L_p discrepancy of generalized two-dimensional Hammersley point sets

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Abstract

We determine the L_p discrepancy of the two-dimensional Hammersley point set in base b. These formulas show that the L_p discrepancy of the Hammersley point set is not of best possible order with respect to the general (best possible) lower bound on L_p discrepancies due to Roth and Schmidt. To overcome this disadvantage we introduce permutations in the construction of the Hammersley point set and show that there always exist permutations such that the L_p discrepancy of the generalized Hammersley point set is of best possible order. For the L_2 discrepancy such permutations are given explicitly.

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Key words: L_p discrepancy, Hammersley point set.

1 Introduction

For a point set $\mathcal{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ of $N \geq 1$ points in the two-dimensional unit-square $[0,1)^2$ the *discrepancy function* is defined as

$$E(x, y, \mathcal{P}) = A([0, x) \times [0, y), \mathcal{P}) - Nxy,$$

where $A([0, x) \times [0, y), \mathcal{P})$ denotes the number of indices $1 \leq M \leq N$ for which $\boldsymbol{x}_M \in [0, x) \times [0, y)$. Hence, the discrepancy function gives "locally" the difference between the number of points from $\mathcal{P}$ in $[0, x) \times [0, y)$ and the expected number of points in $[0, x) \times [0, y)$ if we assume a perfect uniform distribution of $\mathcal{P}$ (which is of course not possible since $\mathcal{P}$ is finite).

If we take a norm of the discrepancy function we obtain a "global" quantitative measure for the irregularity of distribution of a finite point set. Such a measure usually is called a *discrepancy*. An introduction to the theory of discrepancy of sequences can be found in [1, 8, 15, 16]. For an overview see also [20]. In applications, the discrepancy is often normalized by the total number of points N, but in theoretical studies, one keeps the original definition and we will do so as well. Here we will especially consider the so-called L_p discrepancies which can be obtained by taking the L_p norm of the discrepancy function. For any point set $\mathcal{P}$ consisting of N points in $[0,1)^2$ we define,

$$L_p(\mathcal{P}) := \left(\int_0^1 \int_0^1 |E(x, y, \mathcal{P})|^p \,\mathrm{d}x \,\mathrm{d}y \right)^{1/p}.$$

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It is well known that for any $p > 1$ there exists a constant $c_p > 0$ with the following property: for the L_p discrepancy of any point set $\mathcal{P}$ consisting of N points in $[0,1)^2$ we have

$$L_p(\mathcal{P}) \geq c_p \sqrt{\log N}. \tag{1}$$

This was first shown by Roth [18] for $p = 2$ and hence for all $p \geq 2$ and later by Schmidt [19] for all $1 < p < 2$.

In this paper we will consider the L_p discrepancy of the so-called generalized Hammersley point sets in base b with b^n points. These point sets, generalizations of the Hammersley point set in base b (which is also known as Roth net for $b = 2$), can be considered as finite two-dimensional versions of the generalized van der Corput sequences in base b. We use the following definition which was first given by Faure [4].

Definition 1 (generalized van der Corput sequence) Let $b \geq 2$ be an integer and let $\Sigma = (\sigma_r)_{r\geq 0}$ be a sequence of permutations of $\{0, 1, \ldots, b-1\}$. For any integers n and N with $n \geq 0$ and $1 \leq N \leq b^n$, write $N - 1 = \sum_{r=0}^{\infty} a_r(N)\, b^r$ in the b–adic system (so that $a_r(N) = 0$ if $r \geq n$). Then the *generalized van der Corput sequence* S_b^Σ *in base* b *associated to* Σ is defined by

$$S_b^\Sigma(N) := \sum_{r=0}^{\infty} \frac{\sigma_r\big(a_r(N)\big)}{b^{r+1}}, \quad \text{for all } N \geq 1\cdot$$

If $(\sigma_r) = (\sigma)$ is constant, we write $S_b^\Sigma = S_b^\sigma$. The *original van der Corput sequence* in base b, S_b^{id}, is obtained with the identical permutation id. In this case we will simply write S_b instead of S_b^{id}.

Definition 2 (generalized Hammersley point set) Let $b \geq 2$ be an integer, let S_b^Σ be a generalized van der Corput sequence in base b and let $n \geq 0$ be an integer. Then the *generalized two-dimensional Hammersley point set in base* b consisting of b^n points associated to Σ is defined by

$$\mathcal{H}_{b,n}^\Sigma := \left\{ \left(S_b^\Sigma(N), \frac{N-1}{b^n} \right); 1 \leq N \leq b^n \right\}$$

In order to match with the traditional definition of arbitrary (shifted or not) Hammersley point sets which are "n-bits" (i.e. whose b-adic expansions do not exceed n bits), we restrict the infinite sequence of permutations Σ to permutations such that $\sigma_r(0) = 0$ for all $r \geq n$, for instance $\Sigma = (\sigma_0, \ldots, \sigma_{n-1}, id, id, id, \ldots)$. Hence, the behavior of $\mathcal{H}_{b,n}^\Sigma$ will only depend on the finite sequence of n permutations $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1})$, and we will write $\mathcal{H}_{b,n}^\Sigma =: \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$.

Again, if we choose in the above definition $\sigma_j = id$ for every permutation, then we obtain the classical Hammersley point set in base b. In this case we will simply write $\mathcal{H}_{b,n}$ instead of $\mathcal{H}_{b,n}^{\boldsymbol{id}}$.

We first recall the main results obtained for Hammersley point sets in base 2.
For the L_p discrepancy, $p \in \mathbb{N}$, of the classical Hammersley point set $\mathcal{H}_{2,n}$ in base 2 it was shown by Pillichshammer [17, Theorem 1] that

$$(L_p(\mathcal{H}_{2,n}))^p = \frac{n^p}{8^p} + O(n^{p-1}).$$

For $p=1$ we have the exact formula

$$L_1(\mathcal{H}_{2,n}) = \frac{n}{8} + \frac{1}{2} + \frac{1}{2^{n+2}}$$

by [17, Theorem 2], and for $p=2$ we have

$$(L_2(\mathcal{H}_{2,n}))^2 = \frac{n^2}{64} + \frac{29n}{192} + \frac{3}{8} - \frac{n}{16\cdot 2^n} + \frac{1}{4\cdot 2^n} - \frac{1}{72\cdot 2^{2n}}$$

as shown by Vilenkin [21], Halton and Zaremba [9] and Pillichshammer [17].

In Section 2 we will generalize all these results to the b-adic case for arbitrary integers $b\geq 2$.

In many recent papers so-called digitally shifted van der Corput sequences and Hammersley point sets in base 2 are studied. See, for example, [10]–[14].

In our context here, a digitally shifted van der Corput sequence resp. digitally shifted Hammersley point set in base 2 is only a special case of a generalized van der Corput sequence resp. generalized Hammersley point set in base b which can be obtained by allowing permutations $\sigma\in\{id,\tau_b\}$ only. Here id is the identity and τ_b is the permutation of $\{0,1,\ldots,b-1\}$ defined by $\tau_b(k)=b-k-1$ $(0\leq k\leq b-1)$. Note that for $b=2$ we have $\tau_2(k)=k+1 \pmod 2$ which is exactly the "digital shift" as considered in [10]–[14].

The reason for considering generalized Hammersley point sets is that with this concept one can improve the distribution properties dramatically. If we compare the results on the L_p discrepancy of the classical Hammersley point set in base 2 from above with the general lower bound on L_p discrepancy from Roth and Schmidt (1) we find that the Hammersley point set does not show the best possible distribution properties with respect to the order of magnitude in N, the cardinality of the point set.

It were first Halton and Zaremba [9] who showed that for every n there exists a sequence of permutations $\boldsymbol{\sigma}\in\{id,\tau_2\}^n$ such that the L_2 discrepancy of the generalized Hammersley point set $\mathcal{H}_{2,n}^{\boldsymbol{\sigma}}$ in base 2 is given by

$$\left(L_2(\mathcal{H}_{2,n}^{\boldsymbol{\sigma}})\right)^2 = \frac{5n}{192} + \frac{3}{8} - \frac{7\varepsilon_n}{64} + \frac{1}{4\cdot 2^n} + \frac{\varepsilon_n}{16\cdot 2^n} - \frac{1}{72\cdot 2^{2n}}, \tag{2}$$

where ε_n is zero if n is even and one if n is odd. The sequence of permutations given by Halton and Zaremba is

$$\boldsymbol{\sigma} = \begin{cases} (\tau_2, id, \tau_2, id, \ldots, \tau_2, id) & \text{if } n \text{ is even,}\\ (\tau_2, id, \tau_2, id, \ldots, \tau_2, id, \tau_2) & \text{if } n \text{ is odd.}\end{cases}$$

(Of course, Halton and Zaremba did not use our terminology here.) This result was generalized recently by Kritzer and Pillichshammer [13]. They showed that for any $n\in\mathbb{N}$ and any $\boldsymbol{\sigma}\in\{id,\tau_2\}^n$ we have

$$(L_2(\mathcal{H}_{2,n}^{\boldsymbol{\sigma}}))^2 = \frac{n^2}{64} - \frac{19n}{192} - \frac{ln}{16} + \frac{l^2}{16} + \frac{l}{4} + \frac{3}{8} + \frac{n}{16\cdot 2^n} - \frac{l}{8\cdot 2^n} + \frac{1}{4\cdot 2^n} - \frac{1}{72\cdot 4^n},$$

where l is the number of id-permutations in $\boldsymbol{\sigma}$.

