \nu_j$. We now aim to find an upper bound for $\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha L')}(-\widetilde{i}_{j,l} \pmod L)$. We have

$$\begin{aligned}
\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha L')}(-\widetilde{i}_{j,l} \pmod L) &= \sum_{j=1}^{s}\sum_{k=1}^{\min(\nu_j,\alpha)}\sum_{g=1}^{L'}(-i_{j,k}L'-1+g \pmod L) \\
\le \sum_{j=1}^{s}\sum_{k=1}^{\min(\nu_j,\alpha)}\sum_{g=1}^{L'}(-i_{j,k}L' \pmod L) &+ \sum_{j=1}^{s}\sum_{k=1}^{\min(\nu_j,\alpha)}\sum_{g=1}^{L'}(g-1 \pmod L) \\
\le \sum_{j=1}^{s}\sum_{k=1}^{\min(\nu_j,\alpha)}\sum_{g=1}^{L'}(-L' \pmod L)\,i_{j,k} &+ s\alpha(L-1)L'
\end{aligned}$$

$$\leq (-L' \pmod L)L'\left(\frac{\beta}{L'}n' - t''\right) + s\alpha(L-1)L'$$
$$= (-L' \pmod L)(\beta n' - t''L') + s\alpha(L-1)L'.$$

From Equation (13) it follows that

$$t' \leq \min\left\{t'' \,:\, t''L'^2 + \frac{(L'-1)L'}{2} - ((-L' \pmod L))(\beta n' - t''L') + (L-1)L'\alpha s) \geq tL\right\}.$$

This condition is satisfied for all t'' with

$$t'' \geq \left\lceil \frac{tL + s\alpha(L-1)L' - \frac{(L'-1)L'}{2} + (-L' \pmod L)\beta n'}{L'(L' + (-L' \pmod L))} \right\rceil,$$

which gives the first bound. For the second bound, let

$$t'' = \left\lceil \frac{tL + (s\alpha - 1)(L-1) - \frac{(L'-1)L'}{2}}{L'^2} \right\rceil,$$

then, using Equation (12), we have

$$\begin{aligned}
\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha L')} \widetilde{e}_{j,l} &\leq \frac{1}{L}\left(\sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha L')} \widetilde{i}_{j,l} + \sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha L')} r_{j,l}\right) \\
&\leq \frac{1}{L}\left(\beta n'L' - t''L'^2 - \frac{(L'-1)L'}{2} + \sum_{j=1}^{s}\sum_{l=1}^{\min(\nu_j L',\alpha L')} r_{j,l}\right) \\
&\leq \frac{1}{L}\left(\beta n'L' - t''L'^2 - \frac{(L'-1)L'}{2} + s\alpha(L-1)L'\right) \\
&\leq \frac{1}{L}\left(\beta n'L' - Lt - (s\alpha L' - 1)(L-1) + \frac{(L'-1)L'}{2} - \frac{(L'-1)L'}{2} + s\alpha(L-1)L'\right) \\
&= \beta n - t + \frac{L-1}{L}.
\end{aligned}$$

By assumption, βn is an integer, $\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha L')} \widetilde{e}_{j,l}$ is an integer, hence

$$\sum_{j=1}^{s}\sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha L')} \widetilde{e}_{j,l} \leq \beta n - t,$$

which completes the proof. □

We point out that $\alpha L'$ changes to α in Theorem 4.6 . Using propagation rule (ii) from Section 2, we can establish the following corollary to Theorem 4.6, which avoids a change in the parameter α.

Corollary 4.1 *Let $n, n', m, m', s, \alpha, L$ and $L' \in \mathbb{N}$, where $\gcd(L, L') = 1$, $mL = m'L'$, $nL = n'L'$, let $0 < \beta \leq 1$ be a real number and let $0 \leq t \leq \beta n$ and βn be integers. Then a $(t, \alpha, \beta, n, m, s)$-net in base b^L is a $(t', \alpha, \frac{\beta}{L'}, n', m', s)$-net in base $b^{L'}$, where*

$$t' = \min\left(\left\lceil \frac{tL + s\alpha(L-1)L' - \frac{(L'-1)L'}{2} + (-L' \pmod L)\beta n'}{L'(L' + (-L' \pmod L))} \right\rceil,\right.$$

$$\left\lceil \frac{tL + (s\alpha L' - 1)(L-1) - \frac{(L'-1)L'}{2}}{L'^2} \right\rceil \Bigg).$$

However, in some cases it is possible to improve on Corollary 4.1.

Theorem 4.7 *Let $n, n', m, m', s, \alpha, L$ and $L' \in \mathbb{N}$, $L' \geq \alpha$ where $\gcd(L, L') = 1$, $mL = m'L'$, $nL = n'L'$, let $0 < \beta \leq 1$ be a real number and let $0 \leq t \leq \beta n$ and βn be integers. Then a $(t, \alpha, \beta, n, m, s)$-net in base b^L is a $(t', \alpha, \frac{\beta}{\alpha}, n', m', s)$-net in base $b^{L'}$, where*

$$\begin{aligned} t' &= \min\left(\left\lceil \frac{tL + sf(\alpha, L) - \frac{(\alpha-1)\alpha}{2} + (-L' \pmod{L})\beta n'}{\alpha(L' + (-L' \pmod{L}))} \right\rceil, \right. \\ &\qquad \left. \left\lceil \frac{tL + (s\alpha - 1)(L-1) - \frac{(\alpha-1)\alpha}{2}}{\alpha L'} \right\rceil \right), \end{aligned}$$

where $f(\alpha, L) = \sum_{l=1}^{\alpha} (l - 1 \pmod{L}\ )$.

