PART II

COMPLEX ASYMPTOTIC METHODS

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30414093201713378043612608166064768844377641568960512000000000000
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$\left(\frac{N}{e}\right)^N \sqrt{2\pi N}\ \left(1+\frac{1}{12N}\right) + \cdots$

ASYMPTOTICS

$N!$

Some functions are too complicated to be expanded

e.g.: TRAINS

Some GF's are not even explicit

e.g.: unlabelled, non-plane trees

$U(z) = z \exp\left(U(z) + \frac{1}{2} U(z^2) + \frac{1}{3} U(z^3) + \cdots\right)$

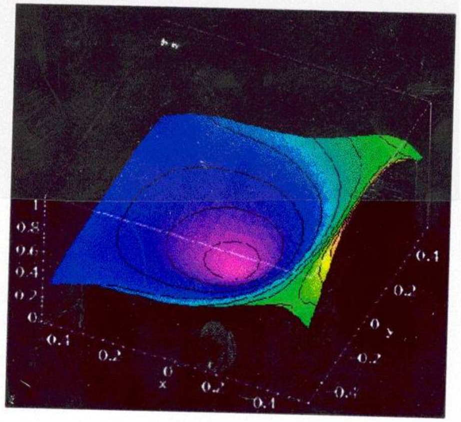
"Universality" phenomena are not apparent

- schemas applying to wide classes
- limit laws shared by — —

PRINCIPLE: Assign to the variable z complex values ⇒

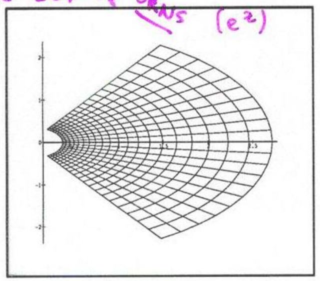
View a GF as a geometric transformation from $\mathbb{C} \to \mathbb{C}$

modulus of GF of balanced trees



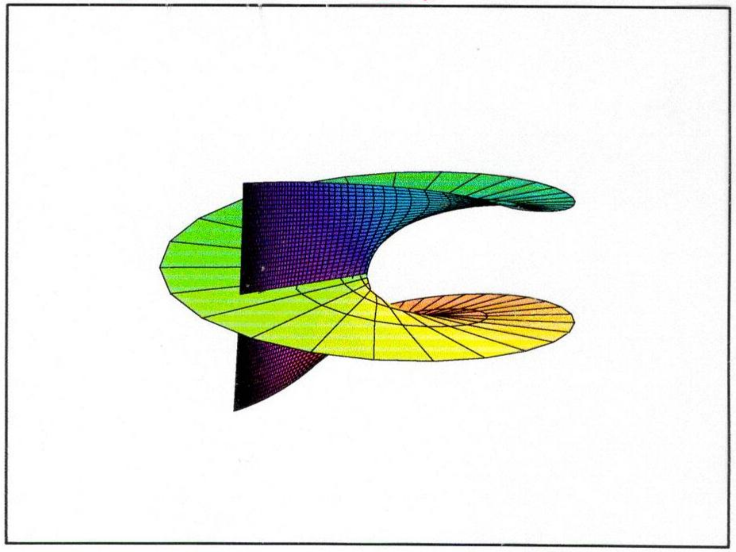
ANALYTIC FUNCTIONS

THE EGF of URNS (e^z)



SINGULARITIES MATTER!

The $\sqrt{\ }$-singularity of the OGF of binary trees.



CHAPTER 4: BASIC COMPLEX ASYMPTOTICS

- Notion of an (analytic / holomorphic) function
- Meromorphic function
- Residue theorem; Cauchy's coefficient theorem
- Singularities and exponential order

Let $f(z)$ be defined from $\mathcal{D}$ to $\mathcal{E}$

(Consider $\mathcal{D}$ = connected open set)

FUNDAMENTAL EQUIVALENCE THEOREM

There is equivalence between the following properties

- $f(z)$ is analytic at all points $z_0 \in \mathcal{D}$
- $f(z)$ is complex-differentiable at all points $z_0 \in \mathcal{D}$

Also say that $f(z)$ is "REGULAR".

