Construction algorithms for good extensible lattice rules

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Dedicated to Ian H. Sloan on the occasion of his 70th birthday

Abstract

Extensible (polynomial) lattice rules have the property that the number N of points in the node set may be increased while retaining the existing points. It was shown by Hickernell and Niederreiter in a nonconstructive manner that there exist generating vectors for extensible integration lattices of excellent quality for $N = b, b^2, b^3, \ldots$, where b is a given integer greater than 1. Similar results were proved by Niederreiter for polynomial lattices. In this paper we provide construction algorithms for good extensible lattice rules. We treat the classical as well as the polynomial case.

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1 Introduction

We are interested in the approximation of a multidimensional integral over the unit cube by a so-called *quasi-Monte Carlo (QMC) algorithm*, which takes the average of function evaluations over well-chosen sample points, i.e.,

$$\int_{[0,1]^s} F(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \frac{1}{N}\sum_{k=0}^{N-1} F(\boldsymbol{x}_k) \quad \text{with} \quad \boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1} \in [0,1)^s.$$

On first sight, this approach looks quite simple, but the crux of this method is the choice of the sample points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}$ to obtain good results for large function classes. In this paper we focus on two methods of constructing point sets which work very well in many practical applications.

The first construction principle is that of integration lattices, introduced independently by Hlawka [16] and Korobov [17] and studied extensively in recent years (see for example [11, 26, 29, 32]). An integer vector $\boldsymbol{a}$, the *generating vector* of the lattice rule, is used

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to generate N points by $\{k\boldsymbol{a}/N + \boldsymbol{\Delta}\}$ for $k = 0, \ldots, N-1$. The braces indicate that we take the fractional part of each component. The shift $\boldsymbol{\Delta} \in [0,1)^s$ is either chosen $\boldsymbol{0}$ (for periodic functions) or i.i.d. (for nonperiodic functions). QMC algorithms which use integration lattices as underlying point sets are called *lattice rules*, or *randomly shifted lattice rules* if $\boldsymbol{\Delta} \in [0,1)^s$ is chosen i.i.d.. In order to distinguish them from polynomial lattice rules introduced below, we will often refer to lattice rules as *classical* or *usual* lattice rules.

The second construction method considered here is that of polynomial lattices, introduced by Niederreiter [25] (see also [26, Section 4.4]). These are similar to integration lattices, but here we use polynomial arithmetic instead of integer arithmetic. Polynomial lattices are special cases of so-called digital (t, m, s)-nets as introduced by Niederreiter in [24] (see also [26, Chapter 4] for an introduction to this topic). We give a detailed definition of polynomial lattices in Section 3. QMC algorithms which use polynomial lattices as underlying point sets are called *polynomial lattice rules*.

The quality of a (polynomial) lattice rule depends on the choice of its generating vector. Until now no explicit constructions of good generating vectors are known (except for dimension $s = 2$), and hence one has to resort to computer search. Several construction algorithms for generating vectors have been introduced and analyzed, where most of them rely on a component-by-component approach. See for example [3, 21, 22, 30] in the classical case and [5, 6, 19] in the polynomial case.

One major disadvantage of these construction methods is their dependence on the cardinality N of the resulting point set. If one constructs a generating vector of a (polynomial) lattice rule of cardinality N with good quality, it does not mean that the same vector can be used to generate a (polynomial) lattice rule of good quality which uses $N' \neq N$ points. Although such a property would be very desirable for practical applications [12], an extension in the number of points has not been shown to be possible with the algorithms known so far.

That an extension of usual lattice rules in the number of points is possible at least in principle was shown in [13]. Such lattice rules are nowadays called *extensible (polynomial) lattice rules*. A cornerstone in this area is the paper [15] by Hickernell and Niederreiter where for the first time the existence of good extensible (usual) lattice rules was proved. In detail, they showed for any integer $b \geq 2$ the existence of a generating vector $\boldsymbol{a}$ which generates a lattice rule which is good for all cardinalities $N = b, b^2, b^3, \ldots$. Furthermore, Niederreiter [27] applied similar techniques to show the existence of good extensible polynomial lattice rules. (We remark here that the (polynomial) lattice rules whose existence was shown in [15] resp. [27] are also extensible in the dimension s.) However, the proofs in [15] and [27] are nonconstructive.

Although the existence of generating vectors yielding good extensible (polynomial) lattice rules is known, it remained an open question how they can be explicitly found in general. Several numerical investigations have been carried out in [2, 10, 14], but a proof that those algorithms yield good results has not been given. A first construction algorithm was provided in [8], but the generating vectors constructed there yield good results only for a finite range of cardinalities $b^u, b^{u+1}, \ldots, b^{u+v}$, where $b \geq 2$ is an integer and $u, v \in \mathbb{N}$ have to be chosen in advance. One could say that these lattice rules are only *finitely* extensible, which does not meet our demand for extensible lattice rules in its full generality. (It should be mentioned that in principle also the choice $v = \infty$ is possible. But in this case the construction cost becomes so large that the algorithm is useless for practical applications. Only for finite v can the fast component-by-component approach be incorporated which makes the algorithm applicable.) Furthermore we note that the algorithm in [8] requires some storage. A similar algorithm for (finitely) extensible polynomial lattice rules was given in [4].

In this paper we present construction algorithms for extensible lattice rules where we treat the classical as well as the polynomial case. These algorithms in principle work as follows: each component of a generating vector $\boldsymbol{a}$ is considered in its p-adic expansion where p is a prime. If we have already constructed the first n digits of each component such that the generating vector yields good results for $N = p, p^2, \ldots, p^n$, we search in each component for the $(n+1)$st digit such that the generator vector yields good results also for $N = p^{n+1}$. In this way we find digit by digit a good generating vector for all $N = p, p^2, p^3, \ldots$. In contrast to other algorithms, we do not have to stop at some a priori fixed p^v. Furthermore we mention already here that in our algorithm the search space is the same in each step and so it does *not* grow with n. The disadvantage of our results is that the bounds we obtain on the considered quality parameters are a bit weaker than those anticipated for example in [15] and [27] (or obtained in [8] and [4]). As far as we know, the only earlier paper where a similar idea was used — but only for the classical case and for a different quality parameter — is that of Korobov [18].

The rest of the paper is organized as follows. In Section 2 we deal with usual integration lattices. We recall the definition of lattice rules and some measures for the quality thereof. Then we introduce two algorithms for the construction of generating vectors of good quality with respect to these measures for all $N = p, p^2, p^3, \ldots$. In Section 3 we treat the polynomial case with similar results. Finally, in Section 4 we present some numerical results for the classical case.

2 Classical integration lattices

In this section we consider classical integration lattices. As already mentioned in the previous section, these are defined as follows.

Let $s \in \mathbb{N}$ (the dimension) and let $N \in \mathbb{N}$. An integer vector $\boldsymbol{a} \in \mathbb{Z}^s$, the *generating vector* of the lattice rule, is used to generate N points by

$$\left\{ \frac{k}{N}\boldsymbol{a} + \boldsymbol{\Delta} \right\} \quad \text{for} \quad k = 0, \ldots, N-1,$$

where we use the notation introduced in Section 1.

Our aim is to construct a generating vector $\boldsymbol{a}$ digit by digit such that the corresponding integration lattice is of good quality for $N = p, p^2, p^3, \ldots$. We consider two quality measures for integration lattices. The first one is the worst-case error for QMC integration in a weighted Korobov space (resp. the root mean-square worst-case error with respect to a shift $\boldsymbol{\Delta}$ in a weighted Sobolev space) and the second one is the discrepancy.

The worst-case error in a weighted Korobov space. Let $s \in \mathbb{N}$, $\alpha > 1$, and $\boldsymbol{\gamma} = (\gamma_j)_{j \geq 1}$. The positive reals γ_j are called *weights*, which are introduced to modify the importance of different coordinate directions [31]. Furthermore, α is a smoothness parameter. As for example in [32] we consider the weighted Korobov space $H(K_{s,\alpha,\boldsymbol{\gamma}})$

with the reproducing kernel

$$K_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y}) := \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y})},$$

where $\cdot$ denotes the standard inner product in $\mathbb{R}^s$ and where for $\boldsymbol{h}=(h_1,\ldots,h_s)\in\mathbb{Z}^s$ we put $r_\alpha(\boldsymbol{h},\boldsymbol{\gamma}) := \prod_{j=1}^s r_\alpha(h_j,\gamma_j)$ with

$$r_\alpha(h,\gamma) := \begin{cases} 1 & \text{if } h=0,\\ \gamma^{-1}|h|^\alpha & \text{if } h\in\mathbb{Z}\setminus\{0\}. \end{cases}$$

In [32] it was shown that the squared worst-case error for integration in $H(K_{s,\alpha,\boldsymbol{\gamma}})$, i.e., the worst performance of a QMC algorithm over the unit ball of $H(K_{s,\alpha,\boldsymbol{\gamma}})$ using a lattice point $\boldsymbol{a}=(a_1,\ldots,a_s)\in\mathbb{Z}^s$, is given by

$$\begin{aligned} e^2_{N,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}) &= -1+\frac{1}{N}\sum_{k=0}^{N-1}\sum_{\boldsymbol{h}\in\mathbb{Z}^s}\frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}\,\mathrm{e}^{2\pi\mathrm{i}k\boldsymbol{a}\cdot\boldsymbol{h}/N}\\ &= -1+\frac{1}{N}\sum_{k=0}^{N-1}\prod_{j=1}^{s}\left(1+\gamma_j\sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}}\frac{\mathrm{e}^{2\pi\mathrm{i}kha_j/N}}{|h|^\alpha}\right). \end{aligned}$$

Remark 1 It is easy to show (or see [26, Chapter 5]) that we have

$$e^2_{N,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}) = \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{a}\equiv 0 \pmod{N}}} \frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}.$$

Thus, in the unweighted case, i.e., if $\gamma_j=1$ for all $j\in\mathbb{N}$, $e^2_{N,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a})$ is the same as the quantity $P_\alpha(\boldsymbol{a},N)$ defined in [26, Definition 5.2].

Remark 2 If $\alpha\geq 2$ is an even integer, then the Bernoulli polynomial B_α of degree α has the Fourier expansion

$$B_\alpha(x) = \frac{(-1)^{(\alpha+2)/2}\alpha!}{(2\pi)^\alpha}\sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}}\frac{\mathrm{e}^{2\pi\mathrm{i}hx}}{|h|^\alpha} \quad \text{for all } x\in[0,1);$$

see for example [29, Appendix C]. Hence in this case we obtain

$$e^2_{N,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}) = -1+\frac{1}{N}\sum_{k=0}^{N-1}\prod_{j=1}^{s}\left(1+\gamma_j\frac{(-1)^{(\alpha+2)/2}(2\pi)^\alpha}{\alpha!}B_\alpha\left(\left\{\frac{ka_j}{N}\right\}\right)\right),$$

so that $e^2_{N,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a})$ can be calculated in $O(Ns)$ operations.

Remark 3 Consider a tensor product Sobolev space $H_{s,\boldsymbol{\gamma}}$ of absolutely continuous functions whose mixed partial derivatives of order 1 in each variable are square integrable. The norm in the unanchored weighted Sobolev space $H_{s,\boldsymbol{\gamma}}$ (see [9]) is given by

$$\|f\|_{H_{s,\gamma}} = \left(\sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}}\prod_{j\in\mathfrak{u}}\gamma_j^{-1}\int_{[0,1]^{|\mathfrak{u}|}}\left(\int_{[0,1]^{s-|\mathfrak{u}|}}\frac{\partial^{|\mathfrak{u}|}}{\partial\boldsymbol{x}_{\mathfrak{u}}}f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_{\{1,\ldots,s\}\setminus\mathfrak{u}}\right)^2\mathrm{d}\boldsymbol{x}_{\mathfrak{u}}\right)^{1/2},$$

where $\partial^{|\mathfrak{u}|} f / \partial \boldsymbol{x}_{\mathfrak{u}}$ denotes the mixed partial derivative with respect to all variables $j \in \mathfrak{u}$. As pointed out in [8], the root mean-square worst-case error $\widehat{e}_{N,s,\boldsymbol{\gamma}}$ for QMC integration in $H_{s,\boldsymbol{\gamma}}$ using randomly shifted lattice rules, i.e.,

$$\widehat{e}_{N,s,\boldsymbol{\gamma}}(\boldsymbol{a}) = \left(\int_{[0,1)^s} e^2_{N,s,\boldsymbol{\gamma}}(\boldsymbol{a}, \boldsymbol{\Delta}) \,\mathrm{d}\boldsymbol{\Delta} \right)^{1/2},$$

where $e_{N,s,\boldsymbol{\gamma}}(\boldsymbol{a}, \boldsymbol{\Delta})$ is the worst-case error for QMC integration in $H_{s,\boldsymbol{\gamma}}$ using a randomly shifted integration lattice, is more or less the same as the worst-case error $e_{N,s,2,\boldsymbol{\gamma}}$ in the weighted Korobov space $H(K_{s,2,\boldsymbol{\gamma}})$ using the unshifted version of the lattice rules. In fact, we have

$$\widehat{e}_{N,s,2\pi^2\boldsymbol{\gamma}}(\boldsymbol{a}) = e_{N,s,2,\boldsymbol{\gamma}}(\boldsymbol{a}), \tag{1}$$

where $2\pi^2\boldsymbol{\gamma}$ denotes the sequence of weights $(2\pi^2\gamma_j)_{j\ge1}$. For more information, see for example [8, Section 2]. Thus, the results that will be shown in the following are valid for the root mean-square worst-case error for numerical integration in the Sobolev space as well as for the worst-case error for numerical integration in the Korobov space. Hence it suffices to state them only for $e_{N,s,\alpha,\boldsymbol{\gamma}}$. Equation (1) can be used to obtain results also for $\widehat{e}_{N,s,\boldsymbol{\gamma}}$.