Proof. Using the same definitions as in the proof of Theorem 4.6, we aim to establish that the assumption

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} \leq \frac{\beta}{\alpha} n' - t''$$

where t'' is a non-negative integer, implies that

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j, \alpha)} \widetilde{e}_{j,l} \leq \beta n - t. \tag{14}$$

We proceed in a manner similar to the proof of Theorem 4.6, i.e. $J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ contains the right number of points if Equation (14) is satisfied which in turn is equivalent to

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j, \alpha)} \widetilde{e}_{j,l} L \leq \beta n L - tL,$$

and hence $J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ still contains the right number of points if

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha)} \widetilde{i}_{j,l} + \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha)} r_{j,l} \leq \beta n L - tL. \tag{15}$$

We now find a bound for $\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha)} \widetilde{i}_{j,l}$. We have

$$\begin{aligned} \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha)} \widetilde{i}_{j,l} &= \sum_{\substack{j=1 \\ \nu_j > 0}}^{s} \sum_{l=1}^{\alpha} \widetilde{i}_{j,l} \\ &= \sum_{\substack{j=1 \\ \nu_j > 0}}^{s} \sum_{l=1}^{\alpha} [i_{j,1} L' + 1 - l] \end{aligned}$$

$$
\begin{aligned}
&= L' \sum_{\substack{j=1\\ \nu_j>0}}^{s} \sum_{l=1}^{\alpha} i_{j,1} + \sum_{\substack{j=1\\ \nu_j>0}}^{s} \sum_{l=1}^{\alpha} (1-l) \\
&= \alpha L' \sum_{\substack{j=1\\ \nu_j>0}}^{s} i_{j,1} - \sum_{\substack{j=1\\ \nu_j>0}}^{s} \frac{(\alpha-1)\alpha}{2} \\
&\leq \alpha L' \left(\frac{\beta}{\alpha} n' - t'' \right) - \frac{(\alpha-1)\alpha}{2}. \qquad (16)
\end{aligned}
$$

Hence, combining Equations (15) and (16), we find that $J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ contains the right number of points if

$$
t''\alpha L' + \frac{(\alpha-1)\alpha}{2} - \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha)} r_{j,l} \geq tL.
$$

We set

$$
t' = \min\left\{ t'' \,:\, t''\alpha L' + \frac{(\alpha-1)\alpha}{2} - M(t'') \geq tL \right\},
$$

where

$$
M(t'') = \max\left\{ \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha)} (-\widetilde{i}_{j,l} \pmod{L}) \,:\, i_{j,1} \geq 0, \sum_{\substack{j=1\\ \nu_j>0}} i_{j,1} \leq \frac{\beta}{\alpha} n' - t'' \right\}.
$$

We now establish a bound for $\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha)} (-\widetilde{i}_{j,l} \pmod{L})$, where we set $f(\alpha, L) = \sum_{l=1}^{\alpha} (l - 1 \pmod{L})$. We have

$$
\begin{aligned}
&\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L', \alpha)} (-\widetilde{i}_{j,l} \pmod{L}) \\
&\quad = \sum_{\substack{j=1\\ \nu_j>0}}^{s} \sum_{l=1}^{\alpha} (-i_{j,1} L' - 1 + l \pmod{L}) \\
&\quad \leq \sum_{\substack{j=1\\ \nu_j>0}}^{s} \sum_{l=1}^{\alpha} (-i_{j,1} L' \pmod{L}) + \sum_{\substack{j=1\\ \nu_j>0}}^{s} \sum_{l=1}^{\alpha} (l - 1 \pmod{L}) \\
&\quad \leq (-L' \pmod{L}) \alpha \left(\frac{\beta}{\alpha} n' - t'' \right) + s f(\alpha, L).
\end{aligned}
$$

Hence

$$
t' \leq \min\left\{ t'' \,:\, t''\alpha L' + \frac{(\alpha-1)\alpha}{2} - ((-L' \pmod{L})(\beta n' - t''\alpha) + s f(\alpha, L)) \geq tL \right\},
$$

which is satisfied for all t'' with

$$
t'' \geq \left\lceil \frac{tL + s f(\alpha, L) + (-L' \pmod{L}) \beta n' - \frac{(\alpha-1)\alpha}{2}}{\alpha(L' + (-L' \pmod{L}))} \right\rceil.
$$

To obtain the second bound, we set

$$t'' = \left\lceil \frac{tL + (s\alpha - 1)(L-1) - \frac{(\alpha-1)\alpha}{2}}{\alpha L'} \right\rceil .$$

Consequently,

$$\begin{aligned}
\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha)} \widetilde{e}_{j,l} &\leq \frac{1}{L} \left(\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L',\alpha)} \widetilde{i}_{j,l} + \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j L',\alpha)} r_{j,l} \right) \\
&\leq \frac{\alpha L'}{L} \left(\frac{\beta}{\alpha} n' - t'' \right) - \frac{(\alpha-1)\alpha}{2L} + \frac{s\alpha(L-1)}{L} \\
&\leq \beta n - t + \frac{L-1}{L},
\end{aligned}$$

hence $\sum_{j=1}^{s} \sum_{l=1}^{\min(\widetilde{\nu}_j,\alpha)} \widetilde{e}_{j,l} \leq \beta n - t$ and the proof is complete. □

In the following corollary, we recover the result due to Pirsic.

Corollary 4.2 *Let m, m', L and $L' \in \mathbb{N}$, $\gcd(L, L') = 1$, $mL = m'L'$ and let $0 \leq t \leq m$ be an integer. Then a (t, m, s)-net in base b^L is a (t', m', s)-net in base $b^{L'}$, with*

$$t' = \min \left(\left\lceil \frac{tL + (-L' \pmod{L})m'}{L' + (-L' \pmod{L})} \right\rceil, \left\lceil \frac{tL + (s-1)(L-1)}{L'} \right\rceil \right).$$

Proof. The proof follows immediately from Theorem 4.7, where we set $\alpha = \beta = 1$, $n = m$ and $n' = m'$ and notice that $f(1, L) = 0$. □

We again remark that in Theorem 4.6, $\alpha L'$ changes to α. However, when considering a base change from b^L to b, there is no need to change α, as the following theorem shows, which can be regarded as a generalization of [9, Theorem 9] and [16, Lemma 9].