DEFINITION: $f(z)$ is analytic at z_0 iff it admits a locally convergent series expansion

$$\boxed{f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n}$$

(Note: such expansions converge in discs!)

$f(z)$ is complex-differentiable (holomorphic) at z_0 iff the limit exists:

$$\boxed{\lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \left(\frac{f(z_0+h)-f(z_0)}{h}\right)} \to f'(z_0) \text{ or } \left.\frac{df}{dz}\right|_{z_0} \cdots$$

- ANALYTICITY (series expansion) is what interests combinatorialists, *a priori*.
- COMPLEX-DIFFERENTIABILITY makes easier the development of the theory.

$$\frac{\Delta f}{\Delta z} \text{ calculations}, \ldots$$

closure under $+, -, \times, \div$ (if denom $\neq 0$)
composition, inversion (cond. on derivatives $\neq 0$)

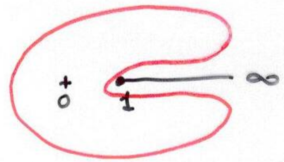
- THE FUNCTION $\sqrt{z}$

$$\sqrt{\rho e^{i\theta}} = \sqrt{\rho}\, e^{i\theta/2}$$

can only be made continuous in ($i = \sqrt{-1}$)

(ANALYTIC / REGULAR) in $\mathbb{C} \setminus (-\infty, 0)$

- $\log z$ $\quad \log(\rho e^{i\theta}) = \log \rho + i\theta$
- $\dfrac{e^z}{\sqrt{1-z}}$ is (regular / analytic) in



- $\dfrac{1-\sqrt{1-4z}}{2}$ is (regular / analytic) in $\mathbb{C} \setminus (\frac{1}{4}, +\infty)$ etc...

- ANY POLYNOMIAL $z^2 + 19z^5$ is (regular / analytic/h.) in whole of $\mathbb{C}$
- A RATIONAL FRACTION $\dfrac{z-3}{(z-1)(z^2-2)}$, $\dfrac{A(z)}{B(z)}$ is (regular / analytic) in $\mathbb{C}$ minus POLES
- THE EXPONENTIAL FUNCTION is REGULAR in $\mathbb{C}$

- $\dfrac{e^{-z}}{1-z}$ regular in $\mathbb{C} \setminus \{1\}$

INTEGRATION

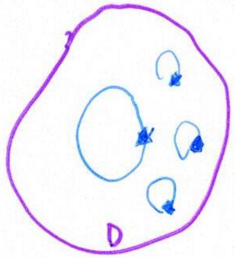
- Define integration along curves $\int_\gamma f(z)\,dz$

FUNDAMENTAL INTEGRAL THEOREM

Let f be analytic in $\mathcal{D}$. ~~that~~ $\underline{\gamma}$ be contractible to a single point in $\mathcal{D}$. Then "a loop"

$$\int_\gamma f(z)\,dz = 0$$

$\Rightarrow \int_A^B f(z)\,dz$ does not depend on path



D is simply connected.

DEFINITION: $g(z)$ is meromorphic in $\mathcal{D}$ iff

near any z_0, one has $g(z) = \frac{A(z)}{B(z)}$ with $A(z)$, $B(z)$ analytic at z_0.

A point z_0 such that $B(z_0) = 0$ while $A(z_0) \neq 0$ is called a pole. Its order is the multiplicity of the zero/root z_0 of $B(z)$.