The discrepancy. For a point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ consisting of N points in $[0,1)^s$, the *discrepancy* is defined by

$$D_N(\mathcal{P}) = \sup_J \left| \frac{1}{N} \sum_{k=0}^{N-1} c_J(\boldsymbol{x}_k) - \lambda_s(J) \right|,$$

where the supremum is extended over all subintervals J of $[0,1)^s$, c_J denotes the characteristic function of J, and $\lambda_s(J)$ is the volume of J. If we take the supremum just over all subintervals J of $[0,1)^s$ with one vertex anchored at the origin, then we speak of the *star discrepancy* and write $D_N^*(\mathcal{P})$. Obviously, we always have $D_N^*(\mathcal{P}) \le D_N(\mathcal{P})$. The star discrepancy of a finite point set is intimately related to the worst-case error of multivariate QMC integration of functions with bounded variation in the sense of Hardy and Krause. Hereby the basic error estimate is provided by the so-called Koksma-Hlawka inequality. For more information, we refer to [20, Chapter 2, Section 5] and [26, Chapter 2].

For a generating vector $\boldsymbol{a} \in \mathbb{Z}^s$, $s \ge 2$, and an integer $N \ge 2$, let $\mathcal{P}$ be the point set consisting of the points

$$\boldsymbol{x}_k = \left\{ \frac{k}{N} \boldsymbol{a} \right\} \quad \text{for} \quad k = 0, 1, \ldots, N-1.$$

Then for the discrepancy $D_N(\mathcal{P})$ of $\mathcal{P}$ we have (see [26, Theorem 5.6])

$$D_N(\mathcal{P}) \le \frac{s}{N} + \frac{1}{2} R(\boldsymbol{a}, N), \tag{2}$$

where

$$R(\boldsymbol{a}, N) := \sum_{\substack{\boldsymbol{h} \in C_s^*(N) \\ \boldsymbol{h}\cdot\boldsymbol{a} \equiv 0 \ (\mathrm{mod}\ N)}} \frac{1}{r(\boldsymbol{h})}$$

with

$$r(\boldsymbol{h}) := \prod_{j=1}^{s} \max\{1, |h_j|\} \qquad \text{for } \boldsymbol{h} = (h_1, \ldots, h_s) \in \mathbb{Z}^s$$

and

$$C_s(N) := \mathbb{Z}^s \cap (-N/2, N/2]^s, \qquad C_s^*(N) := C_s(N) \setminus \{\mathbf{0}\}.$$

To get useful bounds on the discrepancy $D_N(\mathcal{P})$ of $\mathcal{P}$, it suffices to have good bounds on the quantity $R(\boldsymbol{a}, N)$.

2.1 Good extensible integration lattices with respect to the worst-case error

We present an algorithm which constructs digit by digit a generating vector which is good with respect to the worst-case error $e_{N,s,\alpha,\boldsymbol{\gamma}}$ for all $N = p, p^2, p^3, \ldots$.

Algorithm 1 *Let p be a prime number and let $\mathcal{Z}_p := \{0, 1, \ldots, p-1\}$.*

1. *Find $\boldsymbol{a}_1 := \boldsymbol{a}$ by minimizing $e^2_{p,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a})$ over all $\boldsymbol{a} \in \mathcal{Z}_p^s$. For $p = 2$ one can choose $\boldsymbol{a}_1 = (1, \ldots, 1) \in \mathbb{Z}^s$.*
2. *For $n = 2, 3, \ldots$ find $\boldsymbol{a}_n := \boldsymbol{a}_{n-1} + p^{n-1}\boldsymbol{z}$ by minimizing $e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_{n-1} + p^{n-1}\boldsymbol{a})$ over all $\boldsymbol{a} \in \mathcal{Z}_p^s$, i.e., $\boldsymbol{z} = \mathrm{argmin}_{\boldsymbol{a} \in \mathcal{Z}_p^s} e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_{n-1} + p^{n-1}\boldsymbol{a})$.*

Theorem 1 *Let $n, s \in \mathbb{N}$, p be a prime, and $\alpha > 1$. Assume that $\boldsymbol{a}_n \in \mathbb{Z}^s$ is constructed according to Algorithm 1. Then we have*

$$e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_n) \leq \left(\prod_{j=1}^{s}(1 + 2\gamma_j\zeta(\alpha)) - 1\right) \min\left\{n, \frac{p^{\alpha-1}}{p^{\alpha-1} - 1}\right\} \frac{2}{p^n},$$

where in the case $p = 2$ and $0 < \gamma_j \leq 1$ for all $j \in \mathbb{N}$ we can replace the fraction $\frac{2}{p^n}$ by $\frac{1}{2^n}$. Here for $\alpha > 1$, $\zeta(\alpha)$ denotes the Riemann zeta-function defined by $\zeta(\alpha) = \sum_{i \geq 1} i^{-\alpha}$.

Before we prove this result, we state two remarks.

Remark 4 The search for $\boldsymbol{a}_1$ in the first step of Algorithm 1 takes (at least for even α) $O(sp^{s+1})$ operations. With a component-by-component construction this can be reduced to $O(s^2p^2)$ operations. In this case one gets a slightly weaker error bound in Theorem 1 (the term "-1" after the first product must be deleted). Alternatively, one can choose $\boldsymbol{a}_1 = (1, \ldots, 1)$ in all cases. But then the upper bound in Theorem 1 has to be replaced by

$$e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_n) \leq \left(\prod_{j=1}^{s}(1 + 2\gamma_j\zeta(\alpha)) - 1\right) \min\left\{n, \frac{p^{\alpha-1}}{p^{\alpha-1} - 1}\right\} \frac{1}{p^{n-1}}$$

(except in the case $p = 2$ and $0 < \gamma_j \leq 1$ for all $j \in \mathbb{N}$). This follows immediately from the subsequent proof of Theorem 1.

Remark 5 We remark that it is not necessary to start Algorithm 1 with item 1. If one has given an arbitrary generating vector $\boldsymbol{a}_{n_0}$ for some $n_0 \in \mathbb{N}$ with squared worst-case error $e^2_{p^{n_0},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_{n_0})$, then this vector can be extended as well for $n = n_0+1, n_0+2, \ldots$. From the proof below it is easy to see that in this case we obtain

$$e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_n) \le e^2_{p^{n_0},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_{n_0}) \min\left\{n-n_0+1, \frac{p^{\alpha-1}}{p^{\alpha-1}-1}\right\} \frac{1}{p^{n-n_0}}.$$

Now we give the proof of Theorem 1.

Proof. First we show the result for $n = 1$. We have

$$\begin{aligned}
e^2_{p,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_1) &\le \frac{1}{p^s} \sum_{\boldsymbol{a} \in \mathcal{Z}_p^s} e^2_{p,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}) = \frac{1}{p^s} \sum_{\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}} \frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \sum_{\substack{\boldsymbol{a} \in \mathcal{Z}_p^s \\ \boldsymbol{h}\cdot\boldsymbol{a} \equiv 0 \pmod{p}}} 1 \\
&= \sum_{\substack{\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\} \\ \boldsymbol{h} \equiv \boldsymbol{0} \pmod{p}}} \frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} + \frac{1}{p} \sum_{\substack{\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\} \\ \boldsymbol{h} \not\equiv \boldsymbol{0} \pmod{p}}} \frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \\
&= \left(1 - \frac{1}{p}\right) \sum_{\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}} \frac{1}{r_\alpha(p\boldsymbol{h},\boldsymbol{\gamma})} + \frac{1}{p} \sum_{\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}} \frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \\
&\le \frac{2}{p} \sum_{\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\}} \frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} = \frac{2}{p}\left(\prod_{j=1}^s (1 + 2\gamma_j \zeta(\alpha)) - 1\right).
\end{aligned}$$

For $p = 2$ and $0 < \gamma_j \le 1$ for all $j \in \mathbb{N}$ and the choice $\boldsymbol{a}_1 = (1,\ldots,1)$, we can estimate $e^2_{2,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_1)$ directly. In this case we obtain

$$e^2_{2,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_1) = -1 + \sum_{h_1,\ldots,h_{s-1}=-\infty}^{\infty} \prod_{j=1}^{s-1} \frac{1}{r_\alpha(h_j,\gamma_j)} \sum_{\substack{h=-\infty \\ h \equiv h_1+\cdots+h_{s-1} \pmod{2}}}^{\infty} \frac{1}{r_\alpha(h,\gamma_s)}.$$

Denote the innermost sum in the above expression by Σ_1. If $h_1 + \cdots + h_{s-1} \equiv 0 \pmod{2}$, then we have

$$\Sigma_1 = \sum_{h=-\infty}^{\infty} \frac{1}{r_\alpha(2h,\gamma_s)} = 1 + \frac{2}{2^\alpha}\gamma_s \zeta(\alpha).$$

If $h_1 + \cdots + h_{s-1} \equiv 1 \pmod{2}$, then we have

$$\begin{aligned}
\Sigma_1 &= \sum_{h=-\infty}^{\infty} \frac{1}{r_\alpha(2h+1,\gamma_s)} = 2\sum_{h=0}^{\infty} \frac{1}{r_\alpha(2h+1,\gamma_s)} \\
&= 2\gamma_s \left(\sum_{h=1}^{\infty} \frac{1}{h^\alpha} - \sum_{h=1}^{\infty} \frac{1}{(2h)^\alpha}\right) = 2\gamma_s\zeta(\alpha) - \frac{2}{2^\alpha}\gamma_s\zeta(\alpha).
\end{aligned}$$

Altogether we obtain

$$\Sigma_1 = \frac{1}{2} + \gamma_s\zeta(\alpha) + (-1)^{h_1+\cdots+h_{s-1}}\left(\frac{1}{2} - \gamma_s\zeta(\alpha) + \frac{2}{2^\alpha}\gamma_s\zeta(\alpha)\right).$$

Therefore we get

$$
\begin{aligned}
e^2_{2,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_1) &= -1+\left(\frac{1}{2}+\gamma_s\zeta(\alpha)\right)\prod_{j=1}^{s-1}\left(\sum_{h=-\infty}^{\infty}\frac{1}{r_\alpha(h,\gamma_j)}\right)\\
&\quad+\left(\frac{1}{2}-\gamma_s\zeta(\alpha)+\frac{2}{2^\alpha}\gamma_s\zeta(\alpha)\right)\prod_{j=1}^{s-1}\left(\sum_{h=-\infty}^{\infty}\frac{(-1)^h}{r_\alpha(h,\gamma_j)}\right)\\
&= -1+\frac{1}{2}\prod_{j=1}^{s}(1+2\gamma_j\zeta(\alpha))\\
&\quad+\frac{1}{2}\left(1+\frac{4}{2^\alpha}\gamma_s\zeta(\alpha)-2\gamma_s\zeta(\alpha)\right)\prod_{j=1}^{s-1}\left(\sum_{h=-\infty}^{\infty}\frac{(-1)^h}{r_\alpha(h,\gamma_j)}\right).
\end{aligned}
$$

Since

$$
\sum_{h=-\infty}^{\infty}\frac{(-1)^h}{r_\alpha(h,\gamma_j)}=1+\frac{4}{2^\alpha}\gamma_j\zeta(\alpha)-2\gamma_j\zeta(\alpha),
$$

we obtain

$$
e^2_{2,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_1)=-1+\frac{1}{2}\prod_{j=1}^{s}(1+2\gamma_j\zeta(\alpha))+\frac{1}{2}\prod_{j=1}^{s}\left(1-\left(2-\frac{4}{2^\alpha}\right)\gamma_j\zeta(\alpha)\right). \tag{3}
$$

Now we claim that for any $\alpha>1$ we have

$$
1<\left(2-\frac{4}{2^\alpha}\right)\zeta(\alpha)<2. \tag{4}
$$

This inequality is equivalent to

$$
\frac{1}{2}\cdot\frac{1}{1-\frac{2}{2^\alpha}}<\zeta(\alpha)<\frac{1}{1-\frac{2}{2^\alpha}},
$$

which is in turn equivalent to

$$
\frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{2}{2^\alpha}\right)^j<\sum_{i=1}^{\infty}\frac{1}{i^\alpha}<\sum_{j=0}^{\infty}\left(\frac{2}{2^\alpha}\right)^j.
$$

This is now shown by comparing the three series above using a suitable grouping of terms. For instance, to show the upper bound on $\zeta(\alpha)$, we compare the first term of the series for $\zeta(\alpha)$ with the first term of the last series, the 2nd + 3rd term of the series for $\zeta(\alpha)$ with the 2nd term of the last series, the 4th+5th+6th+7th term of the series for $\zeta(\alpha)$ with the 3rd term of the last series, and so on. In this way we find that the inequality (4) is indeed correct.