Furthermore it was shown in [14] that for any even $p\in\mathbb{N}$ and any $n\in\mathbb{N}$ there exists a $\boldsymbol{\sigma}\in\{id,\tau_2\}^n$ such that

$$(L_p(\mathcal{H}_{2,n}^{\boldsymbol{\sigma}}))^p \leq \frac{2S(p,p/2)}{4^p} n^{p/2} + O(n^{p/2-1}), \tag{3}$$

where $S(p, p/2)$ is a Stirling number of the second kind.

As to arbitrary bases, there are papers concerning the star discrepancy of two-dimensional Hammersley point sets [3, 5, 7] but only one, to our knowledge, on the L_2 discrepancy of such sets: the paper of White [22]. It is a remarkable, but too short paper using results obtained by symbolic computations to find exact formulas for the L_2 discrepancy of $\mathcal{H}_{b,n}$ and of a class of generalized Hammersley point sets. We shall refer to these results in the course of the paper to compare them more thoroughly with our results.

The organization of the paper is as follows: In Section 2–3 we will generalize the results above in base 2 to the b-adic case for general integers $b \geq 2$. The proofs of the results are given in Section 5–8. They are based on a detailed study of the discrepancy function of the one-dimensional generalized van der Corput sequence in base b as given in [2, 4, 6] and use also ideas from [13, 14, 17]. In Section 4 we will provide some preliminary results which will be used for the proofs.

2 L_p discrepancy of the classical Hammersley point set

In this section we consider the L_p discrepancy of the classical two-dimensional Hammersley point set $\mathcal{H}_{b,n}$ in base b. We generalize all results from [17] for $b = 2$ to the general b-adic case.

Theorem 1 *For any integer base $b \geq 2$ and any $n \geq 0$ we have*

$$L_1(\mathcal{H}_{b,n}) = n\frac{b^2-1}{12b} + \frac{1}{2} + \frac{1}{4b^n}$$

and

$$\begin{aligned}&(L_2(\mathcal{H}_{b,n}))^2\\ &= n^2\left(\frac{b^2-1}{12b}\right)^2 + n\left(\frac{3b^4+10b^2-13}{720b^2} + \frac{b^2-1}{12b}\left(1-\frac{1}{2b^n}\right)\right) + \frac{3}{8} + \frac{1}{4b^n} - \frac{1}{72b^{2n}}.\end{aligned}$$

The proof for p=1 will be given in Section 5. We also remark that for $p = 1$ the formula, but only for *prime* bases b, was already shown by Kritzer (unpublished) by using Walsh-series analysis (the proof needs about 30 pages).

For $p = 2$ and arbitrary b, the formula was shown by White [22] using recurrence relations and automated symbolic manipulations which should also easily apply to $p = 1$. The paper of White is very terse, with a lot of computations not carried out in detail (in fact, some formulas, which are needed for the proof, are simply omitted for brevity). This makes it very hard to follow White's proof. We don't give an explicit proof since the formula is a special case of Theorem 4 (see Remark 1). It should be longer, but the method allows further investigations which will permit a thorough analysis of generalized Hammersley point sets and improvements on previous results with regard to low discrepancy as assert the subsequent results.

For arbitrary integers $p \in \mathbb{N}$ we obtain a little bit weaker result.

Theorem 2 *Let $p \in \mathbb{N}$. For any integer base $b \geq 2$ and any $n \geq 0$ we have*

$$(L_p(\mathcal{H}_{b,n}))^p = n^p \left(\frac{b^2-1}{12b}\right)^p + O(n^{p-1}),$$

where the constant in the O-notation only depends on p and b (and not on n). Hence

$$\lim_{n\to\infty} \frac{L_p(\mathcal{H}_{b,n})}{\log b^n} = \frac{b^2-1}{12b\log b}.$$

The proof of this result will be given in Section 6.

From Theorem 2 above we find that the L_p discrepancy of the classical Hammersley point set in base b is asymptotically not of best possible order with respect to the lower bound (1) of Roth and Schmidt. This disadvantage of the classical Hammersley point set can be overcome by considering generalized Hammersley point sets.

3 L_p discrepancy of generalized Hammersley point sets

Now we consider the generalized Hammersley point set $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ where $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1})$ is a finite sequence of permutations of $\{0, \ldots, b-1\}$. Here we do not consider arbitrary permutations. We are especially interested in $\boldsymbol{\sigma}$ where some permutations are the identity id and some are the special permutation τ_b given by $\tau_b(k) = b-k-1$. As the base b is always fixed in the following we will simply write τ instead of τ_b.

First we will show that on the average over all such sequences of permutations the L_p discrepancy of a generalized Hammersley point set is of best possible order with respect to (1).

Theorem 3 *Let p be an even positive integer and n a nonnegative integer. Then we have*

$$\frac{1}{2^n} \sum_{\boldsymbol{\sigma}\in\{id,\tau\}^n} \left(L_p(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^p \leq 2\left(\frac{b^2-1}{12}\right)^p \frac{p!}{(p/2)!\,2^{p/2}} n^{p/2} + O(n^{p/2-1}),$$

where the constant in the O-notation only depends on p and b (and not on n).

Note that for $b = 2$ this is an improvement of [14, Theorem 1]. The proof of Theorem 3 is deferred to Section 7.

Using Markov's inequality we immediately find from Theorem 3 that there always exist sequences of permutations which yield the best possible order of L_p discrepancy.

Corollary 1 *Let p be an even integer and n a nonnegative integer. Then there exists a sequence of permutations $\boldsymbol{\sigma}^* \in \{id, \tau\}^n$ such that the L_p discrepancy of the generalized Hammersley point set $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}^*}$ is bounded by*

$$(L_p(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}^*}))^p \leq 2\left(\frac{b^2-1}{12}\right)^p \frac{p!}{(p/2)!\,2^{p/2}} n^{p/2} + O(n^{p/2-1}),$$

where the constant in the O-notation only depends on p and b.

Again, for $b=2$ this result is an improvement of (3).

We can also show that the generalized Hammersley point set yields asymptotically for almost all sequences of permutations from $\{id,\tau\}^n$ better results than the classical Hammersley point set.

Corollary 2 *Let p be an even integer. For any $\varepsilon>0$ and any $c>0$ we have*

$$\lim_{n\to\infty}\frac{\#\left\{\boldsymbol{\sigma}\in\{id,\tau\}^n\ :\ L_p(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})<cn^{1/2+\varepsilon}\right\}}{2^n}=1.$$

Proof. From Theorem 3 it follows that

$$\begin{aligned}
2\left(\frac{b^2-1}{12}\right)^p\frac{p!}{(p/2)!\,2^{p/2}}n^{p/2}+O(n^{p/2-1}) &\geq \frac{1}{2^n}\sum_{\boldsymbol{\sigma}\in\{id,\tau\}^n}\left(L_p(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^p\\
&\geq \frac{1}{2^n}c^pn^{(1/2+\varepsilon)p}\#\left\{\boldsymbol{\sigma}\in\{id,\tau\}^n\ :\ L_p(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\geq cn^{1/2+\varepsilon}\right\}\\
&= \frac{1}{2^n}c^pn^{(1/2+\varepsilon)p}\left(2^n-\#\left\{\boldsymbol{\sigma}\in\{id,\tau\}^n\ :\ L_p(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})<cn^{1/2+\varepsilon}\right\}\right).
\end{aligned}$$

Hence,

$$\frac{\#\left\{\boldsymbol{\sigma}\in\{id,\tau\}^n\ :\ L_p(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})<cn^{1/2+\varepsilon}\right\}}{2^n}\geq 1+O\left(\frac{1}{n^{\varepsilon p}}\right),$$

and the result follows. □

For $p=2$, it is even possible to give the following exact result.

Theorem 4 *Let $\boldsymbol{\sigma}\in\{id,\tau\}^n$ and let l denote the number of components of $\boldsymbol{\sigma}$ which are equal to id. Then we have*

$$(L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}))^2=$$
$$\left(\frac{b^2-1}{12b}\right)^2((n-2l)^2-n)+\frac{b^2-1}{12b}\left(1-\frac{1}{2b^n}\right)(2l-n)+n\frac{b^4-1}{90b^2}+\frac{3}{8}+\frac{1}{4b^n}-\frac{1}{72b^{2n}}.$$

Remark 1 Note that the L_2 discrepancy of $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ with $\boldsymbol{\sigma}\in\{id,\tau\}^n$ only depends on n,b and the number of permutations in $\boldsymbol{\sigma}$ which are equal to id (and not on their distribution). Setting $l=n$ we get the formula for the L_2 discrepancy of the classical Hammersley point set from Theorem 1.

From Theorem 4 we find that among all permutations in $\{id,\tau\}^n$ the one where all components are equal to id yields the worst result with respect to L_2 discrepancy.

Corollary 3 *For any $\boldsymbol{\sigma}\in\{id,\tau\}^n$ we have*

$$L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\leq L_2(\mathcal{H}_{b,n})$$

with equality if and only if $\boldsymbol{\sigma}=(id,\ldots,id)$.

We can also determine the minimal L_2 discrepancy.