Theorem 4.8 *For $L \in \mathbb{N}$, a $(t', \alpha, \beta, n, m, s)$-net in base b^L is a $(t, \alpha, \beta, nL, mL, s)$-net in base b, where*

$$t \leq t'L + (s\alpha - 1)(L-1).$$

Proof. The proof is similar to the proof of Theorem 4.6. □

Finally, we consider a base change from b to $b^{L'}$, which can be considered to be a generalization of [11, Lemma 2.9].

Theorem 4.9 *Let n, m, s, α, $L' \in \mathbb{N}$, let $0 < \beta \leq 1$ be a real number and let $0 \leq t \leq \frac{\beta}{L'} n$ be an integer. Then a $(tL'^2 + \frac{(L'-1)L'}{2}, \alpha L', \beta, nL', mL', s)$-net in base b is a $(t, \alpha, \frac{\beta}{L'}, n, m, s)$-net in base $b^{L'}$.*

Proof. The proof is similar to the proof of Theorem 4.6. □

Furthermore, we point out that $\alpha L'$ changes to α in Theorem 4.9. Using Propagation Rule (ii) from Section 2, we can establish the following corollary to Theorem 4.9, which avoids a change in the parameter α.

Corollary 4.3 *Let $n, m, s, \alpha, L' \in \mathbb{N}$, let $0 < \beta \leq 1$ be a real number and let $0 \leq t \leq \frac{\beta}{L'} n$ be an integer. Then a $(tL'^2 + \frac{(L'-1)L'}{2}, \alpha, \beta, nL', mL', s)$-net in base b is a $(t, \alpha, \frac{\beta}{L'}, n, m, s)$-net in base $b^{L'}$.*

However, in some cases it is possible to improve on Corollary 4.3.

Theorem 4.10 *Let $L' \geq \alpha$, then a $(t\alpha L' + \frac{(\alpha-1)\alpha}{2}, \alpha, \beta, nL', mL', s)$-net in base b is a $(t, \alpha, \frac{\beta}{\alpha}, n, m, s)$-net in base $b^{L'}$.*

Proof. The proof proceeds along the same lines as the proof of Theorem 4.7. □

4.7 A higher order to higher order construction

We next consider a propagation rule which was referred to as "A higher order to higher order construction" in [9]. In [6], it was shown how to construct digital $(t, \alpha, \beta, n \times m, s)$-nets from digital (t, m, sd)-nets. Essentially, the "higher order to higher order construction" in [9] replaces the digital (t, m, sd)-net with a digital $(t, \alpha, \beta, n \times m, sd)$-net, but makes use of the same construction algorithm. We now show that the same idea can be used for $(t, \alpha, \beta, n, m, s)$-nets. Assume we are given a multiset $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ forming a $(t', \alpha', \beta', n, m, sd)$-net in base b. We write $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,sd})$ and $x_{h,j} = \xi_{h,j,1}/b + \xi_{h,j,2}/b^2 + \ldots$ for all $0 \leq h \leq b^m - 1$ and $1 \leq j \leq sd$.

Then we construct a multiset $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ as follows: for $0 \leq h < b^m$ we set $\boldsymbol{y}_h = (y_{h,1}, \ldots, y_{h,s})$ in $[0,1)^s$ where for $1 \leq j \leq s$,

$$y_{h,j} = \sum_{l=1}^{n} \sum_{k=1}^{d} \xi_{h,(j-1)d+k,l} b^{-k-(l-1)d}. \tag{17}$$

Theorem 4.11 *Let $d \in \mathbb{N}$ and let $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a $(t', \alpha', \beta', n, m, sd)$-net in base b, where we assume that $\beta' n$ is an integer. Then for any $\alpha \geq 1$, the multiset $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ defined by Equation (17) forms a $(t, \alpha, \beta' \min(1, \alpha/(\alpha' d)), dn, m, s)$-net in base b with*

$$t = \left\lceil \min\left(d, \frac{\alpha}{\alpha'}\right) \min\left(\beta' n, t' + \left\lfloor \frac{\alpha' s(d-1)}{2} \right\rfloor\right) \right\rceil.$$

Proof. The case where $\beta' n < t' + \lfloor \alpha' s(d-1)/2 \rfloor$ is trivial. Hence assume now we are given an arbitrary generalized elementary interval $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$, for some given values of $\boldsymbol{\nu}$, $\boldsymbol{i}_{\boldsymbol{\nu}}$, $\boldsymbol{a}_{\boldsymbol{\nu}}$, such that $1 \leq i_{j,\nu_j} < \cdots < i_{j,1}$, $\nu_j \geq 0$, for $1 \leq j \leq s$ and

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j, \alpha)} i_{j,l} \leq \beta' \min\left(1, \frac{\alpha}{\alpha' d}\right) dn - t.$$

We need to show that $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$ contains $b^{m-|\boldsymbol{\nu}|_1}$ points. For $\boldsymbol{y}_h$, $0 \leq h \leq b^m - 1$, to be in $J(\boldsymbol{i}_{\boldsymbol{\nu}}, \boldsymbol{a}_{\boldsymbol{\nu}})$, we need for $0 \leq h \leq b^m - 1$, $1 \leq j \leq s$, $1 \leq l \leq n$ and $1 \leq k \leq d$,

$$\eta_{h,j,(l-1)d+k} = a_{j,(l-1)d+k} \quad \text{whenever} \quad (l-1)d + k \in \left\{i_{j,\nu_j}, \ldots, i_{j,1}\right\},$$

where $y_{h,j} := \eta_{h,j,1}/b + \cdots + \eta_{h,j,dn}/b^{dn}$. But from the construction method we find that the condition $\eta_{h,j,(l-1)d+k} = a_{j,(l-1)d+k}$ is equivalent to $\xi_{h,(j-1)d+k,l} = a_{j,(l-1)d+k}$. As