Now, as $0<\gamma_j\le 1$ for all $j\in\mathbb{N}$, (3) and (4) yield

$$
e^2_{2,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_1)=\frac{1}{2}\prod_{j=1}^{s}(1+2\gamma_j\zeta(\alpha))^s-1+\frac{1}{2}\prod_{j=1}^{s}\theta_j(\alpha)\quad\text{with}\quad -1<\theta_j(\alpha)<1,
$$

from which we obtain

$$e^2_{2,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_1) \le \left(\prod_{j=1}^{s}(1+2\gamma_j\zeta(\alpha))-1\right)\frac{1}{2}.$$

Hence the result follows for $n=1$ and any prime p.

Now let $n \ge 2$. With the representation of $e^2_{N,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a})$ from Remark 1 we have

$$\begin{aligned} e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_n) &\le \frac{1}{p^s}\sum_{\boldsymbol{a}\in\mathcal{Z}_p^s} e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_{n-1}+p^{n-1}\boldsymbol{a}) \\ &= \frac{1}{p^s}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s\\ \boldsymbol{h}\neq\boldsymbol{0}}}\frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}\sum_{\substack{\boldsymbol{a}\in\mathcal{Z}_p^s\\ \boldsymbol{h}\cdot(\boldsymbol{a}_{n-1}+p^{n-1}\boldsymbol{a})\equiv 0 \pmod{p^n}}} 1. \end{aligned}$$

The inner sum is equal to the number of $\boldsymbol{a}\in\mathcal{Z}_p^s$ with $p^{n-1}\boldsymbol{h}\cdot\boldsymbol{a}\equiv -\boldsymbol{h}\cdot\boldsymbol{a}_{n-1} \pmod{p^n}$. For this we must have $\boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^{n-1}}$, and then $\boldsymbol{h}\cdot\boldsymbol{a}\equiv -p^{1-n}\boldsymbol{h}\cdot\boldsymbol{a}_{n-1} \pmod{p}$. Thus,

$$e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_n) \le \frac{1}{p^s}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\,\boldsymbol{h}\neq\boldsymbol{0}\\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^{n-1}}}}\frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}\sum_{\substack{\boldsymbol{a}\in\mathcal{Z}_p^s\\ \boldsymbol{h}\cdot\boldsymbol{a}\equiv -p^{1-n}\boldsymbol{h}\cdot\boldsymbol{a}_{n-1} \pmod{p}}} 1.$$

Consider the inner sum. If $\boldsymbol{h}\not\equiv\boldsymbol{0} \pmod{p}$, then the inner sum is equal to p^{s-1}. If $\boldsymbol{h}\equiv\boldsymbol{0} \pmod{p}$, then the inner sum is equal to 0 if $\boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\not\equiv 0 \pmod{p^n}$ and equal to p^s if $\boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^n}$. Thus,

$$\begin{aligned} &e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_n) \\ &\le \frac{1}{p}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\,\boldsymbol{h}\not\equiv\boldsymbol{0} \pmod{p}\\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^{n-1}}}}\frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}+\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\,\boldsymbol{h}\neq\boldsymbol{0},\,\boldsymbol{h}\equiv\boldsymbol{0} \pmod{p}\\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^n}}}\frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \\ &= \frac{1}{p}e^2_{p^{n-1},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_{n-1})-\frac{1}{p}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\,\boldsymbol{h}\neq\boldsymbol{0},\,\boldsymbol{h}\equiv\boldsymbol{0} \pmod{p}\\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^{n-1}}}}\frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}+\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\,\boldsymbol{h}\neq\boldsymbol{0},\,\boldsymbol{h}\equiv\boldsymbol{0} \pmod{p}\\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^n}}}\frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}, \end{aligned}$$

and this holds for all $n \ge 2$.

If we insert this inequality for $e^2_{p^{n-1},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_{n-1})$, then we obtain

$$\begin{aligned} &e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_n) \\ &\le \frac{1}{p^2}e^2_{p^{n-2},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_{n-2})-\frac{1}{p^2}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\,\boldsymbol{h}\neq\boldsymbol{0},\,\boldsymbol{h}\equiv\boldsymbol{0} \pmod{p}\\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-2}\equiv 0 \pmod{p^{n-2}}}}\frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}+\frac{1}{p}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\,\boldsymbol{h}\neq\boldsymbol{0},\,\boldsymbol{h}\equiv\boldsymbol{0} \pmod{p}\\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-2}\equiv 0 \pmod{p^{n-1}}}}\frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \\ &\quad -\frac{1}{p}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\,\boldsymbol{h}\neq\boldsymbol{0},\,\boldsymbol{h}\equiv\boldsymbol{0} \pmod{p}\\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^{n-1}}}}\frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}+\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\,\boldsymbol{h}\neq\boldsymbol{0},\,\boldsymbol{h}\equiv\boldsymbol{0} \pmod{p}\\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^n}}}\frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}. \end{aligned}$$

Assume that $\boldsymbol{h}\in\mathbb{Z}^s$, $\boldsymbol{h}\equiv\boldsymbol{0} \pmod{p}$, i.e., $\boldsymbol{h}=p\widetilde{\boldsymbol{h}}$, and $\boldsymbol{h}\cdot\boldsymbol{a}_{n-2}\equiv 0 \pmod{p^{n-1}}$. Then we have

$$\boldsymbol{h}\cdot\boldsymbol{a}_{n-1}=\boldsymbol{h}\cdot(\boldsymbol{a}_{n-2}+p^{n-2}\boldsymbol{a})=\boldsymbol{h}\cdot\boldsymbol{a}_{n-2}+p^{n-1}\widetilde{\boldsymbol{h}}\cdot\boldsymbol{a}\equiv 0 \pmod{p^{n-1}},$$

with some $\boldsymbol{a} \in \mathcal{Z}_p^s$. Therefore we obtain

$$e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_n) \leq \frac{1}{p^2} e^2_{p^{n-2},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_{n-2}) - \frac{1}{p^2} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\, \boldsymbol{h}\neq\boldsymbol{0},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod p \\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-2}\equiv 0 \pmod{p^{n-2}}}} \frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} + \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\, \boldsymbol{h}\neq\boldsymbol{0},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod p \\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^{n}}}} \frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}.$$

Repeating this argument, we get

$$e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_n) \leq \frac{1}{p^{n-1}} e^2_{p,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_{1}) - \frac{1}{p^{n-1}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\, \boldsymbol{h}\neq\boldsymbol{0},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod p \\ \boldsymbol{h}\cdot\boldsymbol{a}_{1}\equiv 0 \pmod{p}}} \frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} + \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\, \boldsymbol{h}\neq\boldsymbol{0},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod p \\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^{n}}}} \frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}.$$

For the last sum in the above expression, we have

$$\begin{aligned}
\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\, \boldsymbol{h}\neq\boldsymbol{0},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod p \\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^{n}}}} \frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} &= \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\, \boldsymbol{h}\neq\boldsymbol{0} \\ p\boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^{n}}}} \frac{1}{r_\alpha(p\boldsymbol{h},\boldsymbol{\gamma})} = \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\, \boldsymbol{h}\neq\boldsymbol{0} \\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^{n-1}}}} \frac{1}{r_\alpha(p\boldsymbol{h},\boldsymbol{\gamma})} \\
&\leq \frac{1}{p^\alpha} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^s,\, \boldsymbol{h}\neq\boldsymbol{0} \\ \boldsymbol{h}\cdot\boldsymbol{a}_{n-1}\equiv 0 \pmod{p^{n-1}}}} \frac{1}{r_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} = \frac{1}{p^\alpha} e^2_{p^{n-1},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_{n-1}).
\end{aligned}$$

With this upper bound, we obtain

$$e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_n) \leq \frac{1}{p^{n-1}} e^2_{p,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_1) + \frac{1}{p^\alpha} e^2_{p^{n-1},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_{n-1}).$$

With backward induction on n and invoking the upper bound for $e^2_{p,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_1)$, we get

$$\begin{aligned}
e^2_{p^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_n) &\leq e^2_{p,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_1)\left(\frac{1}{p^{n-1}} + \frac{1}{p^{n-2+\alpha}}\right) + \frac{1}{p^{2\alpha}} e^2_{p^{n-2},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_{n-2}) \\
&\vdots \\
&\leq e^2_{p,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_1) \sum_{k=0}^{n-2} \frac{1}{p^{n-1-k+k\alpha}} + \frac{1}{p^{(n-1)\alpha}} e^2_{p,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_1) \\
&= e^2_{p,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_1) \sum_{k=0}^{n-1} \frac{1}{p^{n-1-k+k\alpha}} \\
&\leq \left(\prod_{j=1}^s (1+2\gamma_j\zeta(\alpha)) - 1\right) \min\left\{n, \frac{p^{\alpha-1}}{p^{\alpha-1}-1}\right\} \frac{2}{p^n},
\end{aligned}$$

and

$$e^2_{2^n,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{a}_n) \leq \left(\prod_{j=1}^s (1+2\gamma_j\zeta(\alpha)) - 1\right) \min\left\{n, \frac{2^{\alpha-1}}{2^{\alpha-1}-1}\right\} \frac{1}{2^n}$$

if $0 < \gamma_j \leq 1$ for all $j \in \mathbb{N}$. But this is the desired result. $\square$

2.2 Good extensible integration lattices with respect to the discrepancy

We also present an algorithm which uses the quantity R (and hence the discrepancy) as the quality criterion.

Algorithm 2 *Let p be a prime and let $\mathcal{Z}_p := \{0, \ldots, p-1\}$.*

1. *Find $\boldsymbol{a}_1 := \boldsymbol{a}$ by minimizing $R(\boldsymbol{a}, p)$ over all $\boldsymbol{a} \in \mathcal{Z}_p^s$.*
2. *For $n = 2, 3, \ldots$ find $\boldsymbol{a}_n := \boldsymbol{a}_{n-1} + p^{n-1}\boldsymbol{z}$ by minimizing $R(\boldsymbol{a}_{n-1} + p^{n-1}\boldsymbol{a}, p^n)$ over all $\boldsymbol{a} \in \mathcal{Z}_p^s$, i.e., $\boldsymbol{z} = \mathrm{argmin}_{\boldsymbol{a} \in \mathcal{Z}_p^s} R(\boldsymbol{a}_{n-1} + p^{n-1}\boldsymbol{a}, p^n)$.*

Theorem 2 *Let $n \geq 1$ and $s \geq 2$ be integers and let $p \geq 3$ be a prime. Assume that $\boldsymbol{a}_n \in \mathbb{Z}^s$ is constructed according to Algorithm 2. Then we have*

$$R(\boldsymbol{a}_n, p^n) = O(p^{-n}(\log p + 2)^{sn})$$

with an implied constant depending only on s.

Before we give the proof of this result, we state some remarks and corollaries.

Remark 6 Again we remark that it is not necessary to start Algorithm 2 with item 1. If one has given an arbitrary generating vector $\boldsymbol{a}_{n_0}$ for some $n_0 \in \mathbb{N}$, then this vector can be extended as well for $n = n_0 + 1, n_0 + 2, \ldots$. From the proof of Theorem 2 below it is easy to see that in this case we obtain

$$R(\boldsymbol{a}_n, p^n) \leq \frac{(\log p + 2)^{s(n-n_0)}}{p^{n-n_0}} R(\boldsymbol{a}_{n_0}, p^{n_0}) + O\left(p^{-n}(\log p + 2)^{s(n-n_0+1)}\right),$$

where the implied constant depends only on s.

From (2) and Theorem 2 we immediately obtain a bound on the discrepancy. This bound is good for large values of p.

Corollary 1 *Let $n \geq 1$ and $s \geq 2$ be integers and let $p \geq 3$ be a prime. Assume that $\boldsymbol{a}_n \in \mathbb{Z}^s$ is constructed according to Algorithm 2. Then for the discrepancy $D_{p^n}(\mathcal{P}_n)$ of the point set $\mathcal{P}_n$ consisting of the points $\boldsymbol{x}_k = \{(k/p^n)\boldsymbol{a}_n\}$, $k = 0, 1, \ldots, p^n - 1$, we have*

$$D_{p^n}(\mathcal{P}_n) = O(p^{-n}(\log p + 2)^{sn})$$

with an implied constant depending only on s.

From Theorem 2 we can also obtain a bound on the quantity P_α, i.e., on the squared worst-case error for integration in $H(K_{s,\alpha,\boldsymbol{\gamma}})$ in the unweighted case (cf. Remark 1).