Corollary 4 *The L_2 discrepancy of $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ for $\boldsymbol{\sigma} \in \{id, \tau\}^n$ becomes minimal if we choose a sequence of permutations $\boldsymbol{\sigma}$ in which exactly*

$$l_{\min}(n) := \begin{cases} n/2 - 2 & \text{if } b = 2, \\ n/2 - 1 & \text{if } 3 \leq b \leq 6, \\ n/2 & \text{if } b \geq 7, \end{cases}$$

elements are the identity in the case of even n and

$$l_{\min}(n) := \begin{cases} (n-3)/2 & \text{if } 2 \leq b \leq 3, \\ (n-1)/2 & \text{if } b \geq 4, \end{cases}$$

elements are the identity in the case of odd n. In any case we have

$$\min_{\boldsymbol{\sigma} \in \{id,\tau\}^n} (L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}))^2 = n\frac{(b^2-1)(3b^2+13)}{720b^2} + O(1). \tag{4}$$

Proof. For fixed b and n let

$$f_l := \frac{b^2-1}{12b}(n-2l)^2 + \left(1 - \frac{1}{2b^n}\right)(2l-n).$$

Then we have $f_l > f_{l+1}$ if and only if

$$l < \frac{n-1}{2} - \frac{3b}{b^2-1}\left(1 - \frac{1}{2b^n}\right),$$

and hence f_l becomes minimal for

$$l = \left\lceil \frac{n-1}{2} - \frac{3b}{b^2-1}\left(1 - \frac{1}{2b^n}\right) \right\rceil.$$

From this we find that, for even n, $l = n/2 - 2$ for $b = 2$, $l = n/2 - 1$ for $3 \leq b \leq 6$ and $l = n/2$ for $b \geq 7$. For odd n, $l = (n-3)/2$ for $2 \leq b \leq 3$ and $l = (n-1)/2$ for $b \geq 4$.

Inserting into the formula from Theorem 4 gives the result. $\square$

Remark 2 With the values for $l_{\min}(n)$ one can also determine the exact formula in (4) for every case. Note that White [22] obtained the same constant with another generalization: he considered digital shifts $\sigma_r(k) = (k+r) \pmod b$ for the sequence of permutations $\Sigma = (\sigma_r)$ which, in the case of $b = 2$, reduce to the shift τ_2 like our permutation τ_b (of course, White did not use our terminology here). Hence, in this case, his sequence is $\boldsymbol{\sigma} = (id, \tau, id, \tau, \cdots)$, a special case of our formula in Theorem 4. Of course, the remaining terms in the O-notation are not the same since White needs integers n of the form $n = bm + 1$ with the shifts he uses. It should be interesting to obtain an analog of Theorem 4 for such shifts in base b.

We also obtain a generalization of [13, Corollary 2] for general b. We remark here that there is a little inaccuracy in the statement of [13, Corollary 2]. The correct result can be obtained by setting $b = 2$ in the subsequent corollary.

Corollary 5 *For any real* $x \geq \sqrt{\frac{(b^2-1)(13+3b^2)}{720b^2}}$ *we have*

$$\lim_{n\to\infty} \frac{\#\left\{\boldsymbol{\sigma} \in \{id,\tau\}^n \,:\, L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) \leq x\sqrt{n}\right\}}{2^n} = 2\Phi\left(\sqrt{\frac{720b^2x^2-(b^2-1)(13+3b^2)}{5(b^2-1)^2}}\right)-1,$$

where

$$\Phi(y) = \frac{1}{2\pi}\int_{-\infty}^{y} \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t.$$

Proof. We denote the right hand side of the formula in Theorem 4 by $d_b(n,l)$. Then we have

$$\frac{\#\left\{\boldsymbol{\sigma} \in \{id,\tau\}^n \,:\, L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) \leq x\sqrt{n}\right\}}{2^n} = \frac{1}{2^n}\sum_{\substack{l=0\\ \sqrt{d_b(n,l)}\leq x\sqrt{n}}}^{n} \binom{n}{l}.$$

We have $\sqrt{d_b(n,l)} \leq x\sqrt{n}$ if and only if $a_n(x) \leq l \leq b_n(x)$, where

$$a_n(x) := \frac{n}{2} - \frac{6b-3b^{1-n}}{2-2b^2} - \sqrt{5}\frac{\sqrt{55b^2-360b^{n+2}-90b^{2n+2}+nb^{2n}(720b^2x^2-(b^2-1)(13+3b^2))}}{10(b^2-1)b^n}$$

and

$$b_n(x) := \frac{n}{2} + \frac{6b-3b^{1-n}}{2-2b^2} + \sqrt{5}\frac{\sqrt{55b^2-360b^{n+2}-90b^{2n+2}+nb^{2n}(720b^2x^2-(b^2-1)(13+3b^2))}}{10(b^2-1)b^n}.$$

Therefore (at least for n large enough)

$$\frac{\#\left\{\boldsymbol{\sigma} \in \{id,\tau\}^n \,:\, L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) \leq x\sqrt{n}\right\}}{2^n} = \frac{1}{2^n}\sum_{a_n(x)\leq l\leq b_n(x)} \binom{n}{l}.$$

For $x \geq \sqrt{\frac{(b^2-1)(13+3b^2)}{720b^2}}$ we have

$$\lim_{n\to\infty} \frac{a_n(x)-\frac{n}{2}}{\sqrt{\frac{n}{4}}} = -\sqrt{\frac{720b^2x^2-(b^2-1)(13+3b^2)}{5(b^2-1)^2}}$$

and

$$\lim_{n\to\infty} \frac{b_n(x)-\frac{n}{2}}{\sqrt{\frac{n}{4}}} = +\sqrt{\frac{720b^2x^2-(b^2-1)(13+3b^2)}{5(b^2-1)^2}}$$

and the result follows from the central limit theorem. □

4 Preliminary results

In this section we will provide all necessary tools for the proofs of the presented theorems. As already mentioned they are based on a detailed study of the discrepancy function of generalized van der Corput sequences in [2, 4, 6]. In the following Lemma 1 we will show the basic connection. Before we can state this result we have to introduce some notations.

Like for plane sets, we first define the discrepancy function for (finite or infinite) one-dimensional sequences $X = (x_M)$. Let $N \geq 1$ be an integer and $[\alpha, \beta)$ a sub-interval of $[0,1]$. Then the discrepancy function is the difference $E([\alpha, \beta); N; X) = A([\alpha, \beta); N; X) - N(\beta - \alpha)$ where $A([\alpha, \beta); N; X)$ is the number of indices M $(1 \leq M \leq N)$ such that $x_M \in [\alpha, \beta)$. When $\alpha = 0$, we write $E([0, \beta); N; X) =: E(\beta, N, X)$.

Let σ be a permutation of $\{0, 1, \ldots, b-1\}$ and let $\mathcal{Z}_b^\sigma = (\sigma(0)/b, \sigma(1)/b, \ldots, \sigma(b-1)/b)$. For $h \in \{0, 1, \ldots, b-1\}$ and $x \in [(k-1)/b, k/b)$ where $k \in \{1, \ldots, b\}$ we define

$$\varphi_{b,h}^\sigma(x) = \begin{cases} A([0, h/b); k; \mathcal{Z}_b^\sigma) - hx & \text{if } 0 \leq h \leq \sigma(k-1), \\ (b-h)x - A([h/b, 1); k; \mathcal{Z}_b^\sigma) & \text{if } \sigma(k-1) < h < b. \end{cases}$$

Further, the function $\varphi_{b,h}^\sigma$ is extended to the reals by periodicity. Note that $\varphi_{b,0}^\sigma = 0$ for any σ and that $\varphi_{b,h}^\sigma(0) = 0$ for any σ and any h.

These functions are the basic tool for the study of generalized van der Corput sequences.

For $\sigma = id$ we will simply write $\varphi_{b,h}$ instead of $\varphi_{b,h}^{id}$. In this case we have

$$\varphi_{b,h}(x) = \begin{cases} (b-h)x & \text{if } x \in [0, h/b], \\ h(1-x) & \text{if } x \in [h/b, 1]. \end{cases} \tag{5}$$

Lemma 1 *For integers $1 \leq \lambda, N \leq b^n$ we have*

$$E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right) = \sum_{j=1}^{n} \varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right),$$

where the $\varepsilon_j = \varepsilon_j(\lambda, n, N)$ are given as follows:

For $1 \leq \lambda < b^n$ with b-adic expansion $\lambda = \lambda_1 b^{n-1} + \lambda_2 b^{n-2} + \cdots + \lambda_{n-1} b + \lambda_n$, we define

$$\Lambda_{j-1} = \lambda_j b^{n-j} + \cdots + \lambda_n.$$

Hence $\lambda = \lambda_1 b^{n-1} + \cdots + \lambda_{j-1} b^{n-j+1} + \Lambda_{j-1}$.

Then, for $1 \leq N < b^n$ with b-adic expansion $N = N_{n-1} b^{n-1} + \cdots + N_0$, we define

$$\nu_j = \sigma_j(N_j) b^{n-j-1} + \cdots + \sigma_{n-2}(N_{n-2}) b + \sigma_{n-1}(N_{n-1}).$$

Now we set $\varepsilon_n = \lambda_n$ and for fixed $1 \leq j \leq n-1$ we set

$$\varepsilon_j = \begin{cases} 0 & \text{if } 0 \leq \Lambda_{j-1} \leq \nu_j, \\ h & \text{if } \nu_j + (h-1) b^{n-j} < \Lambda_{j-1} \leq \nu_j + h b^{n-j}, \text{ for } 1 \leq h < b, \\ 0 & \text{if } \nu_j + (b-1) b^{n-j} < \Lambda_{j-1} < b^{n-j+1}. \end{cases}$$

For $\lambda = b^n$ we set $\varepsilon_j = 0$ for all $1 \leq j \leq n$ and also for $N = b^n$.