$\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{b^m-1}\}$ forms a $(t', \alpha', \beta', n, m, sd)$-net, we translate the above condition into a condition on a generalized elementary interval of dimension sd. In particular we set

$$a'_{(j-1)d+k,l} = a_{j,(l-1)d+k}, \text{ if } (l-1)d + k \in \left\{i_{j,\nu_j}, \ldots, i_{j,1}\right\}.$$

Also, for each choice of $1 \le j \le s$ and $1 \le k \le d$ we let $w_{(j-1)d+k}$ denote the largest integer such that there are $e_{(j-1)d+k,1} > \cdots > e_{(j-1)d+k,w_{(j-1)d+k}} > 0$ for which

$$\left\{(e_{(j-1)d+k,u} - 1)d + k : u = 1, \ldots, w_{(j-1)d+k}\right\} \subseteq \left\{i_{j,\nu_j}, \ldots, i_{j,1}\right\},$$

where for $w_{(j-1)d+k} = 0$ we set $\left\{(e_{(j-1)d+k,u} - 1)d + k : u = 1, \ldots, w_{(j-1)d+k}\right\} = \emptyset$. Consequently, for dimension $(j-1)d + k$ with $1 \le j \le s$ and $1 \le k \le d$, the digits $a'_{(j-1)d+k,1}, \ldots, a'_{(j-1)d+k,w_{(j-1)d+k}}$ are specified whenever $w_{(j-1)d+k} > 0$. In particular, $w_{(j-1)d+k}$ gives the number of digits in dimension $(j-1)d+k$ that the generalized elementary interval corresponding to the $(t', \alpha', \beta', n, m, sd)$-net contributes to dimension j of the generalized elementary interval corresponding to the $(t, \alpha, \beta' \min(1, \alpha/(\alpha' d)), dn, m, s)$-net. We hence note that

$$\sum_{k=1}^{d} w_{(j-1)d+k} = \nu_j \quad \text{for } 1 \le j \le s. \tag{18}$$

We hence obtain the following generalized elementary interval $J(\boldsymbol{e}_{\boldsymbol{w}}, \boldsymbol{a}'_{\boldsymbol{w}})$ of dimension sd, where $\boldsymbol{e}_{\boldsymbol{w}} = (e_{1,w_1}, \ldots, e_{1,1}, \ldots, e_{sd,w_{sd}}, \ldots, e_{sd,1})$, $\boldsymbol{a}'_{\boldsymbol{w}} = (a'_{1,w_1}, \ldots, a'_{1,1}, \ldots, a'_{sd,w_{sd}}, \ldots, a'_{sd,1})$ and

$$J(\boldsymbol{e}_{\boldsymbol{w}}, \boldsymbol{a}'_{\boldsymbol{w}}) = \prod_{j=1}^{sd} \bigcup_{\substack{a'_{j,l}=0 \\ l \in \{1,\ldots,n\} \setminus \left\{e_{j,w_j}, \ldots, e_{j,1}\right\}}}^{b-1} \left[\frac{a'_{j,1}}{b} + \frac{a'_{j,2}}{b^2} + \cdots + \frac{a'_{j,n}}{b^n}, \frac{a'_{j,1}}{b} + \frac{a'_{j,2}}{b^2} + \cdots + \frac{a'_{j,n}}{b^n} + \frac{1}{b^n}\right).$$

By the property of the $(t', \alpha', \beta', n, m, sd)$-net, if

$$\sum_{j=1}^{sd} \sum_{l=1}^{\min(w_j, \alpha')} e_{j,l} \le \beta' n - t', \tag{19}$$

then $J(\boldsymbol{e}_{\boldsymbol{w}}, \boldsymbol{a}'_{\boldsymbol{w}})$ contains $b^{m - \sum_{j=1}^{sd} w_j} = b^{m - \sum_{j=1}^{s} \nu_j}$ points, where we used Equation (18), as required. By distinguishing the cases $\alpha' d \le \alpha$ and $\alpha' d > \alpha$, it was shown in [9] that Equation (19) holds, which completes the proof. □

Remark 4.2 Similar to [9, Example 1] one can employ a $(0, m, 2)$-net in base b to show that Theorem 4.11 cannot be improved on in general.

Corollary 4.4 *Let $d \in \mathbb{N}$ and let $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ be a (t', m, sd)-net in base b. Then for every $\alpha \ge 1$, the multiset $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ defined by Equation (17) forms a $(t, \alpha, \min(1, \frac{\alpha}{d}), dm, m, s)$-net in base b with*

$$t = \min(d, \alpha) \min\left(m, t' + \left\lfloor \frac{s(d-1)}{2} \right\rfloor\right).$$

Proof. The proof follows immediately from Remark 2.1 and by setting $\alpha' = \beta' = 1$ and $n = m$ in Theorem 4.11. $\square$

Theorem 4.11 can be improved when $\alpha = \alpha'$, which we show in the following.

Proposition 4.1 *Let $\alpha, d \in \mathbb{N}$ and let $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ form a $(t, \alpha, \beta, n, m, sd)$-net in base b. Then the multiset $\{\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{b^m-1}\}$ defined by Equation* (17) *forms a $(t, \alpha, \beta, n, m, s)$-net in base b.*

Proof. Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_s) \in \{0, \ldots, nd\}^s$ be given and for $j = 1, \ldots, s$ let $dn > i_{j,1} > \cdots > i_{j,\nu_j} > 0$ be such that

$$\sum_{j=1}^{s} \sum_{l=1}^{\min(\alpha,\nu_j)} i_{j,l} \le \beta n - t.$$

Let $\boldsymbol{i_\nu} = (i_{1,1}, \ldots, i_{1,\nu_1}, \ldots, i_{s,1}, \ldots, i_{s,\nu_s})$ and $\boldsymbol{a_\nu} = (a_{1,i_{1,1}}, \ldots, a_{1,i_{1,\nu_1}}, \ldots, a_{s,i_{s,1}}, \ldots, a_{s,i_{s,\nu_s}}) \in \{1, \ldots, nd\}^{|\nu|_1}$. Let a generalized elementary interval

$$J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu}) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l \in \{1,\ldots,nd\} \setminus \{i_{j,1},\ldots,i_{j,\nu_j}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,nd}}{b^{nd}}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,nd}}{b^{nd}} + \frac{1}{b^{nd}}\right),$$

where $\{i_{j,1}, \ldots, i_{j,\nu_j}\} = \emptyset$ in case $\nu_j = 0$ for $1 \le j \le s$ be given.