Corollary 2 *Let $n \geq 1$ and $s \geq 2$ be integers and let $p \geq 3$ be a prime. Assume that $\boldsymbol{a}_n \in \mathbb{Z}^s$ is constructed according to Algorithm 2 and that no component of $\boldsymbol{a}_1$ is zero. Then for $\alpha > 1$ we have*

$$P_\alpha(\boldsymbol{a}_n, p^n) = O(p^{-\alpha n}(\log p + 2)^{s\alpha n})$$

with an implied constant depending only on s and α.

Remark 7 The assumption that no component of $\boldsymbol{a}_1$ is zero is not really a restriction. In Algorithm 2 the first step could be replaced by: *Choose an arbitrary vector $\boldsymbol{a}_1$ in $\mathcal{Z}_p^s$.* It can be seen from the proof of Theorem 2 below that this would not disturb the asymptotic order of $R(\boldsymbol{a}_n, p^n)$. Hence $\boldsymbol{a}_1$ can be chosen such that no component is equal to zero.

We give the proof of Corollary 2.

Proof. As no component of $\boldsymbol{a}_1$ is zero, it follows that for all $\boldsymbol{a}_n := (a_n^{(1)}, \ldots, a_n^{(s)}) \in \mathbb{Z}^s$ which are constructed according to Algorithm 2 we have $\gcd(a_n^{(j)}, p^n) = 1$ for all $1 \leq j \leq s$. Hence from [26, Theorem 5.5] we obtain the bound

$$\begin{aligned} P_\alpha(\boldsymbol{a}_n, p^n) &< R(\boldsymbol{a}_n, p^n)^\alpha + (1 + 2\zeta(\alpha) p^{-\alpha n})^s - 1 \\ &\quad + \frac{1}{p^n}(1 + 2\zeta(\alpha) + 2^\alpha \zeta(\alpha) p^{(1-\alpha)n})^s - \frac{1}{p^n}(1 + 2\zeta(\alpha))^s, \end{aligned}$$

where $\zeta(\alpha)$ denotes the Riemann zeta-function. The desired result follows by using Theorem 2 to estimate $R(\boldsymbol{a}_n, p^n)$ together with

$$(1 + 2\zeta(\alpha) p^{-\alpha n})^s - 1 = O(p^{-\alpha n})$$

and

$$\frac{1}{p^n}(1 + 2\zeta(\alpha) + 2^\alpha \zeta(\alpha) p^{(1-\alpha)n})^s - \frac{1}{p^n}(1 + 2\zeta(\alpha))^s = O(p^{-\alpha n}),$$

with implied constants depending only on s and α. □

For the proof of Theorem 2 we need the following elementary result which is proved by standard arguments (compare with [23]).

Lemma 1 *For any odd integer $k \geq 3$ we have*

$$\sum_{b=1}^{(k-1)/2} \frac{1}{b} \leq \log k.$$

Now we give the proof of Theorem 2.

Proof. For $n = 1$ a standard averaging argument (see [23]) and Lemma 1 yield

$$R(\boldsymbol{a}_1, p) \leq p^{-1}(2 \log p + 1)^s, \tag{5}$$

and so the desired result follows. Now let $n \geq 2$. As in the proof of Theorem 1 we have

$$\begin{aligned} R(\boldsymbol{a}_n, p^n) &\leq \frac{1}{p^s} \sum_{\boldsymbol{a} \in \mathcal{Z}_p^s} R(\boldsymbol{a}_{n-1} + p^{n-1}\boldsymbol{a}, p^n) \\ &= \frac{1}{p^s} \sum_{\boldsymbol{h} \in C_s^*(p^n)} \frac{1}{r(\boldsymbol{h})} \sum_{\substack{\boldsymbol{a} \in \mathcal{Z}_p^s \\ \boldsymbol{h} \cdot (\boldsymbol{a}_{n-1} + p^{n-1}\boldsymbol{a}) \equiv 0 \pmod{p^n}}} 1 \\ &= \frac{1}{p^s} \sum_{\substack{\boldsymbol{h} \in C_s^*(p^n) \\ \boldsymbol{h} \cdot \boldsymbol{a}_{n-1} \equiv 0 \pmod{p^{n-1}}}} \frac{1}{r(\boldsymbol{h})} \sum_{\substack{\boldsymbol{a} \in \mathcal{Z}_p^s \\ \boldsymbol{h} \cdot \boldsymbol{a} \equiv -p^{1-n} \boldsymbol{h} \cdot \boldsymbol{a}_{n-1} \pmod{p}}} 1 \\ &= \frac{1}{p} \sum_{\substack{\boldsymbol{h} \in C_s^*(p^n),\, \boldsymbol{h} \not\equiv \boldsymbol{0} \pmod{p} \\ \boldsymbol{h} \cdot \boldsymbol{a}_{n-1} \equiv 0 \pmod{p^{n-1}}}} \frac{1}{r(\boldsymbol{h})} + \sum_{\substack{\boldsymbol{h} \in C_s^*(p^n),\, \boldsymbol{h} \equiv \boldsymbol{0} \pmod{p} \\ \boldsymbol{h} \cdot \boldsymbol{a}_{n-1} \equiv 0 \pmod{p^n}}} \frac{1}{r(\boldsymbol{h})} \\ &\leq \frac{1}{p} \sum_{\substack{\boldsymbol{h} \in C_s^*(p^n) \\ \boldsymbol{h} \cdot \boldsymbol{a}_{n-1} \equiv 0 \pmod{p^{n-1}}}} \frac{1}{r(\boldsymbol{h})} + \sum_{\substack{\boldsymbol{h} \in C_s^*(p^{n-1}) \\ \boldsymbol{h} \cdot \boldsymbol{a}_{n-1} \equiv 0 \pmod{p^{n-1}}}} \frac{1}{r(p\boldsymbol{h})}. \end{aligned}$$

Using $r(p\boldsymbol{h}) \geq pr(\boldsymbol{h})$ for $\boldsymbol{h} \in \mathbb{Z}^s$ with $\boldsymbol{h} \neq \boldsymbol{0}$, we obtain

$$R(\boldsymbol{a}_n, p^n) \leq \frac{2}{p} R(\boldsymbol{a}_{n-1}, p^{n-1}) + \frac{1}{p} \Sigma \tag{6}$$

with

$$\Sigma := \sum_{\substack{\boldsymbol{h} \in C_s^*(p^n) \setminus C_s^*(p^{n-1}) \\ \boldsymbol{h} \cdot \boldsymbol{a}_{n-1} \equiv 0 \pmod{p^{n-1}}}} \frac{1}{r(\boldsymbol{h})}.$$

To bound Σ, we note that any $\boldsymbol{h} \in C_s^*(p^n) \setminus C_s^*(p^{n-1})$ can be represented uniquely in the form

$$\boldsymbol{h} = \boldsymbol{h}_1 + p^{n-1}\boldsymbol{b} \qquad \text{with } \boldsymbol{h}_1 \in C_s(p^{n-1}),\ \boldsymbol{b} \in C_s^*(p).$$

Therefore

$$\begin{aligned} \Sigma &= \sum_{\substack{\boldsymbol{h} \in C_s(p^{n-1}) \\ \boldsymbol{h} \cdot \boldsymbol{a}_{n-1} \equiv 0 \pmod{p^{n-1}}}} \sum_{\boldsymbol{b} \in C_s^*(p)} \frac{1}{r(\boldsymbol{h} + p^{n-1}\boldsymbol{b})} \\ &= \sum_{\boldsymbol{b} \in C_s^*(p)} \frac{1}{r(p^{n-1}\boldsymbol{b})} + \sum_{\substack{\boldsymbol{h} \in C_s^*(p^{n-1}) \\ \boldsymbol{h} \cdot \boldsymbol{a}_{n-1} \equiv 0 \pmod{p^{n-1}}}} \sum_{\boldsymbol{b} \in C_s^*(p)} \frac{1}{r(\boldsymbol{h} + p^{n-1}\boldsymbol{b})} =: \Sigma_1 + \Sigma_2. \end{aligned}$$

Next we observe that (note that $p \geq 3$ is odd)

$$\begin{aligned} \Sigma_1 &= \sum_{\boldsymbol{b} \in C_s^*(p)} \frac{1}{r(p^{n-1}\boldsymbol{b})} = \left(\sum_{b=-(p-1)/2}^{(p-1)/2} \frac{1}{r(p^{n-1}b)} \right)^s - 1 \\ &= \left(1 + \frac{2}{p^{n-1}} \sum_{b=1}^{(p-1)/2} \frac{1}{b} \right)^s - 1 \leq \frac{2s}{p^{n-1}} \left(\sum_{b=1}^{(p-1)/2} \frac{1}{b} \right) \left(1 + \frac{2}{p^{n-1}} \sum_{b=1}^{(p-1)/2} \frac{1}{b} \right)^{s-1}. \end{aligned}$$

In view of Lemma 1, this yields

$$\Sigma_1 \leq \frac{2s \log p}{p^{n-1}} \left(1 + \frac{2 \log p}{p^{n-1}} \right)^{s-1}. \tag{7}$$

Now we consider

$$\Sigma_2 = \sum_{\substack{\boldsymbol{h} \in C_s^*(p^{n-1}) \\ \boldsymbol{h} \cdot \boldsymbol{a}_{n-1} \equiv 0 \pmod{p^{n-1}}}} \sum_{\boldsymbol{b} \in C_s^*(p)} \frac{1}{r(\boldsymbol{h} + p^{n-1}\boldsymbol{b})}. \tag{8}$$

We write the inner sum in (8) as

$$\sum_{\boldsymbol{b} \in C_s^*(p)} \frac{1}{r(\boldsymbol{h} + p^{n-1}\boldsymbol{b})} = \sum_{\boldsymbol{b} \in C_s(p)} \frac{1}{r(\boldsymbol{h} + p^{n-1}\boldsymbol{b})} - \frac{1}{r(\boldsymbol{h})}. \tag{9}$$

Now for fixed $\boldsymbol{h} = (h_1, \ldots, h_s) \in C_s^*(p^{n-1})$ we have

$$\sum_{\boldsymbol{b} \in C_s(p)} \frac{1}{r(\boldsymbol{h} + p^{n-1}\boldsymbol{b})} = \prod_{j=1}^{s} \left(\sum_{b \in C(p)} \frac{1}{r(h_j + p^{n-1}b)} \right). \tag{10}$$

If $h_j = 0$, then by Lemma 1,

$$\sum_{b\in C(p)} \frac{1}{r(h_j + p^{n-1}b)} = \sum_{b\in C_s(p)} \frac{1}{r(p^{n-1}b)} \le 1 + \frac{2\log p}{p^{n-1}} = \frac{1}{r(h_j)} + \frac{2\log p}{p^{n-1}}.$$

If $1 \le h_j < \frac{1}{2}p^{n-1}$, then

$$\begin{aligned}
\sum_{b\in C(p)} \frac{1}{r(h_j + p^{n-1}b)} &= \frac{1}{r(h_j)} + \sum_{b=1}^{(p-1)/2} \left(\frac{1}{p^{n-1}b + h_j} + \frac{1}{p^{n-1}b - h_j} \right) \\
&= \frac{1}{r(h_j)} + \sum_{b=1}^{(p-1)/2} \frac{2p^{n-1}b}{p^{2n-2}b^2 - h_j^2} \\
&\le \frac{1}{r(h_j)} + \sum_{b=1}^{(p-1)/2} \frac{2p^{n-1}b}{p^{2n-2}b^2 - \frac{p^{2n-2}}{4}} = \frac{1}{r(h_j)} + \frac{2}{p^{n-1}} \sum_{b=1}^{(p-1)/2} \frac{b}{b^2 - \frac{1}{4}}.
\end{aligned}$$

For the last sum we have

$$\begin{aligned}
\sum_{b=1}^{(p-1)/2} \frac{b}{b^2 - \frac{1}{4}} &= \frac{1}{2} \sum_{b=1}^{(p-1)/2} \left(\frac{1}{b - \frac{1}{2}} + \frac{1}{b + \frac{1}{2}} \right) = 1 + \sum_{b=1}^{(p-3)/2} \frac{1}{b + \frac{1}{2}} + \frac{1}{p} \\
&\le 1 + \sum_{b=1}^{(p-1)/2} \frac{1}{b + \frac{1}{2}} \le 1 + \frac{2}{3} + \int_1^{(p-1)/2} \frac{\mathrm{d}x}{x + \frac{1}{2}} \\
&= \log \frac{p}{3} + \frac{5}{3},
\end{aligned}$$

and so

$$\sum_{b\in C(p)} \frac{1}{r(h_j + p^{n-1}b)} \le \frac{1}{r(h_j)} + \frac{2}{p^{n-1}} \left(\log \frac{p}{3} + \frac{5}{3} \right).$$

The case $-\frac{1}{2}p^{n-1} < h_j \le -1$ is symmetric to the above case, and so we obtain the same bound as above. Thus, in all cases we have

$$\sum_{b\in C(p)} \frac{1}{r(h_j + p^{n-1}b)} \le \frac{1}{r(h_j)} + \frac{2}{p^{n-1}} \left(\log \frac{p}{3} + \frac{5}{3} \right) \le \frac{1}{r(h_j)} \left(\log \frac{p}{3} + \frac{8}{3} \right).$$