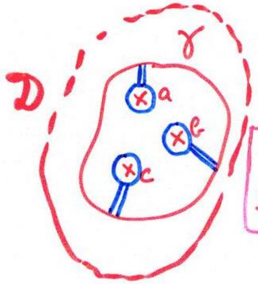
Pole of order m: $g(z) = \frac{c_{-m}}{(z-z_0)^m} + \cdots + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + \cdots$

CAUCHY RESIDUE THEOREM

$f(z)$ has poles

$\mathrm{Res}\,(f(z))_{z=z_0}$ is the coeff. of $\frac{1}{z-z_0}$

$$f(z) = \sum_{n \geq -m} c_n (z-z_0)^n \qquad \Rightarrow \mathrm{Res}(f)_{z=z_0} \equiv c_{-1}$$

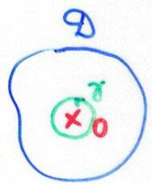


$$\boxed{\frac{1}{2i\pi}\int_\gamma f(z)\,dz = \sum \text{Residues}}$$

Proof: Termwise "local" integration

CAUCHY COEFFICIENT THEOREM

$$\boxed{\text{coeff}\ [z^n]\ f(z) = \frac{1}{2i\pi}\int_\gamma f(z)\,\frac{dz}{z^{n+1}}}$$

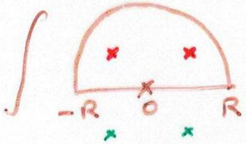


Proof: Residue theorem

COMPLEX ANALYSIS: LOCAL & GLOBAL

- **Computing integrals** global ⇝ local !!!

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^4} = \lim_{R\to\infty} \int$$



$$= 2i\pi \sum_{\zeta \in \{e^{i\pi/4}, e^{3i\pi/4}\}} \mathrm{Res}\left(\frac{1}{1+x^4};\zeta\right)$$

$$= \frac{\pi\sqrt{2}}{2}$$

- **Estimating coefficients** $d_n := \Pr\{\text{derangement}/\mathfrak{S}_n\}$

$$D(z) = \sum \underbrace{d_n}_{(D_n/n!)} z^n = \frac{e^{-z}}{1-z}$$

$$d_n = \frac{1}{2i\pi}\int_{|z|=1/2} \frac{e^{-z}}{1-z}\,\frac{dz}{z^{n+1}}$$

Evaluate instead on $|z|=2$

$$J_n = \frac{1}{2i\pi}\int_{|z|=2} \frac{e^{-z}}{1-z}\,\frac{dz}{z^{n+1}} = O(2^{-n})$$

$$= \mathrm{Res.} @\ z=0 + \mathrm{Res}\ @\ z=1$$

$$= d_n - e^{-1}$$

$$\Rightarrow \boxed{d_n = e^{-1} + O(2^{-n})}$$

ASYMPTOTIC ESTIMATE.

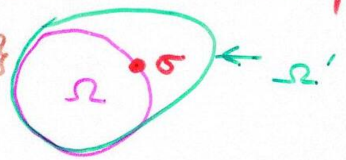
Singularities

– what they are

– why they are important –

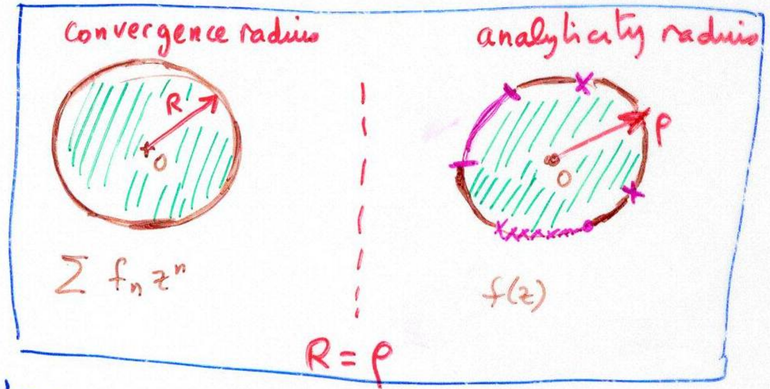
■ $f(z)$ has a SINGULARITY at border point σ iff