Let $\boldsymbol{y}_h = (y_{h,1}, \ldots, y_{h,s})$ with $y_{h,s} = \eta_{h,j,1}/b + \eta_{h,j,2}/b^2 + \cdots$. Then $\boldsymbol{y}_h \in J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ if and only if $\eta_{h,j,l} = a_{j,l}$ for all $l \in \{i_{j,1}, \ldots, i_{j,\nu_j}\}$ and $1 \le j \le s$.

We define now a new generalized elementary interval J' in dimension sd such that $\boldsymbol{y}_h \in J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ if and only if $\boldsymbol{x}_h \in J'$. To this end, for $j = 1, \ldots, s$, let $a'_{(j-1)d+k,l} = a_{j,(l-1)d+k}$ where $1 \le k \le d$ and $1 \le l \le n$ are such that $(l-1)d + k \in \{i_{j,1}, \ldots, i_{j,\nu_j}\}$. For $j' = 1, \ldots, sd$ we have now specified $a'_{j',i'}$ for certain values of $i' \in \{1, \ldots, n\}$. Let $U_{j'}$ be the set of i' for which $a'_{j',i'}$ is specified, i.e.,

$$U_{j'} = \{1 \le i' \le n : (i'-1)d + j' - (j-1)d \in \{i_{j,1}, \ldots, i_{j,\nu_j}\} \text{ for } j = \lceil j'/d \rceil\}.$$

Let $U_{j'} = \{i'_{j',1}, \ldots, i'_{j',\nu'_{j'}}\}$, where we assume that the elements are ordered such that $n \ge i'_{j',1} > \cdots > i'_{j',\nu'_{j'}} > 0$. Define now $\nu' = (\nu'_1, \ldots, \nu'_{sd}) \in \{0, \ldots, n\}^{sd}$, $\boldsymbol{i'_{\nu'}} = (i'_{1,1}, \ldots, i'_{1,\nu'_1}, \ldots, i'_{sd,1}, \ldots, i'_{sd,\nu'_{sd}})$, and $\boldsymbol{a}' = (a'_{1,i'_{1,1}}, \ldots, a'_{1,i'_{1,\nu'_1}}, \ldots, a'_{sd,i'_{sd,1}}, \ldots, a'_{sd,i'_{sd,\nu'_{sd}}})$. Then $J' = J(\boldsymbol{i'_{\nu'}}, \boldsymbol{a'_{\nu'}})$ has the property that $\boldsymbol{y}_h \in J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ if and only if $\boldsymbol{x}_h \in J(\boldsymbol{i'_{\nu'}}, \boldsymbol{a'_{\nu'}})$.

Note that $\nu'_{(j-1)d+1} + \cdots + \nu'_{(j-1)d+d} = \nu_j$ for $1 \le j \le s$ and therefore $|\boldsymbol{\nu}|_1 = |\boldsymbol{\nu}'|_1$. Thus if $J(\boldsymbol{i_\nu}, \boldsymbol{a_\nu})$ contains $b^{m-|\boldsymbol{\nu}|_1}$ points, then J' contains $b^{m-|\boldsymbol{\nu}'|_1}$. The latter will be the case if $\sum_{j=1}^{sd} \sum_{l=1}^{\min(\alpha,\nu'_j)} i'_{j,l} \le \beta n - t$, which we show in the following.

If $\nu_j < \alpha$, then

$$\begin{aligned} \sum_{j'=(j-1)d+1}^{jd} \sum_{l=1}^{\nu'_{j'}} i'_{j',l} &\le \left\lceil \frac{i_{j,1}}{d} \right\rceil + \cdots + \left\lceil \frac{i_{j,\nu_j}}{d} \right\rceil \\ &\le \frac{i_{j,1} + \cdots + i_{j,\nu_j} + \nu_j(d-1)}{d} \end{aligned}$$

$$\leq \; i_{j,1} + \cdots + i_{j,\nu_j}$$

since $i_1 + \cdots i_{\nu_j} \geq \frac{\nu_j(\nu_j+1)}{2}$.

If $\nu_j \geq \alpha$, then

$$\begin{aligned}\sum_{j'=(j-1)d+1}^{jd} \sum_{l=1}^{\min(\nu'_{j'},\alpha)} i'_{j',l} &\leq \left\lceil \frac{i_{j,1}}{d} \right\rceil + \cdots + \left\lceil \frac{i_{j,\alpha}}{d} \right\rceil + \left\lceil \frac{(i_{j,\alpha}-1)_+}{d} \right\rceil + \cdots + \left\lceil \frac{(i_{j,\alpha}-\alpha(d-1))_+}{d} \right\rceil \\ &\leq i_{j,1} + \cdots + i_{j,\alpha},\end{aligned}$$

where $(x)_+ = \max(x, 0)$.

Therefore we have

$$\sum_{j=1}^{sd} \sum_{l=1}^{\min(\alpha,\nu'_j)} i'_{j,l} \leq \sum_{j=1}^{s} \sum_{l=1}^{\min(\nu_j,\alpha)} i_{j,l} \leq \beta n - t.$$

Hence the result follows since $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{b^m-1}\}$ is a $(t, \alpha, \beta, n, m, sd)$-net and therefore J' contains $b^{m-|\boldsymbol{\nu}'|_1}$ points. $\square$

5 Propagation rules for $(t, \alpha, \beta, \sigma, s)$-sequences and an application

Based on results from Section 4 we deduce properties of $(t, \alpha, \beta, \sigma, s)$-sequences in base b.