Proof. A point $\left(S_b^\Sigma(M), \frac{M-1}{b^n}\right)$ of the set $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ belongs to the two-dimensional interval $[0, \lambda/b^n) \times [0, N/b^n)$ if and only if

$$0 \leq S_b^\Sigma(M) < \lambda/b^n \quad \text{and} \quad M \in \{1, \ldots, N\},$$

(recall from the definition of $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ that $\Sigma = (\sigma_0, \ldots, \sigma_{n-1}, id, id, \ldots)$ and $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n-1})$). Hence $A([0, \lambda/b^n) \times [0, N/b^n), \mathcal{H}_{b,n}^{\boldsymbol{\sigma}})$ is exactly the number of elements among the first N elements of S_b^Σ which belong to the one-dimensional interval $[0, \lambda/b^n)$ i.e., from the notations above, $A([0, \lambda/b^n); N; S_b^\Sigma) =: A(\lambda/b^n, N, S_b^\Sigma)$. Now we have

$$\begin{aligned} E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right) &= A(\lambda/b^n, N, S_b^\Sigma) - b^n \frac{\lambda}{b^n}\frac{N}{b^n} \\ &= A(\lambda/b^n, N, S_b^\Sigma) - N\frac{\lambda}{b^n} = E\left(\frac{\lambda}{b^n}, N, S_b^\Sigma\right), \end{aligned}$$

the discrepancy function of the first N elements of the generalized van der Corput sequence in base b. It was shown in [2, Lemme 5.2] that

$$E\left(\frac{\lambda}{b^n}, N, S_b^\Sigma\right) = \sum_{j=1}^{n} \varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)$$

and that the ε_j are as in the statement of the lemma (see [2, Lemme 5.3]). See also [6, Lemma 6.2 and Lemma 6.3].

For $\lambda = b^n$ the formula is trivially true with $\varepsilon_j = 0$ for all $1 \leq j \leq n$ (since $E\left(1, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right) = N - N = 0$) and for $N = b^n$ also since again both terms are naught ($E\left(\frac{\lambda}{b^n}, 1, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right) = \lambda - \lambda = 0$ and the functions φ are zero on the integers). $\square$

Remark 3 Let $0 \leq x, y \leq 1$ be arbitrary. Since all points from $\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}$ have coordinates of the form α/b^n for some $\alpha \in \{0, 1, \ldots, b^n - 1\}$, we have

$$E(x, y, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) = E(x(n), y(n), \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}) + b^n(x(n)y(n) - xy),$$

where for $0 \leq x \leq 1$ we define $x(n) := \min\{\alpha/b^n \geq x \,:\, \alpha \in \{0, \ldots, b^n\}\}$.

Remark 4 It follows from (5), Lemma 1 and Remark 3 that for the classical Hammersley point set for all $0 \leq x, y \leq 1$ we have

$$E(x, y, \mathcal{H}_{b,n}) \geq 0.$$

Lemma 2 *For $1 \leq N \leq b^n$, $1 \leq j_1 < j_2 < \ldots < j_k \leq n$ and $r_1, \ldots, r_k \in \mathbb{N}$ we have*

$$\sum_{\lambda=1}^{b^n} \left(\varphi_{b,\varepsilon_{j_1}}^{\sigma_{j_1}}\left(\frac{N}{b^{j_1}}\right)\right)^{r_1} \cdots \left(\varphi_{b,\varepsilon_{j_k}}^{\sigma_{j_k}}\left(\frac{N}{b^{j_k}}\right)\right)^{r_k} = b^{n-k} \varphi_b^{\sigma_{j_1},(r_1)}\left(\frac{N}{b^{j_1}}\right) \cdots \varphi_b^{\sigma_{j_k},(r_k)}\left(\frac{N}{b^{j_k}}\right),$$

where $\varphi_b^{\sigma,(r)} := \sum_{h=0}^{b-1} (\varphi_{b,h}^{\sigma})^r$.

Proof. We have

$$\begin{aligned} &\sum_{\lambda=1}^{b^n} \left(\varphi_{b,\varepsilon_{j_1}(\lambda)}^{\sigma_{j_1}}\left(\frac{N}{b^{j_1}}\right)\right)^{r_1} \cdots \left(\varphi_{b,\varepsilon_{j_k}(\lambda)}^{\sigma_{j_k}}\left(\frac{N}{b^{j_k}}\right)\right)^{r_k} \\ &= \sum_{h_1,\ldots,h_k=0}^{b-1} \left(\varphi_{b,h_1}^{\sigma_{j_1}}\left(\frac{N}{b^{j_1}}\right)\right)^{r_1} \cdots \left(\varphi_{b,h_k}^{\sigma_{j_k}}\left(\frac{N}{b^{j_k}}\right)\right)^{r_k} \sum_{\substack{\lambda=1 \\ \{\lambda\,;\,\varepsilon_{j_l}(\lambda)=h_l\,,\,1\leq l\leq k\}}}^{b^n} 1. \end{aligned}$$

Hence it remains to show that, for given $h_1, \ldots, h_k \in \{0, \ldots, b-1\}$ we have

$$\sum_{\substack{\lambda=1 \\ \{\lambda\,;\,\varepsilon_{j_l}(\lambda)=h_l\,,\,1\leq l\leq k\}}}^{b^n} 1 = b^{n-k}.$$

This will be shown by induction on k. For $k = 1$ and 2, the proof is given in [2, 5.4] and [6, 6.5]. For the sake of completeness we give the whole proof.

Let $k = 1$ (for short we will write j instead of j_1 and h instead of h_1). For a fixed j, the set $\{1, \ldots, b^n\}$ of λ's is divided into b^{n-j+1} classes, each with b^{j-1} elements, where Λ_{j-1} is constant (see Lemma 1). We have to count the number of λ's for which we have $\varepsilon_j(\lambda) = h$ i.e the number of λ's for which we have $\nu_j + (h-1)b^{n-j} < \Lambda_{j-1} \leq \nu_j + hb^{n-j}$ in the case of $h \neq 0$ and $0 \leq \Lambda_{j-1} \leq \nu_j$ or $\nu_j + (b-1)b^{n-j} < \Lambda_{j-1} < b^{n-j+1}$ in the case of $h = 0$. Hence, for Λ_{j-1} we have b^{n-j} possibilities and for the digits $\lambda_1, \ldots, \lambda_{j-1}$ we have b^{j-1} possible choices. Therefore, the number of $\lambda \in \{1, \ldots, b^n\}$ for which $\varepsilon_j(\lambda) = h$ is given by $b^{n-j}b^{j-1} = b^{n-1}$ independent of the value of h.

Let $k = 2$ (for short, we write j' for j_1 and j for j_2). For fixed $j' < j$, each class above is itself divided into $b^{j-1-j'}$ subclasses with $b^{j'-1}$ elements each, where $\Lambda_{j'-1}$ is constant. Thus for any h and $h' \in \{0, \ldots, b-1\}$, we have $\varepsilon_j(\lambda) = h$ and $\varepsilon_{j'}(\lambda) = h'$ for $b^{n-j}b^{j-1-j'}b^{j'-1} = b^{n-2}$ values of λ, independent of the values of h and h', i.e we have

$$\sum_{\substack{\lambda=1 \\ \{\lambda\,;\,\varepsilon_{j_l}(\lambda)=h_l\,,\,1\leq l\leq 2\}}}^{b^n} 1 = b^{n-2}.$$

$k-1 \to k$: Continuing step by step the decomposition into subclasses as before, from j_k to j_2, we finally divide the subclasses corresponding to j_2 into $b^{j_2-1-j_1}$ more subclasses, each with b^{j_1-1} elements, where Λ_{j_1-1} is constant. Since this decomposition is independent of the values taken by $\varepsilon_{j_1}(\lambda)$ (see Lemma 1), we have

$$\sum_{\substack{\lambda=1 \\ \{\lambda\,;\,\varepsilon_{j_l}(\lambda)=h_l\,,\,2\leq l\leq k\} \\ \varepsilon_{j_1}(\lambda)=u}}^{b^n} 1 = \sum_{\substack{\lambda=1 \\ \{\lambda\,;\,\varepsilon_{j_l}(\lambda)=h_l\,,\,2\leq l\leq k\} \\ \varepsilon_{j_1}(\lambda)=v}}^{b^n} 1.$$

Therefore, by the induction hypothesis, we obtain the formula we looked for:

$$b^{n-k+1} = \sum_{\substack{\lambda=1 \\ \{\lambda\,;\,\varepsilon_{j_l}(\lambda)=h_l\,,\,2\leq l\leq k\}}}^{b^n} 1 = \sum_{u=0}^{b-1} \sum_{\substack{\lambda=1 \\ \{\lambda\,;\,\varepsilon_{j_l}(\lambda)=h_l\,,\,2\leq l\leq k\} \\ \varepsilon_{j_1}(\lambda)=u}}^{b^n} 1 = b \sum_{\substack{\lambda=1 \\ \{\lambda\,;\,\varepsilon_{j_l}(\lambda)=h_l\,,\,2\leq l\leq k\} \\ \varepsilon_{j_1}(\lambda)=h_1}}^{b^n} 1.$$

Ending the proof of Lemma 2 is then straightforward with the definition of $\varphi_b^{\sigma,(r)}$. □

Remark 5 For $r = 1$ we will write φ_b^σ instead of $\varphi_b^{\sigma,(1)}$ for simplicity, and if $\sigma = id$ we will write $\varphi_b^{(r)}$ instead of $\varphi_b^{id,(r)}$. For $r = 2$ the function $\varphi_b^{\sigma,(2)}$ is the same as the function ϕ_b^σ from [2] and [6].