In view of (9) and (10) this yields

$$\sum_{\boldsymbol{b}\in C_s^*(p)} \frac{1}{r(\boldsymbol{h} + p^{n-1}\boldsymbol{b})} \le \frac{1}{r(\boldsymbol{h})} \left(\left(\log \frac{p}{3} + \frac{8}{3} \right)^s - 1 \right),$$

and so

$$\Sigma_2 \le \left(\left(\log \frac{p}{3} + \frac{8}{3} \right)^s - 1 \right) R(\boldsymbol{a}_{n-1}, p^{n-1}).$$

Using $\Sigma = \Sigma_1 + \Sigma_2$ and (7), we obtain

$$\Sigma \le \frac{2s\log p}{p^{n-1}} \left(1 + \frac{2\log p}{p^{n-1}} \right)^{s-1} + \left(\left(\log \frac{p}{3} + \frac{8}{3} \right)^s - 1 \right) R(\boldsymbol{a}_{n-1}, p^{n-1}).$$

Therefore by (6),

$$\begin{aligned} R(\boldsymbol{a}_n, p^n) &\leq \frac{\left(\log \frac{p}{3} + \frac{8}{3}\right)^s + 1}{p} R(\boldsymbol{a}_{n-1}, p^{n-1}) + \frac{2s \log p}{p^n} \left(1 + \frac{2\log p}{p^{n-1}}\right)^{s-1} \\ &\leq \frac{(\log p + 2)^s}{p} R(\boldsymbol{a}_{n-1}, p^{n-1}) + \frac{2s \log p}{p^n} \left(1 + \frac{2\log p}{p^{n-1}}\right)^{s-1}. \end{aligned}$$

By iterating this inequality and using (5), we get for all $n \geq 2$,

$$\begin{aligned} R(\boldsymbol{a}_n, p^n) &\leq \frac{(\log p + 2)^{s(n-1)}}{p^n} (2\log p + 1)^s + \frac{2s \log p}{p^n} \sum_{j=2}^{n} (\log p + 2)^{s(n-j)} \left(1 + \frac{2\log p}{p^{j-1}}\right)^{s-1} \\ &\leq \frac{2^s (\log p + 2)^{sn}}{p^n} + \frac{2s \log p}{p^n} \left(1 + \frac{2\log p}{p}\right)^{s-1} \sum_{j=0}^{n-2} (\log p + 2)^{sj}. \end{aligned}$$

It is clear that the last expression is $O(p^{-n}(\log p + 2)^{sn})$ with an implied constant depending only on s. □

3 Polynomial lattices

The construction of a polynomial lattice is quite similar to the construction of usual integration lattices, but now we use polynomial arithmetic over a finite field. Here we consider only polynomial lattices over the finite field $\mathbb{Z}_p$ of p elements, where p is a prime. For an introduction of polynomial lattices in their full generality, we refer to [25] and [26, Section 4.4].

Let p be a prime and let $\mathbb{Z}_p((x^{-1}))$ be the field of formal Laurent series over $\mathbb{Z}_p$. Elements of $\mathbb{Z}_p((x^{-1}))$ are formal Laurent series of the form

$$L = \sum_{l=w}^{\infty} t_l x^{-l},$$

where w is an arbitrary integer and all $t_l \in \mathbb{Z}_p$. Further let $\mathbb{Z}_p[x]$ be the set of all polynomials over $\mathbb{Z}_p$.

For an integer $m \geq 1$, let υ_m be the map from $\mathbb{Z}_p((x^{-1}))$ to the interval $[0, 1)$ defined by

$$\upsilon_m\left(\sum_{l=w}^{\infty} t_l x^{-l}\right) = \sum_{l=\max(1,w)}^{m} t_l p^{-l},$$

where all $t_l \in \mathbb{Z}_p = \{0, \ldots, p-1\}$.

With the above notation we can now introduce polynomial lattices. For a given dimension $s \geq 1$, we choose $f \in \mathbb{Z}_p[x]$ with $\deg(f) \geq 1$ and $\boldsymbol{g} = (g_1, \ldots, g_s) \in \mathbb{Z}_p[x]^s$. Then $\mathcal{P}(\boldsymbol{g}, f)$ is defined as the point set consisting of the $p^{\deg(f)}$ points

$$\boldsymbol{x}_h = \left(\upsilon_{\deg(f)}\left(\frac{hg_1}{f}\right), \ldots, \upsilon_{\deg(f)}\left(\frac{hg_s}{f}\right)\right) \in [0, 1)^s,$$

where h runs through all polynomials in $\mathbb{Z}_p[x]$ with $\deg(h) < \deg(f)$. The point set $\mathcal{P}(\boldsymbol{g}, f)$ is called a *polynomial lattice* and $\boldsymbol{g}$ is called the *generating vector* of the polynomial lattice. QMC algorithms which use polynomial lattices as underlying point sets are called *polynomial lattice rules*. It is clear that $g_1, \ldots, g_s$ are relevant only modulo f. Furthermore, it was shown by Niederreiter [25] that a polynomial lattice $\mathcal{P}(\boldsymbol{g}, f)$ is a digital (t, m, s)-net over $\mathbb{Z}_p$, where $m = \deg(f)$. For the determination of the quality parameter t and the corresponding generating matrices, we refer to [25] and [26, Section 4.4].

We could also consider randomly digitally shifted polynomial lattices. Let $x = \frac{x_1}{p} + \frac{x_2}{p^2} + \cdots$ and $\sigma = \frac{\sigma_1}{p} + \frac{\sigma_2}{p^2} + \cdots$ be the base p representation of x resp. σ in $[0, 1)$. Then the digitally shifted point $y = x \oplus \sigma$ is given by $y = \frac{y_1}{p} + \frac{y_2}{p^2} + \cdots$, where $y_i = x_i + \sigma_i \in \mathbb{Z}_p$ with addition modulo p. For vectors $\boldsymbol{x}$ and $\boldsymbol{\sigma}$ in $[0, 1)^s$, we define the digitally shifted point $\boldsymbol{x} \oplus \boldsymbol{\sigma}$ componentwise. Obviously, the shift depends on the base p. If the shift $\boldsymbol{\sigma} \in [0, 1)^s$ is chosen i.i.d. and the same shift is applied to all points of $\mathcal{P}(\boldsymbol{g}, f)$, then we speak of a *randomly digitally shifted polynomial lattice*. QMC algorithms which use randomly digitally shifted polynomial lattices as underlying point sets are called *randomly digitally shifted polynomial lattice rules*.

Our aim is to construct a generating vector $\boldsymbol{g}$ coefficient by coefficient such that the corresponding polynomial lattice $\mathcal{P}(\boldsymbol{g}, f')$ is of good quality for $f' = f, f^2, f^3, \ldots$, i.e., for cardinalities $N = p^{\deg(f)}, p^{2\deg(f)}, p^{3\deg(f)}, \ldots$. (We remark here that the existence result of Niederreiter [27] is more general. Instead of the sequence $f, f^2, f^3, \ldots$, Niederreiter investigated arbitrary divisibility chains of polynomials in $\mathbb{Z}_p[x]$.)

We consider two quality measures for polynomial lattices. The first one is the worst-case error for QMC integration in a weighted Hilbert space of functions which is based on Walsh functions (resp. the root mean-square worst-case error with respect to a digital shift $\boldsymbol{\sigma}$ in a weighted Sobolev space) and the second one is the star discrepancy.

The worst-case error in a weighted Hilbert space. Let $s \in \mathbb{N}$, $\alpha > 1$, and $\boldsymbol{\gamma} = (\gamma_j)_{j \geq 1}$ with positive real numbers γ_j. Further let p be a prime and denote by $\mathrm{wal}_{\boldsymbol{k}}$, $\boldsymbol{k} \in \mathbb{N}_0^s$, the p-adic Walsh functions defined as follows. Let $\omega_p = \mathrm{e}^{2\pi \mathtt{i}/p} \in \mathbb{C}$. For a nonnegative integer k with base p representation $k = \kappa_0 + \kappa_1 p + \cdots + \kappa_a p^a$, the function $\mathrm{wal}_k : \mathbb{R} \to \mathbb{C}$, periodic with period 1, is defined by

$$\mathrm{wal}_k(x) = \omega_p^{\kappa_0 x_1 + \cdots + \kappa_a x_{a+1}},$$

where $x \in [0, 1)$ has base p representation $x = x_1/p + x_2/p^2 + \cdots$ (unique in the sense that infinitely many of the x_j must be different from $p-1$). For dimensions $s \geq 1$ and for $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, the s-dimensional p-adic Walsh function $\mathrm{wal}_{\boldsymbol{k}} : \mathbb{R}^s \to \mathbb{C}$ is defined as

$$\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{j=1}^{s} \mathrm{wal}_{k_j}(x^{(j)}) \quad \text{for all} \quad \boldsymbol{x} = (x^{(1)}, \ldots, x^{(s)}) \in \mathbb{R}^s.$$

For any integer $s \geq 1$, the system $\{\mathrm{wal}_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{N}_0^s\}$ is a complete orthonormal system in $L_2([0, 1)^s)$. More information on Walsh functions can be found in [1, 28, 33].

As in [5, 7, 19], we consider the weighted Hilbert function space $H_{\mathrm{wal}, s, \alpha, \boldsymbol{\gamma}}$ with reproducing kernel given by

$$K_{\mathrm{wal}, s, \alpha, \boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \frac{1}{\rho_\alpha(\boldsymbol{k}, \boldsymbol{\gamma})} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{y})},$$

where for $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ we put $\rho_\alpha(\boldsymbol{k}, \boldsymbol{\gamma}) = \prod_{j=1}^s \rho_\alpha(k_j, \gamma_j)$ with

$$\rho_\alpha(k, \gamma) = \begin{cases} 1 & \text{if } k = 0, \\ \gamma^{-1} p^{\alpha \psi_p(k)} & \text{if } k \in \mathbb{N}. \end{cases}$$

Here, for $k = \kappa_0 + \kappa_1 p + \cdots + \kappa_a p^a \in \mathbb{N}$ with $\kappa_a \neq 0$, we put $\psi_p(k) = a$.

Throughout this section we use the following notation. For arbitrary $\boldsymbol{h} = (h_1, \ldots, h_s)$ and $\boldsymbol{g} = (g_1, \ldots, g_s)$ in $\mathbb{Z}_p[x]^s$, we define the inner product

$$\boldsymbol{h} \cdot \boldsymbol{g} := \sum_{j=1}^s h_j g_j,$$

and we write $g \equiv 0 \pmod f$ if f divides g in $\mathbb{Z}_p[x]$. Furthermore, we define for $f \in \mathbb{Z}_p[x]$ with $\deg(f) \geq 1$,

$$G_p(f) := \{h \in \mathbb{Z}_p[x] : \deg(h) < \deg(f)\} \quad \text{and} \quad G_p^s(f) := (G_p(f))^s.$$

In [5] it was shown that the squared worst-case error of integration in $H_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}$ using a polynomial lattice $\mathcal{P}(\boldsymbol{g}, f)$, with $f \in \mathbb{Z}_p[x]$, $\deg(f) \geq 1$, and $\boldsymbol{g} \in G_p^s(f)$, is given by

$$e^2_{p^{\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}, f) = \sum_{\substack{\boldsymbol{h} \in \mathbb{Z}_p[x]^s \setminus \{\boldsymbol{0}\} \\ \boldsymbol{h} \cdot \boldsymbol{g} \equiv 0 \ (\mathrm{mod}\ f)}} \frac{1}{\rho_\alpha(\boldsymbol{h}, \boldsymbol{\gamma})}.$$

Here, for $\boldsymbol{h} = (h_1, \ldots, h_s) \in \mathbb{Z}_p[x]^s$, we put $\rho_\alpha(\boldsymbol{h}, \boldsymbol{\gamma}) = \prod_{j=1}^s \rho_\alpha(h_j, \gamma_j)$ with

$$\rho_\alpha(h, \gamma) = \begin{cases} 1 & \text{if } h = 0, \\ \gamma^{-1} p^{\alpha \deg(h)} & \text{if } h \in \mathbb{Z}_p[x] \setminus \{0\}. \end{cases}$$

Remark 8 It was shown in [7] (see also [5]) that $e^2_{p^{\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}, f)$ can be computed in $O(sp^{\deg(f)})$ operations.