5.1 A higher order to higher order construction for $(t, \alpha, \beta, \sigma, s)$-sequences

We use the higher order to higher order construction from Section 4.7 to construct $(t, \alpha, \beta, \sigma, s)$-sequences in base b.

Assume we are given an infinite sequence $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$ forming a $(t', \alpha', \beta', \sigma, sd)$-sequence in base b. We write $\boldsymbol{x}_h = (x_{h,1}, \ldots, x_{h,sd})$ and $x_{h,j} = \xi_{h,j,1}/b + \xi_{h,j,2}/b^2 + \ldots$ for all $h \geq 0$ and $1 \leq j \leq sd$.

Then we construct an infinite sequence $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots\}$ as follows: for $h \geq 0$ we set $\boldsymbol{y}_h = (y_{h,1}, \ldots, y_{h,s})$ in $[0,1)^s$ where

$$y_{h,j} = \sum_{l=1}^{\infty} \sum_{k=1}^{d} \xi_{h,(j-1)d+k,l} b^{-k-(l-1)d}. \tag{20}$$

Theorem 5.1 *Let $\alpha', d, s, \sigma \in \mathbb{N}$, $0 < \beta' \leq 1$ be such that $\beta'\sigma$ is an integer, and $t' \geq 0$ be an integer. Let $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots\}$ be a $(t', \alpha', \beta', \sigma, sd)$-sequence in base b. Then for any $\alpha \geq 1$, the infinite sequence $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots\}$ defined by Equation (20) forms a $(t, \alpha, \beta' \min(1, \alpha/(\alpha' d)), d\sigma, s)$-sequence in base b with*

$$t = \left\lceil \min\left(d, \frac{\alpha}{\alpha'}\right) \left(t' + \left\lfloor \frac{\alpha' s(d-1)}{2} \right\rfloor\right) \right\rceil.$$

Proof. We need to show that for all $k \geq 0$ and all $m > \frac{t}{\beta' \min(1, \frac{\alpha}{\alpha' d}) d\sigma}$ the multiset $\{\boldsymbol{y}_{kb^m}, \dots, \boldsymbol{y}_{(k+1)b^m-1}\}$ forms a $(t, \alpha, \beta' \min(1, \frac{\alpha}{\alpha' d}), d\sigma m, m, s)$-net in base b. It is clear that $m > \frac{t'}{\beta'\sigma}$ and hence $\{\boldsymbol{x}_{kb^m}, \dots, \boldsymbol{x}_{(k+1)b^m-1}\}$ forms a $(t', \alpha', \beta', \sigma m, m, sd)$-net in base b. But $\beta'\sigma m$ is an integer, hence $\{\boldsymbol{y}_{kb^m}, \dots, \boldsymbol{y}_{(k+1)b^m-1}\}$ forms a $(t, \alpha, \beta' \min(1, \frac{\alpha}{\alpha' d}), d\sigma m, m, s)$-net in base b, by Theorem 4.11, where $t \leq \lceil \min(d, \frac{\alpha}{\alpha'})(t' + \lfloor \frac{\alpha' s(d-1)}{2} \rfloor) \rceil$. Hence Equation (20) defines a $(t, \alpha, \beta' \min(1, \alpha/(\alpha' d)), d\sigma, s)$-sequence. □

Remark 5.1 *As in Remark 4.2 and [9, Example 1] one can employ a $(0, 2)$-sequence in base b to show that Theorem 5.1 cannot be improved on in general.*

Similar to Corollary 4.4 in Subsection 4.7, we consider the following special case.

Corollary 5.1 *Let $\alpha', d, s, \sigma \in \mathbb{N}$, $0 < \beta' \leq 1$ be such that $\beta'\sigma$ is an integer, and $t' \geq 0$ be an integer. Let $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots\}$ be a (t', sd)-sequence in base b. Then for any $\alpha \geq 1$, the infinite sequence $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \dots\}$ defined by Equation (20) forms a $(t, \alpha, \min(1, \frac{\alpha}{d}), d, s)$-sequence in base b with*

$$t = \min(d, \alpha) \left(t' + \left\lfloor \frac{s(d-1)}{2} \right\rfloor \right).$$

The following result is analogous to Proposition 4.1.

Proposition 5.1 *Let $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots\}$ be a $(t, \alpha, \beta, \sigma, sd)$-sequence in base b. Then the infinite sequence $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \dots\}$ defined by Equation (20) forms a $(t, \alpha, \beta, \sigma, s)$-sequence in base b.*

Proof. We need to show that for $m > t/(\beta\sigma)$, $k \geq 0$, the multiset $\{\boldsymbol{y}_{kb^m}, \dots, \boldsymbol{y}_{(k+1)b^m-1}\}$ forms a $(t, \alpha, \beta, \sigma m, m, s)$-net in base b. But for $m > t/(\beta\sigma)$, $k \geq 0$, $\{\boldsymbol{x}_{kb^m}, \dots, \boldsymbol{x}_{(k+1)b^m-1}\}$ forms a $(t, \alpha, \beta, \sigma m, m, sd)$-net in base b, hence, by Proposition 4.1, $\{\boldsymbol{y}_{kb^m}, \dots, \boldsymbol{y}_{(k+1)b^m-1}\}$ forms a $(t, \alpha, \beta, \sigma m, m, s)$-net in base b. □

5.2 A base reduction for $(t, \alpha, \beta, \sigma, s)$-sequences

We show that a $(t', \alpha, \beta, \sigma, s)$-sequence in base b^L can be considered as a $(t, \alpha, \beta, \sigma, s)$-sequence in base b with some quality parameter t. The following theorem generalizes [16, Proposition 4].