Lemma 3 *For $x \in \left[\frac{k}{b}, \frac{k+1}{b}\right]$, $0 \le k \le b-1$, we have*

$$\varphi_b(x) = \frac{b(b-2k-1)}{2}\left(x - \frac{k}{b}\right) + \frac{k(b-k)}{2}$$

and

$$\varphi_b^{(2)}(x) = (1-x)^2\frac{k(k+1)(2k+1)}{6} + x^2\frac{(b-k)(b-k-1)(2b-2k-1)}{6}.$$

Proof. Let $x \in [k/b, (k+1)/b]$. Then we have $x \in [0, h/b]$ for $h \in \{k+1, \ldots, b-1\}$ and $x \in [h/b, 1]$ for $h \in \{0, \ldots, k\}$. Hence

$$\varphi_b(x) = \sum_{h=0}^{k} h(1-x) + \sum_{h=k+1}^{b-1}(b-h)x = \frac{b(b-2k-1)}{2}\left(x - \frac{k}{b}\right) + \frac{k(b-k)}{2}$$

and

$$\begin{aligned}\varphi_b^{(2)}(x) &= \sum_{h=0}^{k} h^2(1-x)^2 + \sum_{h=k+1}^{b-1}(b-h)^2x^2 \\ &= (1-x)^2\frac{k(k+1)(2k+1)}{6} + x^2\frac{(b-k)(b-k-1)(2b-2k-1)}{6}.\end{aligned}$$

□

Lemma 4 *For any $h \in \{0, \ldots, b-1\}$ we have $\varphi_{b,h}^{\tau} = -\varphi_{b,b-h}^{id}$. Furthermore we have $\varphi_b^{id,(r)} = (-1)^r \varphi_b^{\tau,(r)}$.*

Proof. First we note that $\mathcal{Z}_b^{\tau} = \left(\frac{b-1}{b}, \ldots, \frac{1}{b}, 0\right)$. Hence, for the counting function $A(Y; k; \mathcal{Z}_b^{\tau})$ we have

$$A(Y; k; \mathcal{Z}_b^{\tau}) = \begin{cases} 0 & \text{if } Y = \left[0, \frac{h}{b}\right) \text{ and } 0 \le h \le b-k, \\ b-h & \text{if } Y = \left[\frac{h}{b}, 1\right) \text{ and } b-k < h < b. \end{cases}$$

Therefore

$$\varphi_{b,h}^{\tau}(x) = \begin{cases} -hx & \text{if } x \in \left[0, \frac{b-h}{b}\right], \\ (b-h)x - (b-h) & \text{if } x \in \left[\frac{b-h}{b}, 1\right], \end{cases}$$

and the first result follows from a comparison with Eq. (5). The second result follows easily from the first one (note that $\varphi_{b,0}^{\sigma} = 0$ for any σ). □

Lemma 5 *1. For $1 \le j_1 < j_2 < \ldots < j_k \le n$ and $r_1, \ldots, r_k \in \mathbb{N}$ we have*

$$\sum_{N=1}^{b^n} \varphi_b^{(r_1)}\left(\frac{N}{b^{j_1}}\right) \cdots \varphi_b^{(r_k)}\left(\frac{N}{b^{j_k}}\right) \le \left(\frac{b(b^2-1)}{12}\right)^{r_1+\cdots+r_k} b^{n-k}$$

with equality if $r_1 = \cdots = r_k = 1$.

2. We have

$$\sum_{N=1}^{b^n} \varphi_b^{(2)}\left(\frac{N}{b^j}\right) = b^n(b^2-1)\frac{5b^2 + 2b^{2j} + 2b^{2(j+1)}}{180b^{2j+1}}.$$

Proof. 1. We show this part of the lemma by induction on k.

Let $k=1$ (for short we will write r instead of r_1 and j instead of j_1). Let $N = N_0 + N_1 b + \cdots + N_{n-1} b^{n-1}$ be the b-adic expansion of $N \in \{0, \ldots, b^n - 1\}$. Then we have (recall that $\varphi_b^{(r)}(0) = 0$ and $\varphi_b^{(r)}$ is 1-periodic)

$$\begin{aligned}
\sum_{N=1}^{b^n} \varphi_b^{(r)}\left(\frac{N}{b^j}\right) &= \sum_{N=0}^{b^n-1} \varphi_b^{(r)}\left(\frac{N}{b^j}\right) \\
&= \sum_{N_0,\ldots,N_{n-1}=0}^{b-1} \varphi_b^{(r)}\left(\frac{N_0 + \cdots + N_{n-1}b^{n-1}}{b^j}\right) \\
&= b^{n-j} \sum_{N_0,\ldots,N_{j-1}=0}^{b-1} \varphi_b^{(r)}\left(\frac{N_0 + \cdots + N_{j-1}b^{j-1}}{b^j}\right) \\
&= b^{n-j} \sum_{N_0,\ldots,N_{j-2}=0}^{b-1} \sum_{l=0}^{b-1} \varphi_b^{(r)}\left(\frac{l}{b} + \frac{N_0 + \cdots + N_{j-2}b^{j-2}}{b^j}\right) \\
&\leq b^{n-j} \sum_{N_0,\ldots,N_{j-2}=0}^{b-1} \left(\sum_{l=0}^{b-1} \varphi_b\left(\frac{l}{b} + \frac{N_0 + \cdots + N_{j-2}b^{j-2}}{b^j}\right)\right)^r,
\end{aligned}$$

with equality if $r=1$. For any $N_0, \ldots, N_{j-2} \in \{0, \ldots, b-1\}$ we have $\frac{l}{b} + \frac{N_0 + \cdots + N_{j-2}b^{j-2}}{b^j} \in [l/b, (l+1)/b]$ and hence, with Lemma 3 and $x := \frac{N_0 + \cdots + N_{j-2}b^{j-2}}{b^j}$,

$$\sum_{l=0}^{b-1} \varphi_b\left(\frac{l}{b} + x\right) = \sum_{l=0}^{b-1} \left(\frac{b(b-2l-1)}{2}x + \frac{l(b-l)}{2}\right) = \frac{b(b^2-1)}{12}.$$

Therefore we have

$$\sum_{N=1}^{b^n} \varphi_b^{(r)}\left(\frac{N}{b^j}\right) \leq \left(\frac{b(b^2-1)}{12}\right)^r b^{n-1}$$

with equality if $r=1$.

$k-1 \to k$: We have

$$\begin{aligned}
&\sum_{N=1}^{b^n} \varphi_b^{(r_1)}\left(\frac{N}{b^{j_1}}\right) \cdots \varphi_b^{(r_k)}\left(\frac{N}{b^{j_k}}\right) = \sum_{N=0}^{b^n-1} \varphi_b^{(r_1)}\left(\frac{N}{b^{j_1}}\right) \cdots \varphi_b^{(r_k)}\left(\frac{N}{b^{j_k}}\right) \\
&= b^{n-j_k} \sum_{N_0,\ldots,N_{j_k-1}=0}^{b-1} \varphi_b^{(r_1)}\left(\frac{N_0 + \cdots + N_{j_1-1}b^{j_1-1}}{b^{j_1}}\right) \cdots \varphi_b^{(r_k)}\left(\frac{N_0 + \cdots + N_{j_k-1}b^{j_k-1}}{b^{j_k}}\right) \\
&= b^{n-j_k} \sum_{N_0,\ldots,N_{j_k-2}=0}^{b-1} \varphi_b^{(r_1)}\left(\frac{N_0 + \cdots + N_{j_1-1}b^{j_1-1}}{b^{j_1}}\right) \cdots \varphi_b^{(r_{k-1})}\left(\frac{N_0 + \cdots + N_{j_{k-1}-1}b^{j_{k-1}-1}}{b^{j_{k-1}}}\right) \\
&\qquad \times \sum_{l=0}^{b-1} \varphi_b^{(r_k)}\left(\frac{l}{b} + \frac{N_0 + \cdots + N_{j_k-2}b^{j_k-2}}{b^{j_k}}\right).
\end{aligned}$$

As above we have

$$\sum_{l=0}^{b-1} \varphi_b^{(r_k)} \left(\frac{l}{b} + \frac{N_0 + \cdots + N_{j_k-2} b^{j_k-2}}{b^{j_k}} \right) \le \left(\frac{b(b^2-1)}{12} \right)^{r_k}$$

with equality if $r_k = 1$. Hence

$$\begin{aligned}
&\sum_{N=1}^{b^n} \varphi_b^{(r_1)} \left(\frac{N}{b^{j_1}} \right) \cdots \varphi_b^{(r_k)} \left(\frac{N}{b^{j_k}} \right) \\
&\quad \le \; b^{n-j_k} \left(\frac{b(b^2-1)}{12} \right)^{r_k} \\
&\qquad \times \sum_{N_0,\ldots,N_{j_k-2}=0}^{b-1} \varphi_b^{(r_1)} \left(\frac{N_0 + \cdots + N_{j_1-1} b^{j_1-1}}{b^{j_1}} \right) \cdots \varphi_b^{(r_{k-1})} \left(\frac{N_0 + \cdots + N_{j_{k-1}-1} b^{j_{k-1}-1}}{b^{j_{k-1}}} \right) \\
&\quad = \; b^{-1} \left(\frac{b(b^2-1)}{12} \right)^{r_k} \sum_{N=0}^{b^n-1} \varphi_b^{(r_1)} \left(\frac{N}{b^{j_1}} \right) \cdots \varphi_b^{(r_{k-1})} \left(\frac{N}{b^{j_{k-1}}} \right).
\end{aligned}$$

Now the result follows by using the induction hypothesis.