$\nexists\, \Omega' \supset \Omega \cup \{\sigma\}$



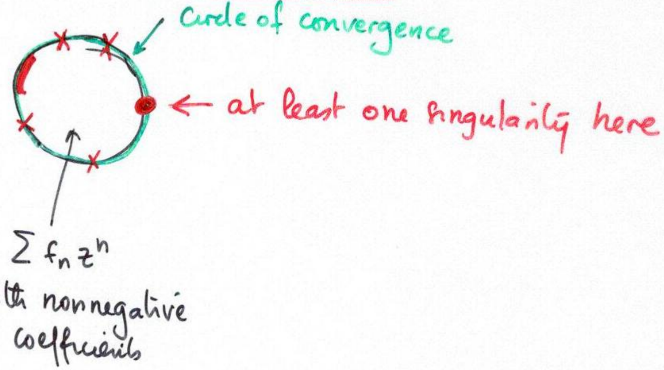
e.g.: $f(\sigma) = \infty$, $f'(\sigma) = \infty$, + other causes

$\boxed{\text{FACT A}}$ A series $f(z) = \sum_n f_n z^n$ always has at least one singularity on its circle of convergence but NONE inside.



Thm: convergence radius = analyticity radius

Help: PRINGSHEIM'S THEOREM



$\boxed{\text{Fact}}$ Let $f(z)$ be analytic at 0 with radius of convergence exactly R. Then for any ε

- $f_n (R-\varepsilon)^n \to 0$
- $f_n (R+\varepsilon)^n$ is unbounded.

In other words $\boxed{\limsup |f_n|^{1/n} = \frac{1}{R}}$

$f_n = R^{-n}\theta(n) \longrightarrow \limsup |\theta(n)|^{1/n} = 1$

where $\theta(n)$ is "subexponential"

The exponential order of growth of f_n is R^{-n}. $\boxed{f_n \bowtie R^{-n}}$

Once you find the singularities nearest to the origin, you KNOW the exponential growth of the function's coefficients.

Examples (singularities)

$\frac{e^{-z}}{1-z}$ is singular at $z=1$ [derangements]

$\frac{1}{\sqrt{1-2z^2}}$ is singular at $z=\frac{\pm 1}{\sqrt{2}}$ [binary mappings]

$\frac{1-\sqrt{1-4z}}{2z}$ is singular at $z=\frac{1}{4}$ [Catalan trees]

- Words

$$\frac{1}{1-2z} \Longrightarrow W_n \bowtie 2^n$$

- derangements

$$\frac{e^{-z}}{1-z} \Longrightarrow \frac{D_n}{n!} \bowtie 1^n$$

- (1,2)-coverings

$$\frac{1}{1-(z+z^2)} \Longrightarrow C_n \bowtie \phi^n \qquad \phi = \frac{1+\sqrt{5}}{2} = 1.6..$$

- general trees

$$\frac{1-\sqrt{1-4z}}{2} \Longrightarrow G_n \bowtie 4^n$$

In fact

$$G_n = \frac{1}{n}\binom{2n-2}{n-1}$$

$$\sim \frac{4^{n-1}}{\sqrt{\pi n^3}} \quad \text{[by Stirling]}$$

Exercise: unary-binary trees $\mathcal{U}$

$$U = z(1+U+U^2)$$

$$U(z) = \frac{1-z - \sqrt{1-2z-3z^2}}{2z}$$

Show that $U_n \bowtie 3^n$.

ASYMPTOTIC EXPONENTIAL ORDER is COMPUTABLE AUTOMATICALLY for POSITIVE FUNCTIONS / CONSTRUCTIONS

$$\rho(f+g) = \min(\rho(f), \rho(g))$$

$$\rho(f \times g) = \min(\rho(f), \rho(g))$$

$$\rho\left(\frac{1}{1-f}\right) = \min(\rho(f), \{|z| \mid f(z)=1\})$$

$$\rho(e^f) = \rho(f)$$

$$\rho\left(\log\frac{1}{1-f}\right) = \text{cf } 1/(1-f)$$

$(f(0)=0)$

+ Recursive structures can be approached via Implicit function theorem.

CHAPTER 5

Rational and meromorphic function asymptotics.

Find subexponential factors

$$f_n \bowtie R^{-n}$$

$$f_n = \theta(n) \cdot R^{-n}$$

where $\theta(n)$ is (usually) like n^α, $(\log n)^\beta$, $e^{\sqrt{n}}$, etc.