Remark 9 Consider again the weighted Sobolev space $H_{s,\boldsymbol{\gamma}}$ from Remark 3 in Section 2. It was shown in [7] (see also [5]) that the mean-square worst-case error $\widehat{e}^2_{p^{\deg(f)},s,\boldsymbol{\gamma}}$ for QMC integration in $H_{s,\boldsymbol{\gamma}}$ using a randomly digitally shifted polynomial lattice $\mathcal{P}(\boldsymbol{g}, f)$ is given by

$$\widehat{e}^2_{p^{\deg(f)},s,\boldsymbol{\gamma}}(\boldsymbol{g}, f) = \sum_{\substack{\boldsymbol{h} \in \mathbb{Z}_p[x]^s \setminus \{\boldsymbol{0}\} \\ \boldsymbol{h} \cdot \boldsymbol{g} \equiv 0 \ (\mathrm{mod}\ f)}} \frac{1}{\widehat{\rho}(\boldsymbol{h}, \boldsymbol{\gamma})},$$

where for $\boldsymbol{h} = (h_1, \ldots, h_s) \in \mathbb{Z}_p[x]^s$ we put $\widehat{\rho}(\boldsymbol{h}, \boldsymbol{\gamma}) = \prod_{j=1}^s \widehat{\rho}(h_j, \gamma_j)$ with

$$\widehat{\rho}(h, \gamma) = \begin{cases} 1 & \text{if } h = 0, \\ 2\gamma^{-1} p^{2(r+1)} \left(\frac{1}{\sin^2(\kappa_r \pi / p)} - \frac{1}{3} \right)^{-1} & \text{if } h = \kappa_0 + \kappa_1 x + \cdots + \kappa_r x^r \text{ with } \kappa_r \neq 0. \end{cases}$$

Note that because of the similarities between the worst-case error $e_{p^{\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}, f)$ in the weighted Hilbert space $H_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}$ and the root mean-square worst-case error $\widehat{e}_{p^{\deg(f)},s,\boldsymbol{\gamma}}(\boldsymbol{g}, f)$ in the Sobolev space $H_{s,\boldsymbol{\gamma}}$, the following results will apply to both cases, though we will state them only for the Hilbert space $H_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}$.

The star discrepancy. The star discrepancy (resp. discrepancy) has already been used in Section 2 as a quality criterion for usual integration lattices.

For $f \in \mathbb{Z}_p[x]$ with $\deg(f) \geq 1$ and $\boldsymbol{g} \in G_p^s(f)$, we define the quantity

$$R_p(\boldsymbol{g}, f) = \sum_{\substack{\boldsymbol{h} \in G_p^s(f) \setminus \{\boldsymbol{0}\} \\ \boldsymbol{h} \cdot \boldsymbol{g} \equiv 0 \pmod{f}}} \frac{1}{\rho(\boldsymbol{h})},$$

where for $\boldsymbol{h} = (h_1, \ldots h_s) \in \mathbb{Z}_p[x]^s$ we put $\rho(\boldsymbol{h}) = \prod_{j=1}^s \rho(h_j)$ with

$$\rho(h) = \begin{cases} 1 & \text{if } h = 0, \\ p^{r+1} \sin^2\left(\frac{\pi \kappa_r}{p}\right) & \text{if } h = \kappa_0 + \kappa_1 x + \cdots + \kappa_r x^r \text{ with } \kappa_r \neq 0. \end{cases}$$

Then for the star discrepancy $D^*_{p^{\deg(f)}}(\mathcal{P}(\boldsymbol{g}, f))$ of the polynomial lattice $\mathcal{P}(\boldsymbol{g}, f)$ we have (see [6, Proposition 2.1])

$$D^*_{p^{\deg(f)}}(\mathcal{P}(\boldsymbol{g}, f)) \leq \frac{s}{p^{\deg(f)}} + R_p(\boldsymbol{g}, f). \tag{11}$$

Hence good bounds on the quantity $R_p(\boldsymbol{g}, f)$ yield good bounds on the star discrepancy of the polynomial lattice generated by $\boldsymbol{g}$ and f.

We remark that the bound (11) even holds without the square at the sine in the definition of the factors $\rho(h)$. However, as shown in [6, Section 4], the slightly weaker version used here allows the computation of $R_p(\boldsymbol{g}, f)$ in $O(sp^{\deg(f)})$ operations.

3.1 Good extensible polynomial lattices with respect to the worst-case error

We now present an algorithm which constructs coefficient by coefficient a generating vector which is good with respect to the worst-case error $e_{p^{\deg(f')},s,\alpha,\boldsymbol{\gamma}}$ for all $f' = f, f^2, f^3, \ldots$.

Algorithm 3 *Let $f \in \mathbb{Z}_p[x]$ be irreducible.*

1. *Find $\boldsymbol{g}_1 := \boldsymbol{g}$ by minimizing $e^2_{p^{\deg(f)},s,\alpha,\boldsymbol{\gamma}}$ over all $\boldsymbol{g} \in G_p^s(f)$.*
2. *For $n = 2, 3, \ldots$ find $\boldsymbol{g}_n := \boldsymbol{g}_{n-1} + f^{n-1}\boldsymbol{a}$ by minimizing $e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_{n-1} + f^{n-1}\boldsymbol{g}, f^n)$ over all $\boldsymbol{g} \in G_p^s(f)$, i.e., $\boldsymbol{a} = \mathrm{argmin}_{\boldsymbol{g} \in G_p^s(f)} e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_{n-1} + f^{n-1}\boldsymbol{g}, f^n)$.*

Theorem 3 *Let $s, n \in \mathbb{N}$, p be a prime, $\alpha > 1$, and $f \in \mathbb{Z}_p[x]$ irreducible. Assume that $\boldsymbol{g}_n \in G_p^s(f^n)$ is constructed according to Algorithm 3. Then we have*

$$e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_n, f^n) \leq \left(\prod_{j=1}^s (1 + \gamma_j \mu_p(\alpha)) - 1\right) \min\left\{n, \frac{p^{(\alpha-1)\deg(f)}}{p^{(\alpha-1)\deg(f)} - 1}\right\} \frac{2}{p^{n\deg(f)}},$$

where $\mu_p(\alpha) := \frac{p^\alpha(p-1)}{p^\alpha - p}$.

Before we prove this result, we state two remarks.

Remark 10 The search for $\boldsymbol{g}_1$ in the first step of Algorithm 3 takes $O(sp^{\deg(f)(s+1)})$ operations. With a component-by-component construction (see [5]) this can be reduced to $O(s^2 p^{2\deg(f)})$ operations. In this case we obtain a slightly weaker error bound in Theorem 3 (the term "-1" after the first product must be deleted).

Alternatively, in the first step of Algorithm 3 one can choose an arbitrary vector $\boldsymbol{g}_1 \in G_p^s(f)$. But then the error bound in Theorem 3 has to be replaced by

$$e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_n, f^n) \leq \left(\prod_{j=1}^{s}(1+\gamma_j\mu_p(\alpha))-1\right)\min\left\{n, \frac{p^{(\alpha-1)\deg(f)}}{p^{(\alpha-1)\deg(f)}-1}\right\}\frac{1}{p^{(n-1)\deg(f)}}.$$

This follows immediately from the subsequent proof of Theorem 3.

Remark 11 We remark that it is not necessary to start Algorithm 3 with item 1. If one has given an arbitrary generating vector $\boldsymbol{g}_{n_0}$ for some $n_0 \in \mathbb{N}$ with squared worst-case error $e^2_{p^{n_0\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_{n_0}, f^{n_0})$, then this vector can be extended as well for $n = n_0+1, n_0+2, \ldots$. From the proof of Theorem 3 below it is easy to see that in this case we obtain

$$\begin{aligned} &e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_n, f^n) \\ &\quad \leq e^2_{p^{n_0\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_{n_0}, f^{n_0})\min\left\{n-n_0+1, \frac{p^{(\alpha-1)\deg(f)}}{p^{(\alpha-1)\deg(f)}-1}\right\}\frac{1}{p^{(n-n_0)\deg(f)}}. \end{aligned}$$

Now we give the proof of Theorem 3.

Proof. First we show the result for $n=1$. We have

$$\begin{aligned} e^2_{p^{\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_1, f) &\leq \frac{1}{p^{s\deg(f)}}\sum_{\boldsymbol{g}\in G_p^s(f)} e^2_{p^{\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}, f) \\ &= \frac{1}{p^{s\deg(f)}}\sum_{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\}}\frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}\sum_{\substack{\boldsymbol{g}\in G_p^s(f)\\ \boldsymbol{h}\cdot\boldsymbol{g}\equiv 0 \pmod{f}}} 1 \\ &= \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\equiv\boldsymbol{0} \pmod{f}}}\frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} + \frac{1}{p^{\deg(f)}}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\not\equiv\boldsymbol{0} \pmod{f}}}\frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \\ &= \left(1-\frac{1}{p^{\deg(f)}}\right)\sum_{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\}}\frac{1}{\rho_\alpha(f\boldsymbol{h},\boldsymbol{\gamma})} + \frac{1}{p^{\deg(f)}}\sum_{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\}}\frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \\ &\leq \frac{2}{p^{\deg(f)}}\sum_{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\}}\frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} = \left(\prod_{j=1}^{s}(1+\gamma_j\mu_p(\alpha))-1\right)\frac{2}{p^{\deg(f)}}, \end{aligned}$$

which is the desired bound for $n=1$.

Let $n \geq 2$. Then we have

$$\begin{aligned} e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_n, f^n) &\leq \frac{1}{p^{s\deg(f)}}\sum_{\boldsymbol{g}\in G_p^s(f)} e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_{n-1}+f^{n-1}\boldsymbol{g}, f^n) \\ &= \frac{1}{p^{s\deg(f)}}\sum_{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\}}\frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}\sum_{\substack{\boldsymbol{g}\in G_p^s(f)\\ \boldsymbol{h}\cdot(\boldsymbol{g}_{n-1}+f^{n-1}\boldsymbol{g})\equiv 0 \pmod{f^n}}} 1. \end{aligned}$$

The inner sum is equal to the number of $\boldsymbol{g} \in G_p^s(f)$ with $f^{n-1}\boldsymbol{h}\cdot\boldsymbol{g} \equiv -\boldsymbol{h}\cdot\boldsymbol{g}_{n-1} \pmod{f^n}$. For this we must have $\boldsymbol{h}\cdot\boldsymbol{g}_{n-1} \equiv 0 \pmod{f^{n-1}}$, and then $\boldsymbol{h}\cdot\boldsymbol{g} \equiv -\frac{1}{f^{n-1}}\boldsymbol{h}\cdot\boldsymbol{g}_{n-1} \pmod{f}$. Thus,

$$e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_n, f^n) \le \frac{1}{p^{s\deg(f)}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \sum_{\substack{\boldsymbol{g}\in G_p^s(f) \\ \boldsymbol{h}\cdot\boldsymbol{g}\equiv -\frac{1}{f^{n-1}}\boldsymbol{h}\cdot\boldsymbol{g}_{n-1} \pmod{f}}} 1.$$

Consider the inner sum. If $\boldsymbol{h} \not\equiv \boldsymbol{0} \pmod{f}$, then the inner sum is equal to $p^{(s-1)\deg(f)}$. If $\boldsymbol{h} \equiv \boldsymbol{0} \pmod{f}$, then the inner sum is equal to 0 if $\boldsymbol{h}\cdot\boldsymbol{g}_{n-1} \not\equiv 0 \pmod{f^n}$ and equal to $p^{s\deg(f)}$ if $\boldsymbol{h}\cdot\boldsymbol{g}_{n-1} \equiv 0 \pmod{f^n}$. Thus,

$$\begin{aligned}
e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_n, f^n) &\le \frac{1}{p^{\deg(f)}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\},\, \boldsymbol{h}\not\equiv\boldsymbol{0} \pmod{f} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} + \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod{f} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n}}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \\
&= \frac{1}{p^{\deg(f)}} e^2_{p^{(n-1)\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_{n-1}, f^{n-1}) - \frac{1}{p^{\deg(f)}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod{f} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \\
&\quad + \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod{f} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n}}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})},
\end{aligned}$$

and this holds for all $n \ge 2$.