2. By using the periodicity of $\varphi_b^{(2)}$ we obtain

$$\sum_{N=1}^{b^n} \varphi_b^{(2)} \left(\frac{N}{b^j} \right) = b^{n-j} \sum_{N=1}^{b^j} \varphi_b^{(2)} \left(\frac{N}{b^j} \right) = b^{n-j} \sum_{k=0}^{b-1} \sum_{N=kb^{j-1}+1}^{(k+1)b^{j-1}} \varphi_b^{(2)} \left(\frac{N}{b^j} \right).$$

For $kb^{j-1} + 1 \le N \le (k+1)b^{j-1}$ we have $k/b < N/b^j \le (k+1)/b$ and hence we can use Lemma 3 to obtain

$$\begin{aligned}
\sum_{N=1}^{b^n} \varphi_b^{(2)} \left(\frac{N}{b^j} \right) &= b^{n-j} \sum_{k=0}^{b-1} \sum_{N=kb^{j-1}+1}^{(k+1)b^{j-1}} \left[\left(1 - \frac{N}{b^j} \right)^2 \frac{k(k+1)(2k+1)}{6} \right. \\
&\qquad \left. + \left(\frac{N}{b^j} \right)^2 \frac{(b-k)(b-k-1)(2b-2k-1)}{6} \right] \\
&= b^n (b^2-1) \frac{5b^2 + 2b^{2j} + 2b^{2(j+1)}}{180 b^{2j+1}}.
\end{aligned}$$

□

5 The proof of Theorem 1

The following key-lemma can be considered as a discrete version of Theorem 1.

Lemma 6 *We have*

$$\frac{1}{b^{2n}} \sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n} \right) = n \frac{b^2-1}{12b} \tag{6}$$

and

$$\frac{1}{b^{2n}} \sum_{\lambda,N=1}^{b^n} \left(E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n} \right) \right)^2 = n^2 \left(\frac{b^2-1}{12b} \right)^2 + n \frac{3b^4 + 10b^2 - 13}{720 b^2} + \frac{1}{36} - \frac{1}{36 b^{2n}}. \tag{7}$$

Proof. We have

$$\begin{aligned}\sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}\right) &= \sum_{j=1}^{n}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_j}\left(\frac{N}{b^j}\right) = b^{n-1}\sum_{j=1}^{n}\sum_{N=1}^{b^n}\varphi_b\left(\frac{N}{b^j}\right)\\ &= b^{n-1}\sum_{j=1}^{n}\frac{b(b^2-1)}{12}b^{n-1} = b^{2n}n\frac{b^2-1}{12b},\end{aligned}$$

where we used Lemma 1, Lemma 2 and Lemma 5. Thus Eq. (6) follows.

Eq. (7) is a special case of the second assertion in the subsequent Lemma 9 (just choose $l=n$) and so we omit a direct proof at this place. □

Now we can give the proof of Theorem 1.

Proof. Using Remark 4, Remark 3 and Lemma 6 we have

$$\begin{aligned}L_1(\mathcal{H}_{b,n}) &= \int_0^1\int_0^1 E(x,y,\mathcal{H}_{b,n})\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^1\int_0^1 E(x(n),y(n),\mathcal{H}_{b,n})+b^n(x(n)y(n)-xy)\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}\right)+\frac{1}{b^n}\sum_{\lambda,N=1}^{b^n}\frac{\lambda}{b^n}\frac{N}{b^n}-b^n\int_0^1 x\,\mathrm{d}x\int_0^1 y\,\mathrm{d}y\\ &= n\frac{b^2-1}{12b}+\frac{1}{2}+\frac{1}{4b^n}.\end{aligned}$$

The result for the L_2 discrepancy is a special case of Theorem 4 (just choose $l=n$) and so we omit a direct proof at this place. □

6 The proof of Theorem 2

Again, we provide a discrete version of Theorem 2.

Lemma 7 *For any $p\in\mathbb{N}$ we have*

$$\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}\right)\right)^p = n^p\left(\frac{b^2-1}{12b}\right)^p+O(n^{p-1}).$$

Proof. With Lemma 1 we have

$$\begin{aligned}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}\right)\right)^p &= \sum_{\lambda,N=1}^{b^n}\prod_{u=1}^{p}\sum_{j_u=1}^{n}\varphi_{b,\varepsilon_{j_u}}\left(\frac{N}{b^{j_u}}\right)\\ &= \sum_{j_1,\ldots,j_p=1}^{n}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_{j_1}}\left(\frac{N}{b^{j_1}}\right)\cdots\varphi_{b,\varepsilon_{j_p}}\left(\frac{N}{b^{j_p}}\right).\end{aligned}$$

For $(j_1,\dots,j_p)\in\{1,\dots,n\}^p$ let $v_1,\dots,v_k$ be the different j_u's, where $k=k(j_1,\dots,j_p)$, such that v_1 appears r_1 times, $\dots$, v_k appears r_k times. With Lemma 2 and Lemma 5 and invoking the fact that $r_1+\dots+r_k=p$ we obtain

$$\begin{aligned}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_{j_1}}\left(\frac{N}{b^{j_1}}\right)\cdots\varphi_{b,\varepsilon_{j_p}}\left(\frac{N}{b^{j_p}}\right) &= \sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_{v_1}}^{r_1}\left(\frac{N}{b^{v_1}}\right)\cdots\varphi_{b,\varepsilon_{v_k}}^{r_k}\left(\frac{N}{b^{v_k}}\right)\\ &= b^{n-k}\sum_{N=1}^{b^n}\varphi_b^{(r_1)}\left(\frac{N}{b^{v_1}}\right)\cdots\varphi_b^{(r_k)}\left(\frac{N}{b^{v_k}}\right)\\ &\le b^{2n-2k}\left(\frac{b(b^2-1)}{12}\right)^p.\end{aligned}$$

Hence

$$\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}\right)\right)^p\le\left(\frac{b(b^2-1)}{12}\right)^p b^{2n}\sum_{j_1,\dots,j_p=1}^{n}\frac{1}{b^{2k(j_1,\dots,j_p)}}. \tag{8}$$

Let us denote the number of tuples $(j_1,\dots,j_p)\in\{1,\dots,n\}^p$ such that k different j_u's occur by $\#(p,k,n)$. This is the number of mappings from $\{1,\dots,p\}$ to $\{1,\dots,n\}$ such that the range has cardinality k. It is well known from combinatorics that the $\#(p,k,n)$ are closely related to Stirling numbers of the second kind, $S(p,k)$, via

$$\#(p,k,n)=k!\binom{n}{k}S(p,k).$$

This follows easily from the fact that the number of surjective mappings from $\{1,\dots,p\}$ to $\{1,\dots,k\}$ is given by $k!S(p,k)$. (Recall that for $p,k\in\mathbb{N}$ the Stirling number $S(p,k)$ of the second kind is defined as the number of partitions of an p-element set into k non-empty subsets (hence $S(p,p)=1$). A well known formula states that $S(p,k)=\frac{1}{k!}\sum_{r=0}^{k-1}(-1)^r\binom{k}{r}(k-r)^p$.)

Therefore we obtain

$$\begin{aligned}\sum_{j_1,\dots,j_p=1}^{n}\frac{1}{b^{2k(j_1,\dots,j_p)}} &= \sum_{k=0}^{p}\frac{1}{b^{2k}}\#(p,k,n)\\ &= \sum_{k=0}^{p}\frac{1}{b^{2k}}k!\binom{n}{k}S(p,k)\\ &= \frac{1}{b^{2p}}p!\binom{n}{p}S(p,p)+\sum_{k=0}^{p-1}\frac{1}{b^{2k}}k!\binom{n}{k}S(p,k)\\ &\le \frac{1}{b^{2p}}(n(n-1)\cdots(n-p+1))+(n(n-1)\cdots(n-p+2))c_b(p)\\ &= \frac{n^p}{b^{2p}}+O(n^{p-1}),\end{aligned}$$

where $c_b(p):=\sum_{k=0}^{p-1}\frac{1}{b^{2k}}S(p,k)$. Inserting this bound in (8) leads to

$$\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}\right)\right)^p\le n^p\left(\frac{b^2-1}{12b}\right)^p+O(n^{p-1}). \tag{9}$$

On the other hand, using Lemma 2 and Lemma 5, we have

$$\begin{aligned}
\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}\right)\right)^p &\geq \frac{1}{b^{2n}}\sum_{\substack{j_1,\ldots,j_p\\ j_u\neq j_v}}^{n}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_{j_1}}\left(\frac{N}{b^{j_1}}\right)\cdots\varphi_{b,\varepsilon_{j_p}}\left(\frac{N}{b^{j_p}}\right)\\
&= \frac{1}{b^{2n}}\sum_{\substack{j_1,\ldots,j_p\\ j_u\neq j_v}}^{n}\sum_{N=1}^{b^n} b^{n-p}\varphi_b\left(\frac{N}{b^{j_1}}\right)\cdots\varphi_b\left(\frac{N}{b^{j_p}}\right)\\
&= \frac{1}{b^{2n}}\sum_{\substack{j_1,\ldots,j_p\\ j_u\neq j_v}}^{n} b^{2n-2p}\left(\frac{b(b^2-1)}{12}\right)^p\\
&= \left(\frac{b^2-1}{12b}\right)^p \#(p,p,n)\\
&= \left(\frac{b^2-1}{12b}\right)^p p!\binom{n}{p}S(p,p)\\
&= n^p\left(\frac{b^2-1}{12b}\right)^p + O(n^{p-1}). \qquad (10)
\end{aligned}$$

Now the result follows from (9) and (10). □

Now we can give the proof of Theorem 2.