(A) COEFFICIENTS of RATIONAL FUNCTIONS

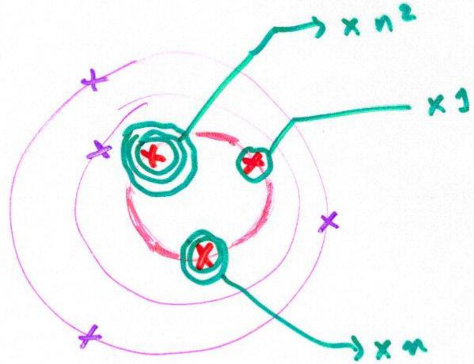
$$[z^n] \frac{1}{(z-\zeta)^m} = (-\zeta)^{-m} \binom{n+m-1}{m-1} \zeta^{-n}$$

$$\sim 6 \frac{n^{m-1}}{(m-1)!} \zeta^{-n}$$

Theorem:

$$\boxed{\begin{array}{l}\text{EACH POLE } \zeta \text{ WITH MULTIPLICITY } r \text{ CONTRIBUTES A TERM} \\ \qquad \zeta^{-n} P(n) \text{ with } \deg(P) = r-1.\end{array}}$$

Poles are arranged in order of increasing modulus. Dominant one matter ⇝ exponential rate; next, multiplicities give polynomial growth



Denumerants = ways of changing n cents given 1¢, 2¢, 5¢, 10¢ coins.

$$D(z) = \frac{1}{(1-z)(1-z^2)(1-z^5)(1-z^{10})}$$

→ Q1: Find A, k in formula

$$D_n \sim C \cdot A^n \cdot n^k$$

Q2: Find constant C.

→ Q3 $[z^n]\, D(z) e^z$; $\frac{D(z)}{1-2z}$; $\frac{D(z)}{1-z/2}$

Problem: Denumerants. How many integer solutions to $\sum_{j=1}^{m} x_j s_j = n$ where $\{s_j\}$ is fixed? Assume freely $\gcd(\{s_j\}) = 1$.

= Integer partitions with fixed summands
= MONEY CHANGER's PROBLEM

$$\mathcal{D} = \mathrm{Seq}(\mathbf{1}^{s_1}) \times \mathrm{Seq}(\mathbf{1}^{s_2}) \times \cdots \times \mathrm{Seq}(\mathbf{1}^{s_m})$$

$$D(z) = \frac{1}{1-z^{s_1}} \times \frac{1}{1-z^{s_2}} \times \cdots \times \frac{1}{1-z^{s_m}}$$

Poles at various roots of unity w/order < m
at z = 1 with order exactly m

$$D_n \equiv [z^n]\, D(z) \sim [z^n] \frac{1}{(1-z)^m} \frac{1}{\prod s_j}$$

[Schur]

$$\sim \frac{n^{m-1}}{(m-1)!} \times \frac{1}{\prod s_j}$$

Eg: $\#\left\{ \sum_{j=1}^{m} j x_j = n \right\} \sim \frac{n^m}{(m-1)!\, m!}$.

OPJ

Largest RUN in a random string

b b b $\boxed{a\,b\,b}$ $\boxed{a}$ $\boxed{a}$ $\boxed{a\,b\,b\,b}$ $\boxed{a\,b\,b\,b}$

$$\mathrm{Seq}_{<m}(b) \times \mathrm{Seq}(a \times \mathrm{Seq}_{<m} b)$$

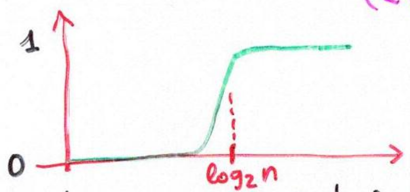
$$\frac{1-z^m}{1-z} \times \frac{1}{1-z\frac{1-z^m}{1-z}} = \frac{1-z^m}{1-2z+z^{m+1}}$$