If we insert this inequality for $e^2_{p^{(n-1)\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_{n-1}, f^{n-1})$, then we obtain

$$\begin{aligned}
& e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_n, f^n) \\
&\le \frac{1}{p^{2\deg(f)}} e^2_{p^{(n-2)\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_{n-2}, f^{n-2}) - \frac{1}{p^{2\deg(f)}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod{f} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-2}\equiv 0 \pmod{f^{n-2}}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \\
&\quad + \frac{1}{p^{\deg(f)}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod{f} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-2}\equiv 0 \pmod{f^{n-1}}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} - \frac{1}{p^{\deg(f)}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod{f} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \\
&\quad + \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod{f} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n}}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}.
\end{aligned}$$

Assume that $\boldsymbol{h} \in \mathbb{Z}_p[x]^s$, $\boldsymbol{h} \equiv \boldsymbol{0} \pmod{f}$, i.e., $\boldsymbol{h} = f\widetilde{\boldsymbol{h}}$, and $\boldsymbol{h}\cdot\boldsymbol{g}_{n-2} \equiv 0 \pmod{f^{n-1}}$. Then we have

$$\boldsymbol{h}\cdot\boldsymbol{g}_{n-1} = \boldsymbol{h}\cdot(\boldsymbol{g}_{n-2} + f^{n-2}\boldsymbol{g}) = \boldsymbol{h}\cdot\boldsymbol{g}_{n-2} + f^{n-1}\widetilde{\boldsymbol{h}}\cdot\boldsymbol{g} \equiv 0 \pmod{f^{n-1}},$$

with some $\boldsymbol{g} \in G_p^s(f)$. Therefore we obtain

$$\begin{aligned}
e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_n, f^n) &\le \frac{1}{p^{2\deg(f)}} e^2_{p^{(n-2)\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_{n-2}, f^{n-2}) - \frac{1}{p^{2\deg(f)}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod{f} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-2}\equiv 0 \pmod{f^{n-2}}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \\
&\quad + \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod{f} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n}}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}.
\end{aligned}$$

Repeating this argument, we get

$$\begin{aligned}
e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_n, f^n) \;&\leq\; \frac{1}{p^{(n-1)\deg(f)}} e^2_{p^{\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_1, f) - \frac{1}{p^{(n-1)\deg(f)}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod{f} \\ \boldsymbol{h}\cdot\boldsymbol{g}_1\equiv 0 \pmod{f}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \\
&\quad + \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod{f} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^n}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})}.
\end{aligned}$$

For the last sum in the above expression, we have

$$\begin{aligned}
\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\},\, \boldsymbol{h}\equiv\boldsymbol{0} \pmod{f} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^n}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \;&=\; \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}} \frac{1}{\rho_\alpha(f\boldsymbol{h},\boldsymbol{\gamma})} \\
&\leq\; \frac{1}{p^{\alpha\deg(f)}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}_p[x]^s\setminus\{\boldsymbol{0}\} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}} \frac{1}{\rho_\alpha(\boldsymbol{h},\boldsymbol{\gamma})} \\
&=\; \frac{1}{p^{\alpha\deg(f)}} e^2_{p^{(n-1)\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_{n-1}, f^{n-1}).
\end{aligned}$$

With this upper bound, we obtain

$$e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_n, f^n) \leq \frac{1}{p^{(n-1)\deg(f)}} e^2_{p^{\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_1, f) + \frac{1}{p^{\alpha\deg(f)}} e^2_{p^{(n-1)\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_{n-1}, f^{n-1}).$$

With backward induction on n and invoking the upper bound for $e^2_{p^{\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_1, f)$, we get

$$\begin{aligned}
e^2_{p^{n\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_n, f^n) \;&\leq\; e^2_{p^{\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_1, f) \left(\frac{1}{p^{(n-1)\deg(f)}} + \frac{1}{p^{(n-2+\alpha)\deg(f)}} \right) \\
&\quad + \frac{1}{p^{2\alpha\deg(f)}} e^2_{p^{(n-2)\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_{n-2}, f^{n-2}) \\
&\;\;\vdots \\
&\leq\; e^2_{p^{\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_1, f) \sum_{k=0}^{n-2} \frac{1}{p^{(n-1-k+k\alpha)\deg(f)}} + \frac{1}{p^{(n-1)\alpha\deg(f)}} e^2_{p^{\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_1, f) \\
&=\; e^2_{p^{\deg(f)},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{g}_1, f) \sum_{k=0}^{n-1} \frac{1}{p^{(n-1-k+k\alpha)\deg(f)}} \\
&\leq\; \left(\prod_{j=1}^{s} (1+\gamma_j \mu_p(\alpha)) - 1 \right) \min\left\{ n, \frac{p^{(\alpha-1)\deg(f)}}{p^{(\alpha-1)\deg(f)} - 1} \right\} \frac{2}{p^{n\deg(f)}},
\end{aligned}$$

which is the desired bound. □

3.2 Good extensible polynomial lattices with respect to the star discrepancy

We also present an algorithm which uses the quantity R_p (and hence the star discrepancy) as a quality criterion.

Algorithm 4 *Let $f \in \mathbb{Z}_p[x]$ be monic and irreducible.*

1. *Find $\boldsymbol{g}_1 := \boldsymbol{g}$ by minimizing $R_p(\boldsymbol{g}, f)$ over all $\boldsymbol{g} \in G_p^s(f)$.*
2. *For $n = 2, 3, \ldots$ find $\boldsymbol{g}_n := \boldsymbol{g}_{n-1} + f^{n-1}\boldsymbol{a}$ by minimizing $R_p(\boldsymbol{g}_{n-1} + f^{n-1}\boldsymbol{g}, f^n)$ over all $\boldsymbol{g} \in G_p^s(f)$, i.e., $\boldsymbol{a} = \mathrm{argmin}_{\boldsymbol{g}\in G_p^s(f)} R_p(\boldsymbol{g}_{n-1} + f^{n-1}\boldsymbol{g}, f^n)$.*

Theorem 4 *Let $s, n \in \mathbb{N}$, p be a prime, $f \in \mathbb{Z}_p[x]$ monic and irreducible, and define $\nu_{p,f} := \deg(f)\frac{p^2-1}{3p}$. Assume that $\boldsymbol{g}_n \in G_p^s(f^n)$ is constructed according to Algorithm 4. Then we have*

$$R_p(\boldsymbol{g}_n, f^n) = O\left(p^{-\deg(f)n}\left((1+\nu_{p,f})^s + 1\right)^n\right)$$

with an implied constant depending only on s.

Before we give the proof of this result, we state a remark and a corollary.

Remark 12 Again we remark that it is not necessary to start Algorithm 4 with item 1. If one has given an arbitrary generating vector $\boldsymbol{g}_{n_0}$ for some $n_0 \in \mathbb{N}$, then this vector can be extended as well for $n = n_0 + 1, n_0 + 2, \ldots$. From the proof of Theorem 4 below it is easy to see that in this case we obtain

$$R(\boldsymbol{g}_n, f^n) \leq \frac{((1+\nu_{p,f})^s + 1)^{n-n_0}}{p^{\deg(f)(n-n_0)}} R(\boldsymbol{g}_{n_0}, f^{n_0}) + O\left(p^{-\deg(f)n}\left((1+\nu_{p,f})^s + 1\right)^{n-n_0+1}\right),$$

where the implied constant depends only on s.

From (11) and Theorem 4 we immediately obtain a bound on the star discrepancy. This bound is good only for small p and polynomials f of large degree.

Corollary 3 *Let $s, n \in \mathbb{N}$, p be a prime, and $f \in \mathbb{Z}_p[x]$ monic and irreducible. Assume that $\boldsymbol{g}_n \in G_p^s(f^n)$ is constructed according to Algorithm 4. Then we have*

$$D^*_{p^{\deg(f)n}}(\mathcal{P}(\boldsymbol{g}_n, f^n)) = O\left(p^{-\deg(f)n}\left((1+\nu_{p,f})^s + 1\right)^n\right)$$

with an implied constant depending only on s.

For the proof of Theorem 4 we need the following result.

Lemma 2 *We have*

$$\sum_{\boldsymbol{h}\in G_p^s(f)} \frac{1}{\rho(\boldsymbol{h})} = (1+\nu_{p,f})^s,$$

where $\nu_{p,f} := \deg(f)\frac{p^2-1}{3p}$.

Proof. We have

$$\sum_{\boldsymbol{h}\in G_p^s(f)} \frac{1}{\rho(\boldsymbol{h})} = \left(\sum_{h\in G_p(f)} \frac{1}{\rho(h)}\right)^s$$

and further

$$\sum_{h\in G_p(f)} \frac{1}{\rho(h)} = 1 + \sum_{u=0}^{\deg(f)-1} \frac{1}{p^{u+1}} p^u \sum_{\kappa=1}^{p-1} \frac{1}{\sin^2\left(\frac{\pi\kappa}{p}\right)} = 1 + \deg(f)\frac{p^2-1}{3p},$$

since $\sum_{\kappa=1}^{p-1} \frac{1}{\sin^2\left(\frac{\pi\kappa}{p}\right)} = \frac{p^2-1}{3}$ as shown, for example, in [7, Appendix C]. □

Now we give the proof of Theorem 4.

Proof. First we show the result for $n=1$. We have

$$\begin{aligned} R_p(\boldsymbol{g}_1,f) &\leq \frac{1}{p^{s\deg(f)}}\sum_{\boldsymbol{g}\in G_p^s(f)} R_p(\boldsymbol{g},f) = \frac{1}{p^{s\deg(f)}}\sum_{\boldsymbol{h}\in G_p^s(f)\setminus\{\boldsymbol{0}\}}\frac{1}{\rho(\boldsymbol{h})}\sum_{\substack{\boldsymbol{g}\in G_p^s(f)\\ \boldsymbol{h}\cdot\boldsymbol{g}\equiv 0 \pmod f}} 1 \\ &= \frac{1}{p^{\deg(f)}}\sum_{\boldsymbol{h}\in G_p^s(f)\setminus\{\boldsymbol{0}\}}\frac{1}{\rho(\boldsymbol{h})} = \frac{1}{p^{\deg(f)}}\left((1+\nu_{p,f})^s-1\right), \end{aligned}$$

where we used Lemma 2 for the last equality. Hence the desired result follows for $n=1$.

Now let $n\geq 2$. As in the proof of Theorem 3 we have

$$\begin{aligned} R_p(\boldsymbol{g}_n,f^n) &\leq \frac{1}{p^{s\deg(f)}}\sum_{\boldsymbol{g}\in G_p^s(f)} R_p(\boldsymbol{g}_{n-1}+f^{n-1}\boldsymbol{g},f^n) \\ &= \frac{1}{p^{s\deg(f)}}\sum_{\boldsymbol{h}\in G_p^s(f^n)\setminus\{\boldsymbol{0}\}}\frac{1}{\rho(\boldsymbol{h})}\sum_{\substack{\boldsymbol{g}\in G_p^s(f)\\ \boldsymbol{h}\cdot(\boldsymbol{g}_{n-1}+f^{n-1}\boldsymbol{g})\equiv 0 \pmod{f^n}}} 1 \\ &= \frac{1}{p^{s\deg(f)}}\sum_{\substack{\boldsymbol{h}\in G_p^s(f^n)\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}}\frac{1}{\rho(\boldsymbol{h})}\sum_{\substack{\boldsymbol{g}\in G_p^s(f)\\ \boldsymbol{h}\cdot\boldsymbol{g}\equiv -\frac{1}{f^{n-1}}\boldsymbol{h}\cdot\boldsymbol{g}_{n-1} \pmod f}} 1 \\ &= \frac{1}{p^{\deg(f)}}\sum_{\substack{\boldsymbol{h}\in G_p^s(f^n)\setminus\{\boldsymbol{0}\},\,\boldsymbol{h}\not\equiv\boldsymbol{0} \pmod f\\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}}\frac{1}{\rho(\boldsymbol{h})} + \sum_{\substack{\boldsymbol{h}\in G_p^s(f^n)\setminus\{\boldsymbol{0}\},\,\boldsymbol{h}\equiv\boldsymbol{0} \pmod f\\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^n}}}\frac{1}{\rho(\boldsymbol{h})} \\ &\leq \frac{1}{p^{\deg(f)}}\sum_{\substack{\boldsymbol{h}\in G_p^s(f^n)\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}}\frac{1}{\rho(\boldsymbol{h})} + \sum_{\substack{\boldsymbol{h}\in G_p^s(f^{n-1})\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}}\frac{1}{\rho(f\boldsymbol{h})}. \end{aligned}$$

Since f is monic, we find that $\rho(f\boldsymbol{h})\geq p^{\deg(f)}\rho(\boldsymbol{h})$ for $\boldsymbol{h}\in\mathbb{Z}_p[x]^s$ with $\boldsymbol{h}\neq\boldsymbol{0}$. Hence we obtain

$$R_p(\boldsymbol{g}_n,f^n)\leq\frac{2}{p^{\deg(f)}}R_p(\boldsymbol{g}_{n-1},f^{n-1})+\frac{1}{p^{\deg(f)}}\Sigma \tag{12}$$

with

$$\Sigma := \sum_{\substack{\boldsymbol{h}\in G_p^s(f^n)\setminus G_p^s(f^{n-1})\\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}}\frac{1}{\rho(\boldsymbol{h})}.$$

Any $\boldsymbol{h}\in G_p^s(f^n)\setminus G_p^s(f^{n-1})$ can be represented uniquely in the form

$$\boldsymbol{h}=\widetilde{\boldsymbol{h}}+f^{n-1}\boldsymbol{b}\quad\text{with}\quad\widetilde{\boldsymbol{h}}\in G_p^s(f^{n-1}),\ \boldsymbol{b}\in G_p^s(f)\setminus\{\boldsymbol{0}\}.$$

Therefore

$$\begin{aligned} \Sigma &= \sum_{\substack{\boldsymbol{h}\in G_p^s(f^{n-1})\\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}}\sum_{\boldsymbol{b}\in G_p^s(f)\setminus\{\boldsymbol{0}\}}\frac{1}{\rho(\boldsymbol{h}+f^{n-1}\boldsymbol{b})} \\ &= \sum_{\boldsymbol{b}\in G_p^s(f)\setminus\{\boldsymbol{0}\}}\frac{1}{\rho(f^{n-1}\boldsymbol{b})} + \sum_{\substack{\boldsymbol{h}\in G_p^s(f^{n-1})\setminus\{\boldsymbol{0}\}\\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}}\sum_{\boldsymbol{b}\in G_p^s(f)\setminus\{\boldsymbol{0}\}}\frac{1}{\rho(\boldsymbol{h}+f^{n-1}\boldsymbol{b})} \\ &=: \Sigma_1+\Sigma_2. \end{aligned}$$

First we deal with Σ_1. By Lemma 2 we have

$$\begin{aligned}\Sigma_1 &= \left(\sum_{b\in G_p(f)} \frac{1}{\rho(f^{n-1}b)}\right)^s - 1 = \left(1 + \frac{1}{p^{\deg(f^{n-1})}} \sum_{b\in G_p(f)\setminus\{0\}} \frac{1}{\rho(b)}\right)^s - 1 \\ &= \left(1 + \frac{\nu_{p,f}}{p^{\deg(f^{n-1})}}\right)^s - 1 \le \frac{s\nu_{p,f}}{p^{\deg(f^{n-1})}} \left(1 + \frac{\nu_{p,f}}{p^{\deg(f^{n-1})}}\right)^{s-1}.\end{aligned}$$

Now we turn to Σ_2. We have

$$\Sigma_2 = \sum_{\substack{\boldsymbol{h}\in G_p^s(f^{n-1})\setminus\{\boldsymbol{0}\} \\ \boldsymbol{h}\cdot\boldsymbol{g}_{n-1}\equiv 0 \pmod{f^{n-1}}}} \sum_{\boldsymbol{b}\in G_p^s(f)\setminus\{\boldsymbol{0}\}} \frac{1}{\rho(\boldsymbol{h}+f^{n-1}\boldsymbol{b})}.$$

Denoting the inner sum by Σ_3, we get

$$\Sigma_3 = \sum_{\boldsymbol{b}\in G_p^s(f)} \frac{1}{\rho(\boldsymbol{h}+f^{n-1}\boldsymbol{b})} - \frac{1}{\rho(\boldsymbol{h})} = \prod_{j=1}^s \left(\sum_{b\in G_p(f)} \frac{1}{\rho(h_j+f^{n-1}b)}\right) - \frac{1}{\rho(\boldsymbol{h})},$$

where $\boldsymbol{h} = (h_1,\ldots,h_s)$.