Proof. From Lemma 1, Remark 3 and Remark 4 we obtain

$$\begin{aligned}
(L_p(\mathcal{H}_{b,n}))^p &= \int_0^1\int_0^1 (E(x,y,\mathcal{H}_{b,n}))^p \,\mathrm{d}x\,\mathrm{d}y\\
&= \int_0^1\int_0^1 (E(x(n),y(n),\mathcal{H}_{b,n}) + b^n(x(n)y(n)-xy))^p \,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

With Lemma 7 we find

$$\begin{aligned}
(L_p(\mathcal{H}_{b,n}))^p &\geq \int_0^1\int_0^1 (E(x(n),y(n),\mathcal{H}_{b,n}))^p \,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}\right)\right)^p = n^p\left(\frac{b^2-1}{12b}\right)^p + O(n^{p-1}). \qquad (11)
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
L_p(\mathcal{H}_{b,n}))^p &= \int_0^1\int_0^1\sum_{k=0}^{p}\binom{p}{k}(E(x(n),y(n),\mathcal{H}_{b,n}))^k\cdot(b^n(x(n)y(n)-xy))^{p-k}\,\mathrm{d}x\,\mathrm{d}y\\
&\leq \int_0^1\int_0^1 (E(x(n),y(n),\mathcal{H}_{b,n}))^p \,\mathrm{d}x\,\mathrm{d}y\\
&\quad + \int_0^1\int_0^1\sum_{k=0}^{p-1}\binom{p}{k}(E(x(n),y(n),\mathcal{H}_{b,n}))^k\cdot 2^{p-k}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

where we used the fact $0 \le x(n)y(n) - xy \le \frac{2}{b^n} - \frac{1}{b^{2n}}$. With Lemma 7 we obtain

$$\begin{aligned} L_p(\mathcal{H}_{b,n}))^p &\le n^p \left(\frac{b^2-1}{12b}\right)^p + O(n^{p-1}) + \sum_{k=0}^{p-1} \binom{p}{k} \left(n^k \left(\frac{b^2-1}{12b}\right)^k + O(n^{k-1})\right) 2^{p-k} \\ &= n^p \left(\frac{b^2-1}{12b}\right)^p + O(n^{p-1}). \end{aligned} \tag{12}$$

Now the result follows from (11) and (12). $\square$

7 The proof of Theorem 3

The following lemma is a discrete version of Theorem 3.

Lemma 8 *Let $p \in \mathbb{N}$. Then we have*

$$\frac{1}{2^n} \sum_{\boldsymbol{\sigma} \in \{id,\tau\}^n} \frac{1}{b^{2n}} \sum_{\lambda,N=1}^{b^n} \left(E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^p = 0$$

for odd p, and

$$\frac{1}{2^n} \sum_{\boldsymbol{\sigma} \in \{id,\tau\}^n} \frac{1}{b^{2n}} \sum_{\lambda,N=1}^{b^n} \left(E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^p \le \left(\frac{b^2-1}{12}\right)^p \frac{p!}{(p/2)!\, 2^{p/2}} n^{p/2} + O(n^{p/2-1}),$$

for even p.

Proof. We have

$$\begin{aligned} &\frac{1}{2^n} \sum_{\boldsymbol{\sigma} \in \{id,\tau\}^n} \frac{1}{b^{2n}} \sum_{\lambda,N=1}^{b^n} \left(E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^p \\ &= \frac{1}{2^n} \sum_{(\sigma_0,\ldots,\sigma_{n-1}) \in \{id,\tau\}^n} \frac{1}{b^{2n}} \sum_{\lambda,N=1}^{b^n} \sum_{j_1,\ldots,j_p=1}^{n} \varphi_{b,\varepsilon_{j_1}}^{\sigma_{j_1-1}}\left(\frac{N}{b^{j_1}}\right) \cdots \varphi_{b,\varepsilon_{j_p}}^{\sigma_{j_p-1}}\left(\frac{N}{b^{j_p}}\right) \\ &= \frac{1}{b^{2n}} \sum_{j_1,\ldots,j_p=1}^{n} \frac{1}{2^n} \sum_{(\sigma_0,\ldots,\sigma_{n-1}) \in \{id,\tau\}^n} \sum_{N=1}^{b^n} \sum_{\lambda=1}^{b^n} \varphi_{b,\varepsilon_{j_1}}^{\sigma_{j_1-1}}\left(\frac{N}{b^{j_1}}\right) \cdots \varphi_{b,\varepsilon_{j_p}}^{\sigma_{j_p-1}}\left(\frac{N}{b^{j_p}}\right). \end{aligned}$$

For $(j_1,\ldots,j_p) \in \{1,\ldots,n\}^p$ define $v_1,\ldots,v_k$, $k = k(j_1,\ldots,j_p)$ and $r_1,\ldots,r_k$ as in the proof of Lemma 7. Then, using Lemma 2, we obtain

$$\begin{aligned} \sum_{\lambda=1}^{b^n} \varphi_{b,\varepsilon_{j_1}}^{\sigma_{j_1-1}}\left(\frac{N}{b^{j_1}}\right) \cdots \varphi_{b,\varepsilon_{j_p}}^{\sigma_{j_p-1}}\left(\frac{N}{b^{j_p}}\right) &= \sum_{\lambda=1}^{b^n} \left(\varphi_{b,\varepsilon_{v_1}}^{\sigma_{v_1-1}}\left(\frac{N}{b^{v_1}}\right)\right)^{r_1} \cdots \left(\varphi_{b,\varepsilon_{v_k}}^{\sigma_{v_k-1}}\left(\frac{N}{b^{v_k}}\right)\right)^{r_k} \\ &= b^{n-k} \varphi_b^{\sigma_{v_1-1},(r_1)}\left(\frac{N}{b^{v_1}}\right) \cdots \varphi_b^{\sigma_{v_k-1},(r_k)}\left(\frac{N}{b^{v_k}}\right). \end{aligned}$$

From Lemma 4 it follows that $\varphi_b^{\tau,(r)} = (-1)^r \varphi_b^{(r)}$. For $1 \le i \le n$ we define

$$s_i := \begin{cases} 1 & \text{if } \sigma_{i-1} = \tau \\ 0 & \text{if } \sigma_{i-1} = id. \end{cases}$$

Then we have

$$b^{n-k}\varphi_b^{\sigma_{v_1-1},(r_1)}\left(\frac{N}{b^{v_1}}\right)\cdots\varphi_b^{\sigma_{v_k-1},(r_k)}\left(\frac{N}{b^{v_k}}\right)=b^{n-k}(-1)^{s_{v_1}r_1+\cdots+s_{v_k}r_k}\varphi_b^{(r_1)}\left(\frac{N}{b^{v_1}}\right)\cdots\varphi_b^{(r_k)}\left(\frac{N}{b^{v_k}}\right).$$

We have

$$\frac{1}{2^n}\sum_{(\sigma_0,\ldots,\sigma_{n-1})\in\{id,\tau\}^n}(-1)^{s_{v_1}r_1+\cdots+s_{v_k}r_k}=\frac{1}{2^k}\prod_{i=1}^{k}\sum_{\sigma_{v_i-1}\in\{id,\tau\}}(-1)^{s_{v_i}r_i}$$

and for each $1\le i\le k$,

$$\sum_{\sigma_{v_i-1}\in\{id,\tau\}}(-1)^{s_{v_i}r_i}=\begin{cases}2 & \text{if } r_i\equiv 0 \pmod 2,\\ 0 & \text{if } r_i\not\equiv 0 \pmod 2.\end{cases}$$

Thus,

$$\frac{1}{2^n}\sum_{(\sigma_0,\ldots,\sigma_{n-1})\in\{id,\tau\}^n}(-1)^{s_{v_1}r_1+\cdots+s_{v_k}r_k}=f(r_1,\ldots,r_k),$$

where $f(r_1,\ldots,r_k)$ is one if r_i is even for all $i\in\{1,\ldots,k\}$ and zero otherwise.

If p is odd, it is impossible that all r_i, $i\in\{1,\ldots,k\}$ are even, since $r_1+\cdots+r_k=p$. Thus the result follows for odd p.