- Dominant pole at $\rho_m \approx \frac{1}{2}$

$$\rho_m \approx \frac{1}{2}\left(1+\left(\frac{1}{2}\right)^{m+1}\right)$$

- Check error from dominant pole is good by $\int_{|z|=3/4}$

$$\Pr(\text{Longest run} \le m) \approx \left(\frac{1}{2\rho_m}\right)^n \approx e^{-n/2^{m+1}}$$

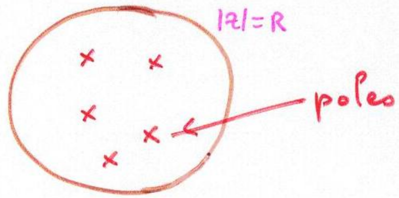


Guibas-Odlyzko: Correlation Polynomials

Normality of strings; longest repeated substring...

(B). COEFFICIENTS of MEROMORPHIC FNS.

Assumption: $g(z)$ is meromorphic in $|z|<R$ and analytic on $|z|=R$



Theorem

EACH POLE ζ WITH MULTIPLICITY r CONTRIBUTES A TERM

$$\zeta^{-n} P(n) \quad \text{with} \quad \deg(P) = r-1.$$

AND ERROR TERM $O(R^{-n})$

Proof: 1) Let $h(z)$ gather contributions of poles. Then $[g(z)-h(z)]$ is analytic in $|z|\le R$. Cauchy coeff. formula + trivial bounds

2) Estimate $\frac{1}{2i\pi}\int g$ on $|z|=R$ by residues.

A WORKED OUT EXAMPLE DERANGEMENTS

$$\mathcal{D} = \mathrm{Set}(\mathrm{Cycle}(\mathcal{Z}, \mathrm{card} \geq 2))$$

$$D(z) = \exp\left(\log \frac{1}{1-z} - z\right)$$

$$D(z) = \frac{e^{-z}}{1-z}$$

$D(z) \sim \dfrac{e^{-1}}{1-z}$ at singularity $z = 1$

$$\Rightarrow \quad \frac{D_n}{n!} \sim [z^n] \frac{e^{-1}}{1-z} = e^{-1}$$

PROP. A perm is a derangement with probability $e^{-1} = 0.345\ldots$

■ Generalized derangements $\mathcal{D}^*$

$$D^*(z) = \frac{e^{-z-\frac{z^2}{2}}}{1-z}$$

$$\frac{D^*_n}{n!} \sim e^{-3/2}$$

in general, get

$$\boxed{\text{proba} \sim e^{-H_k} \text{ of all cycles of length} > k.}$$

Theorem: Any «path in graph» model [finite automaton] leads to a **rational generating function** that has a unique **dominant pole**, so that

$$f_n \sim c \cdot \rho^{-n}$$

provided [Perron Frobenius theory]

- graph is strongly connected.
- aperiodicity condition $\equiv \forall n \geq n_0$, there exists paths of length n from source to destination.

cf: words containing a fixed pattern (abb)

APPLICATION: Supercritical schema (sequences)

Assume $\mathcal{F} = \mathrm{Seq}(\mathcal{G}) \Rightarrow F(z) = \frac{1}{1-G(z)}$

H_1: $G(z)$ reaches 1 before it becomes singular

(H_2: the schema is aperiodic: $F_n > 0$ for all $n \geq n_0$.)

Then $F_n \sim C \cdot \rho^{-n}$ $\quad \rho = G^{(-1)}(1)$

Also: number of $\mathcal{G}$-components in random $\mathcal{F}$-structure has mean $\sim \mu n$; variance $\sim \sigma^2 n \Longrightarrow$ concentration

Preferential arrangements / surjections

$\mathcal{R} = \mathrm{Seq}(\mathrm{Set}_{\geq 1}(\mathcal{Z}))$ $\quad R(z) = \frac{1}{1-(e^z-1)} = \frac{1}{2-e^z}$

Pole at $z = \log 2$; others at $\log 2 \pm 2i\pi$, $\log 2 \pm 3i\pi, \ldots$

$\frac{R_n}{n!} \sim c.(\log 2)^{-n}$ $\quad \left(c = \frac{1}{2\log 2}\right)$