If $h_j = 0$, then by Lemma 2 and since f is monic,

$$\sum_{b\in G_p(f)} \frac{1}{\rho(h_j+f^{n-1}b)} = \sum_{b\in G_p(f)} \frac{1}{\rho(f^{n-1}b)} = 1 + \frac{\nu_{p,f}}{p^{\deg(f^{n-1})}} = \frac{1}{\rho(h_j)} + \frac{\nu_{p,f}}{p^{\deg(f^{n-1})}}.$$

If $h_j \neq 0$, then $0 \le \deg(h_j) < \deg(f^{n-1})$ and

$$\begin{aligned}\sum_{b\in G_p(f)} \frac{1}{\rho(h_j+f^{n-1}b)} &= \frac{1}{\rho(h_j)} + \sum_{b\in G_p(f)\setminus\{0\}} \frac{1}{\rho(h_j+f^{n-1}b)} = \frac{1}{\rho(h_j)} + \sum_{b\in G_p(f)\setminus\{0\}} \frac{1}{\rho(f^{n-1}b)} \\ &= \frac{1}{\rho(h_j)} + \frac{\nu_{p,f}}{p^{\deg(f^{n-1})}},\end{aligned}$$

where we again used Lemma 2 and the assumption that f is monic.

Hence we obtain

$$\Sigma_3 = \prod_{j=1}^s \left(\frac{1}{\rho(h_j)} + \frac{\nu_{p,f}}{p^{\deg(f^{n-1})}}\right) - \frac{1}{\rho(\boldsymbol{h})}.$$

Since $\frac{1}{\rho(h_j)} \ge \frac{1}{p^{\deg(f^{n-1})}}$, it follows now that

$$\Sigma_3 \le \prod_{j=1}^s \left(\frac{1}{\rho(h_j)} + \frac{\nu_{p,f}}{\rho(h_j)}\right) - \frac{1}{\rho(\boldsymbol{h})} = \frac{1}{\rho(\boldsymbol{h})}\left((1+\nu_{p,f})^s - 1\right)$$

and hence

$$\Sigma_2 \le \left((1+\nu_{p,f})^s - 1\right) R_p(\boldsymbol{g}_{n-1}, f^{n-1}).$$

Altogether we find that

$$\Sigma = \Sigma_1 + \Sigma_2 \le \frac{s\nu_{p,f}}{p^{\deg(f^{n-1})}} \left(1 + \frac{\nu_{p,f}}{p^{\deg(f^{n-1})}}\right)^{s-1} + \left((1+\nu_{p,f})^s - 1\right) R_p(\boldsymbol{g}_{n-1}, f^{n-1})$$

and hence

$$R_p(\boldsymbol{g}_n, f^n) \leq \frac{1}{p^{\deg(f)}} \left((1+\nu_{p,f})^s + 1\right) R_p(\boldsymbol{g}_{n-1}, f^{n-1}) + \frac{s\nu_{p,f}}{p^{\deg(f^n)}} \left(1 + \frac{\nu_{p,f}}{p^{\deg(f^{n-1})}}\right)^{s-1}.$$

Iterating this inequality, we get for all $n \geq 2$,

$$\begin{aligned} R_p(\boldsymbol{g}_n, f^n) &\leq \frac{1}{p^{\deg(f)(n-1)}} \left((1+\nu_{p,f})^s + 1\right)^{n-1} R_p(\boldsymbol{g}_1, f) \\ &\quad + s\nu_{p,f} \sum_{j=0}^{n-2} \left(\frac{(1+\nu_{p,f})^s + 1}{p^{\deg(f)}}\right)^j \frac{1}{p^{\deg(f)(n-j)}} \left(1 + \frac{\nu_{p,f}}{p^{\deg(f)(n-1-j)}}\right)^{s-1} \\ &\leq \frac{1}{p^{\deg(f)n}} \left((1+\nu_{p,f})^s + 1\right)^n \\ &\quad + \frac{s\nu_{p,f}}{p^{\deg(f)n}} \left(1 + \frac{\nu_{p,f}}{p^{\deg(f)}}\right)^{s-1} \sum_{j=0}^{n-2} \left((1+\nu_{p,f})^s + 1\right)^j. \end{aligned}$$

Hence

$$R_p(\boldsymbol{g}_n, f^n) = O\left(\frac{\left((1+\nu_{p,f})^s + 1\right)^n}{p^{\deg(f)n}}\right)$$

with an implied constant depending only on s. □

4 Numerical Results

In this section we present some numerical results for the classical case. We wrote a program for Algorithm 1 with Mathematica. In each of the following examples we have chosen $p = 2$.

In Figure 1 we considered the unweighted case in dimension $s = 5$, whereas in Figures 2 and 3 we investigated the weighted case in dimensions $s = 10$ and $s = 15$ with weights $\gamma_j = j^{-2}$ in both cases. In all cases we tried different values of α.

In contrast to our theoretical bound from Theorem 1, in all experiments higher values of α show higher convergence rates for the worst-case errors. This reflects the widely held view that smoother functions are in general much easier to integrate than nonsmooth functions. However, the numerical results seem to suggest that the integration lattices constructed with Algorithm 1 do not achieve the optimal rate for the worst-case error $e_{N,s,\alpha,\boldsymbol{\gamma}}$ in the first place which would be of order $O(N^{-\alpha/2+\varepsilon})$ for $\varepsilon > 0$. For example, for $\alpha = 2$, the observed α (for the convergence rate) seems to be around 1.25 or for $\alpha = 6$, the observed α seems to be around 2.5.

In each of the following figures we plotted on the x-axes the value $\log_2 2^n = n$ and on the y-axes the base 2 logarithm of the corresponding squared worst-case error $e^2_{2^n,s,\alpha,\boldsymbol{\gamma}}$.

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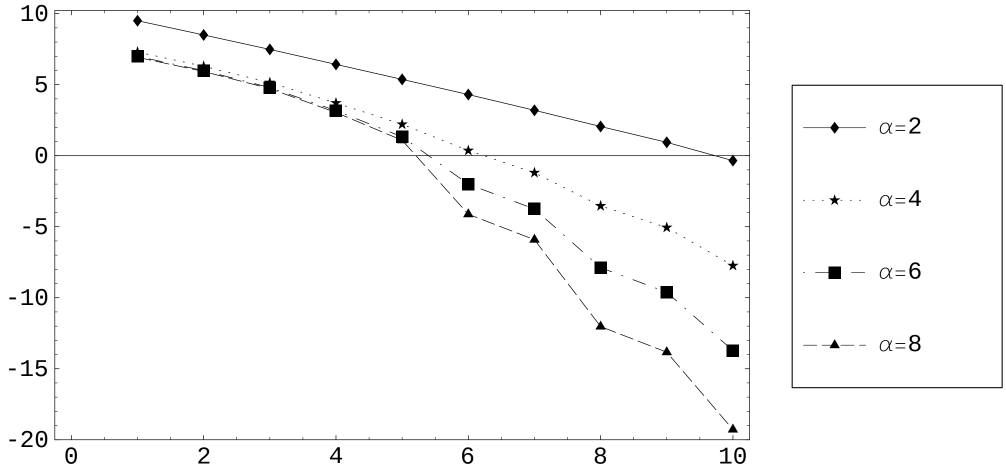

Figure 1: $\log_2(e^2_{2^n,s,\alpha,\boldsymbol{\gamma}})$ with $s = 5$, $\alpha \in \{2, 4, 6, 8\}$, $\boldsymbol{\gamma} = (1)_{j\geq 1}$, and $n = 2, \ldots, 15$.

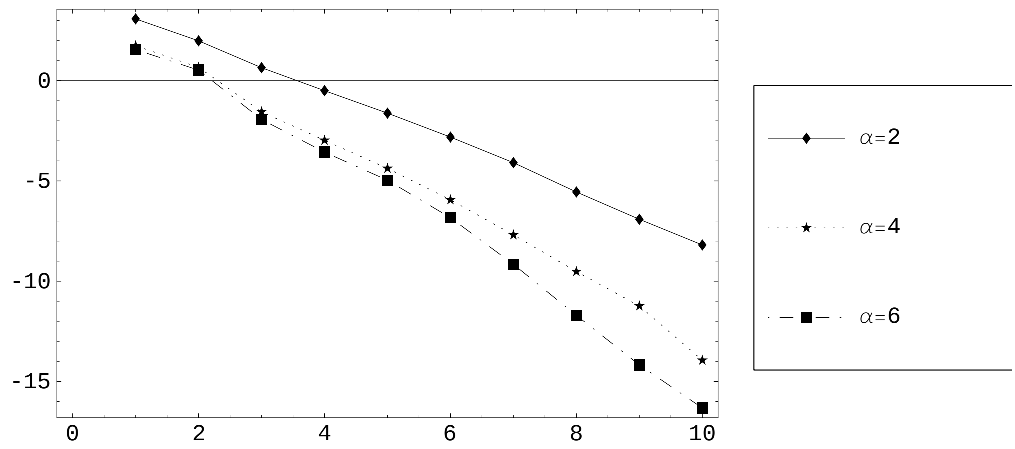

Figure 2: $\log_2(e^2_{2^n,s,\alpha,\boldsymbol{\gamma}})$ with $s = 10$, $\alpha \in \{2, 4, 6\}$, $\boldsymbol{\gamma} = (j^{-2})_{j\geq 1}$, and $n = 2, \ldots, 10$.

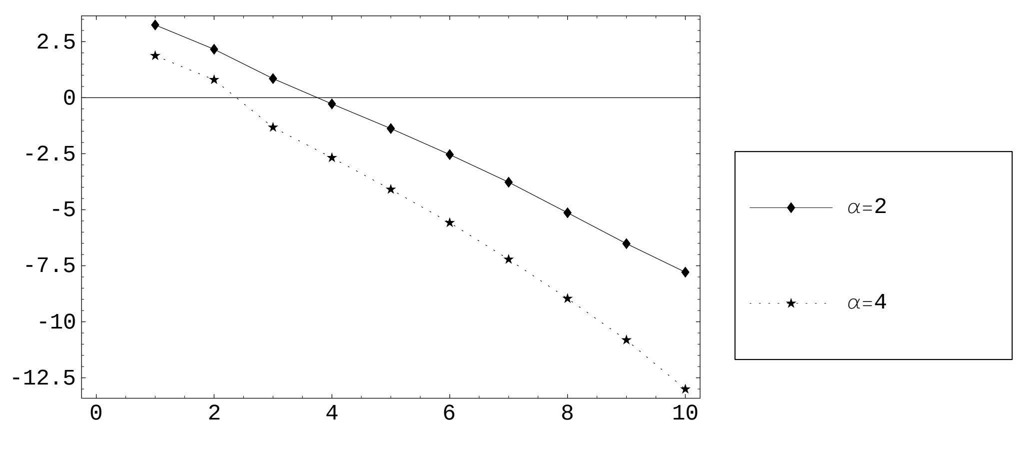

Figure 3: $\log_2(e^2_{2^n,s,\alpha,\boldsymbol{\gamma}})$ with $s = 15$, $\alpha \in \{2, 4\}$, $\boldsymbol{\gamma} = (j^{-2})_{j\geq 1}$, and $n = 2, \ldots, 10$.

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