Theorem 5.2 *Let $\sigma, s, \alpha, L' \in \mathbb{N}$, let $0 < \beta \leq 1$ be a real number and $t \geq 0$ and $\beta\sigma$ be integers. A $(t', \alpha, \beta, \sigma, s)$-sequence in base b^L is a $(t, \alpha, \beta, \sigma, s)$-sequence in base b with*

$$t = t'L + (s\alpha - 1 + \beta\sigma)(L - 1).$$

Proof. Let $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots\}$ be a $(t', \alpha, \beta, \sigma, s)$-sequence in base b^L, t as above and fix $m > \frac{t}{\beta\sigma}$ and write it in the form $m = pL + r$ with integers p and r such that $0 \leq r < L$. Note that $p > \frac{t'}{\beta\sigma}$. For a fixed integer $k \geq 0$, we consider the multiset $\mathcal{P} = \{\boldsymbol{x}_{kb^m}, \dots, \boldsymbol{x}_{(k+1)b^m-1}\}$. Then $\mathcal{P}$ can be split up into b^r multisets $\{\boldsymbol{x}_{lb^{pL}}, \dots, \boldsymbol{x}_{(l+1)b^{pL}-1}\}$ where $kb^r \leq l < (k+1)b^r$. As $p > \frac{t'}{\beta\sigma}$, each of these subsequences forms a $(t', \alpha, \beta, \sigma p, p, s)$-net in base b^L, which by Theorem 4.8 is a $(t'L + (s\alpha - 1)(L-1), \alpha, \beta, \sigma pL, pL, s)$-net in base b. A $(t'L + (s\alpha - 1)(L-$

$1), \alpha, \beta, \sigma pL, pL, s)$-net in base b is also a $(t'L + (s\alpha - 1 + \beta\sigma)(L-1), \alpha, \beta, \sigma m, pL, s)$-net in base b, as the strength of the latter is smaller than the strength of the former. An application of Propagation rule (vi) from Section 2 shows that $\mathcal{P}$ is a $(t'L + (s\alpha - 1 + \beta\sigma)(L-1), \alpha, \beta, \sigma m, pL + r, s)$-net in base b, and hence a $(t, \alpha, \beta, \sigma m, m, s)$-net in base b. $\square$

5.3 A base expansion for $(t, \alpha, \beta, \sigma, s)$-sequences

Here we consider a base change in the opposite direction: we show that a $(t, \alpha, \beta, \sigma, s)$-sequence in base b can be interpreted as a $(t', \alpha', \beta', \sigma, s)$-sequence in base $b^{L'}$. The following theorem generalizes Theorem 4.9 from Subsection 4.6 to $(t, \alpha, \beta, \sigma, s)$-sequences (see also [16, Proposition 5]).

Theorem 5.3 *Let $\sigma, s, \alpha, L' \in \mathbb{N}$, let $0 < \beta \leq 1$ be a real number and $t \geq 0$ and $\beta\sigma$ be integers. Then a $(u, \alpha L', \beta, \sigma, s)$-sequence in base b is a $(t, \alpha, \frac{\beta}{L'}, \sigma, s)$-sequence in base $b^{L'}$, with*

$$t = \left\lceil \frac{u}{L'^2} - \frac{(L'-1)}{2L'} \right\rceil .$$

Proof. Denote the $(u, \alpha L', \beta, \sigma, s)$-sequence in base b by $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots\}$, which is of course also a $(tL'^2 + \frac{(L'-1)L'}{2}, \alpha L', \beta, \sigma, s)$-sequence in base b. By Definition 2.2 for all integers $k \geq 0$ and $m \geq 1$ the finite subsequence

$$\{\boldsymbol{x}_{kb^{mL'}}, \dots, \boldsymbol{x}_{(k+1)b^{mL'}-1}\} \tag{21}$$

forms a $(\min(tL'^2 + \frac{(L'-1)L'}{2}, \beta\sigma mL'), \alpha L', \beta, \sigma mL', mL', s)$-net in base b. We consider two cases:

1. Assume first that m is such that $tL'^2 + \frac{(L'-1)L'}{2} \leq \beta\sigma mL'$, then by Theorem 4.9, the multiset given by Equation (21) forms a $(t, \alpha, \frac{\beta}{L'}, \sigma m, m, s)$-net in base $b^{L'}$. Furthermore, $tL'^2 + \frac{(L'-1)L'}{2} \leq \beta\sigma mL'$ implies that $t \leq \lfloor \frac{\beta}{L'}\sigma m \rfloor$.
2. Now assume $\beta\sigma mL' < tL'^2 + \frac{(L'-1)L'}{2}$. According to Remark 2.2, the multiset given by Equation (21) forms a $(\lfloor \frac{\beta}{L'}\sigma m \rfloor, \alpha, \frac{\beta}{L'}, \sigma m, m, s)$-net in base $b^{L'}$. Furthermore, $\beta\sigma mL' < tL'^2 + \frac{(L'-1)L'}{2}$ implies that $\lfloor \frac{\beta}{L'}\sigma m \rfloor \leq t$.

Hence the multiset given in Equation (21) is a $\big(\min(t, \lfloor \frac{\beta}{L'}\sigma m \rfloor), \alpha, \frac{\beta}{L'}, \sigma m, m, s\big)$-net in base $b^{L'}$. We conclude that for all m such that $\frac{\beta}{L'}\sigma m > t$ we obtain a $(t, \alpha, \frac{\beta}{L'}, \sigma m, m, s)$-net in base $b^{L'}$ and therefore a $(t, \alpha, \frac{\beta}{L'}, \sigma, s)$-sequence in base $b^{L'}$. $\square$

We also consider a special case based on Theorem 4.10.

Theorem 5.4 *Let $\sigma, s, \alpha, L' \in \mathbb{N}$, $L' \geq \alpha$, let $0 < \beta \leq 1$ be a real number and $t \geq 0$ and $\beta\sigma$ be integers. Then a $(t\alpha L' + \frac{(\alpha-1)\alpha}{2}, \alpha, \beta, \sigma, s)$-sequence in base b is a $(t, \alpha, \frac{\beta}{\alpha}, \sigma, s)$-sequence in base $b^{L'}$.*

Proof. We denote the $(t\alpha L' + \frac{(\alpha-1)\alpha}{2}, \alpha, \beta, \sigma, s)$-sequence in base b by $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots\}$. Then by Definition 2.2 for all integers $k \geq 0$ and $m \geq 1$ the finite subsequence

$$\{\boldsymbol{x}_{kb^{mL'}}, \dots, \boldsymbol{x}_{(k+1)b^{mL'}-1}\} \tag{22}$$

forms a $(\min(t\alpha L' + \frac{(\alpha-1)\alpha}{2}, \beta\sigma m L'), \alpha, \beta, \sigma m L', m L', s)$-net in base b. We consider two cases:

1. Assume that $t\alpha L' + \frac{(\alpha-1)\alpha}{2} \leq \beta\sigma m L'$. Then by Theorem 4.10 the multiset given in Equation (22) is a $(t, \alpha, \frac{\beta}{\alpha}, \sigma m, m, s)$-net in base $b^{L'}$. Furthermore, $t\alpha L' + \frac{(\alpha-1)\alpha}{2} \leq \beta\sigma m L'$ implies that $t \leq \lfloor \frac{\beta}{\alpha}\sigma m \rfloor$.

2. Assume that $\beta\sigma m L' < t\alpha L' + \frac{(\alpha-1)\alpha}{2}$. According to Remark 2.2, the multiset given in Equation (22) forms a $(\lfloor \frac{\beta}{\alpha}\sigma m \rfloor, \alpha, \frac{\beta}{\alpha}, \sigma m, m, s)$-net in base $b^{L'}$. Furthermore, $\beta\sigma m L' < t\alpha L' + \frac{(\alpha-1)\alpha}{2}$ implies that $\lfloor \frac{\beta}{\alpha}\sigma m \rfloor \leq t$.

Hence the multiset given in Equation (22) is a $(\min(t, \lfloor \frac{\beta}{\alpha}\sigma m \rfloor), \alpha, \frac{\beta}{\alpha}, \sigma m, m, s)$-net in base $b^{L'}$. We conclude that for all m such that $\frac{\beta}{\alpha}\sigma m > t$ we obtain a $(t, \alpha, \frac{\beta}{\alpha}, \sigma m, m, s)$-net in base $b^{L'}$ and therefore a $(t, \alpha, \frac{\beta}{\alpha}, \sigma, s)$-sequence in base $b^{L'}$. $\square$

5.4 An explicit bound for $t_b(\alpha, s)$ for prime-powers b

In this subsection the least value t such that there exists a $(t, \alpha, \beta, \sigma, s)$-sequence in base b is studied.

Definition 5.1 For integers $b \geq 2$, $s \geq 1$, $\alpha \geq 1$, let $t_b(\alpha, s)$ denote the least value t such that there exists a $(t, \alpha, \beta, \sigma, s)$-sequence in base b with $\alpha = \beta\sigma$.

Remark 5.2 In [8, Definition 6] the analogous quantity for the digital case has been introduced: let b be a prime-power, then $d_b(\alpha, s)$ denotes the smallest value of t such that there exists a digital $(t, \alpha, \beta, \sigma, s)$-sequence over the finite field $\mathbb{F}_b$ with $\alpha = \beta\sigma$.

In this case it is known (see [8, Theorem 7]) that for all $s \geq 1$ and $\alpha \geq 2$ we have

$$s\frac{\alpha(\alpha-1)}{2} - \alpha < d_q(\alpha, s) \leq s\alpha^2 \frac{c}{\log q} + \alpha + \alpha \left\lfloor \frac{s(\alpha-1)}{2} \right\rfloor,$$

where $c > 0$ is an absolute constant. Note that these bounds also apply to (non-digital) $(t, \alpha, \beta, \sigma, s)$-sequences where $\alpha = \beta\sigma$.

The following corollary follows from Theorem 5.2 and Theorem 5.3. Setting $\alpha = \beta = \sigma = 1$ and making use of Theorem 5.2 and Theorem 5.4, we could even recover [16, Corollary 4].

Corollary 5.2 *For all integers $b \geq 2$, $s \geq 1$, $\alpha \geq 1$, $\alpha = \beta\sigma$, we have*

$$\frac{t_b(\alpha, s) - (s\alpha - 1 + \beta\sigma)(L-1)}{L} \leq t_{b^L}(\alpha, s) \leq \left\lceil \frac{t_b(\alpha L, s) - \frac{(L-1)L}{2}}{L^2} \right\rceil.$$

The next theorem provides an explicit bound for $t_b(\alpha, s)$ for prime-powers b. Setting $\alpha = \beta = \sigma = 1$, this result recovers [16, Proposition 6].

Theorem 5.5 *For every prime-power b, we have*

$$t_b(\alpha, s) \leq \frac{2bs\alpha^2}{b-1} - 2\frac{b\alpha^{3/2}s^{1/2}}{(b^2-1)^{1/2}} + 2\alpha \left\lfloor \frac{s(\alpha-1)}{2} \right\rfloor + s\alpha - 1 + \alpha.$$

Proof. We use Theorem 5.2 with $L = 2$ to obtain

$$t_b(\alpha, s) \leq 2t_{b^2}(\alpha, s) + (s\alpha - 1 + \alpha).$$

By Corollary 5.1, where we set $d = \alpha$,

$$t_{b^2}(\alpha, s) \leq \alpha t_{b^2}(1, s\alpha) + \alpha \left\lfloor \frac{s(\alpha - 1)}{2} \right\rfloor,$$

where $t_{b^2}(1, s\alpha)$ corresponds to the least value t such that there exists a $(t, s\alpha)$-sequence in base b^2. From [16, Theorem 5] we obtain

$$t_{b^2}(1, s\alpha) \leq \frac{bs\alpha}{b-1} - \frac{b(s\alpha)^{1/2}}{(b^2-1)^{1/2}}$$

and the result follows. □

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