From now on assume that p is even. For $(j_1,\ldots,j_p)\in\{1,\ldots n\}^p$ we have

$$\begin{aligned}&\frac{1}{2^n}\sum_{(\sigma_0,\ldots,\sigma_{n-1})\in\{id,\tau\}^n}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_{j_1}}^{\sigma_{j_1-1}}\left(\frac{N}{b^{j_1}}\right)\cdots\varphi_{b,\varepsilon_{j_p}}^{\sigma_{j_p-1}}\left(\frac{N}{b^{j_p}}\right)\\ =\;& f(r_1,\ldots,r_k)\sum_{N=1}^{b^n}b^{n-k}\varphi_b^{(r_1)}\left(\frac{N}{b^{v_1}}\right)\cdots\varphi_b^{(r_k)}\left(\frac{N}{b^{v_k}}\right)\\ \le\;& f(r_1,\ldots,r_k)b^{2n-2k}\left(\frac{b(b^2-1)}{12}\right)^p,\end{aligned}$$

where we used Lemma 5 for the last inequality. Therefore we obtain

$$\begin{aligned}&\frac{1}{2^n}\sum_{\boldsymbol{\sigma}\in\{id,\tau\}^n}\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^p\\ \le\;&\frac{1}{b^{2n}}\sum_{\substack{j_1,\ldots,j_p=1\\ r_1,\ldots,r_k\equiv 0\ (\bmod\ 2)}}^{n}b^{2n-2k(j_1,\ldots,j_p)}\left(\frac{b(b^2-1)}{12}\right)^p\\ =\;&\left(\frac{b(b^2-1)}{12}\right)^p\sum_{\substack{j_1,\ldots,j_p=1\\ r_1,\ldots,r_k\equiv 0\ (\bmod\ 2)}}^{n}\frac{1}{b^{2k(j_1,\ldots,j_p)}}.\end{aligned}$$

Note that $r_1,\ldots,r_k$ can only be even if $k\le p/2$. If $k=p/2$, then the number of tuples $(j_1,\ldots,j_p)\in\{1,\ldots,n\}^p$ such that $p/2$ different j_u's occur and each of them occurs two times is given by $\frac{p!}{2^{p/2}}\binom{n}{p/2}$. (This follows from the fact that the number of mappings $f:\{1,\ldots,p\}\to\{1,\ldots,p/2\}$ such that $|f^{-1}(\{i\})|=2$ for all $i\in\{1,\ldots,p/2\}$ is given by $\frac{p!}{2^{p/2}}$.)

Again we denote the number of tuples $(j_1,\ldots,j_p)\in\{1,\ldots,n\}^p$ such that k different j_u's occur by $\#(p,k,n)$. Then we obtain

$$\begin{aligned}
&\frac{1}{2^n}\sum_{\boldsymbol{\sigma}\in\{id,\tau\}^n}\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^p\\
&\le \left(\frac{b(b^2-1)}{12}\right)^p\left(\frac{1}{b^p}\frac{p!}{2^{p/2}}\binom{n}{p/2}+\sum_{k=0}^{p/2-1}\frac{1}{b^{2k}}\#(p,k,n)\right)\\
&= \left(\frac{b^2-1}{12}\right)^p\frac{p!}{2^{p/2}}\binom{n}{p/2}+\left(\frac{b(b^2-1)}{12}\right)^p\sum_{k=0}^{p/2-1}\frac{1}{b^{2k}}k!\binom{n}{k}S(p,k)\\
&\le \left(\frac{b^2-1}{12}\right)^p n(n-1)\cdots(n-p/2+1)\frac{p!}{(p/2)!\,2^{p/2}}\\
&\quad+\left(\frac{b(b^2-1)}{12}\right)^p n(n-1)\cdots(n-p/2+2)c_b(p)\\
&= \left(\frac{b^2-1}{12}\right)^p\frac{p!}{(p/2)!\,2^{p/2}}n^{p/2}+O(n^{p/2-1}),
\end{aligned}$$

where $c_b(p)=\sum_{k=0}^{p/2-1}\frac{1}{b^{2k}}S(p,k)$. □

Now, we can give the proof of Theorem 3.

Proof. From Remark 3 it follows that for all $0\le x,y\le 1$ and all $n\in\mathbb{N}$ we have

$$E(x(n),y(n),\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\le E(x,y,\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\le E(x(n),y(n),\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})+2.$$

For even integers p, the function $z\mapsto z^p$ is convex and hence it follows that

$$\begin{aligned}
\left(E(x,y,\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^p &\le \max\left\{\left(E(x(n),y(n),\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^p,\left(E(x(n),y(n),\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})+2\right)^p\right\}\\
&\le \left(E(x(n),y(n),\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^p+\left(E(x(n),y(n),\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})+2\right)^p.
\end{aligned}$$

Now we have

$$\begin{aligned}
&(L_p(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}))^p=\int_0^1\int_0^1(E(x,y,\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}))^p\,\mathrm{d}x\,\mathrm{d}y\\
&\le \int_0^1\int_0^1\left[\left(E(x(n),y(n),\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^p+\left(E(x(n),y(n),\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})+2)\right)^p\right]\mathrm{d}x\,\mathrm{d}y\\
&= \int_0^1\int_0^1\left[2\left(E(x(n),y(n),\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^p+\sum_{l=0}^{p-1}\binom{p}{l}\left(E(x(n),y(n),\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})\right)^l2^{p-l}\right]\mathrm{d}x\,\mathrm{d}y\\
&= \frac{2}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^p+\sum_{l=0}^{p-1}\binom{p}{l}2^{p-l}\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^l.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{2^n}\sum_{\boldsymbol{\sigma}\in\{id,\tau\}^n}(L_p(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}))^p &\leq \frac{2}{2^n}\sum_{\boldsymbol{\sigma}\in\{id,\tau\}^n}\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^p \\
&\quad +\sum_{l=0}^{p-1}\binom{p}{l}2^{p-l}\left(\frac{1}{2^n}\sum_{\boldsymbol{\sigma}\in\{id,\tau\}^n}\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^l\right) \\
&\leq 2\left(\frac{b^2-1}{12}\right)^p\frac{p!}{(p/2)!\,2^{p/2}}n^{p/2}+O(n^{p/2-1})
\end{aligned}$$

by Lemma 8. $\square$

8 The proof of Theorem 4

The following lemma is a discrete version of Theorem 4.

Lemma 9 *Let $\boldsymbol{\sigma}\in\{id,\tau\}^n$ and let l to denote the number of components of $\boldsymbol{\sigma}$ which are equal to id. Then we have*

$$\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)=\frac{b^2-1}{12b}(2l-n)$$

and

$$\begin{aligned}
\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^2 &= \frac{2nb^{2(n+2)}-2nb^{2n}+5b^{2(n+1)}-5b^2}{180b^{2+2n}} \\
&\quad +\left(\frac{b^2-1}{12b}\right)^2((n-2l)^2-n).
\end{aligned}$$

Proof. We just give the (more involved) proof of the second formula. Using Lemma 1 we have

$$\begin{aligned}
\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^2 &= \frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\sum_{i,j=1}^{n}\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) \\
&= \frac{1}{b^{2n}}\sum_{i=1}^{n}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\left(\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\right)^2 \\
&\quad +\frac{1}{b^{2n}}\sum_{\substack{i,j=1\\i\neq j}}^{n}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right).
\end{aligned}$$

By Lemma 2 we have

$$\sum_{\lambda=1}^{b^n}\left(\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\right)^2=b^{n-1}\varphi_b^{\sigma_{i-1},(2)}\left(\frac{N}{b^i}\right)$$

and for $i \neq j$,

$$\sum_{\lambda=1}^{b^n} \varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) = b^{n-2}\varphi_b^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_b^{\sigma_{i-1}}\left(\frac{N}{b^i}\right).$$

Therefore we obtain

$$\begin{aligned}\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^2 &= \frac{1}{b^{2n}}\sum_{i=1}^{n}\sum_{N=1}^{b^n} b^{n-1}\varphi_b^{\sigma_{i-1},(2)}\left(\frac{N}{b^i}\right)\\ &\quad +\frac{1}{b^{2n}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{N=1}^{b^n} b^{n-2}\varphi_b^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right).\end{aligned}$$

From Lemma 4 we find that $\varphi_b^{id,(2)} = \varphi_b^{\tau,(2)}$ and $\varphi_b^{id} = -\varphi_b^{\tau}$. Let again

$$s_i = \begin{cases} 1 & \text{if } \sigma_{i-1} = \tau,\\ 0 & \text{if } \sigma_{i-1} = id.\end{cases}$$

Then we obtain

$$\begin{aligned}\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^2 &= \frac{1}{b^{2n}}\sum_{i=1}^{n}\sum_{N=1}^{b^n} b^{n-1}\varphi_b^{(2)}\left(\frac{N}{b^i}\right)\\ &\quad +\frac{1}{b^{2n}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(-1)^{s_i+s_j}\sum_{N=1}^{b^n} b^{n-2}\varphi_b\left(\frac{N}{b^i}\right)\varphi_b\left(\frac{N}{b^j}\right).\end{aligned}$$

Using Lemma 5 we obtain

$$b^{n-1}\sum_{i=1}^{n}\sum_{N=1}^{b^n}\varphi_b^{(2)}\left(\frac{N}{b^i}\right) = \frac{2nb^{2(n+2)} - 2nb^{2n} + 5b^{2(n+1)} - 5b^2}{180b^2}$$

and

$$\sum_{N=1}^{b^n}\varphi_b\left(\frac{N}{b^i}\right)\varphi_b\left(\frac{N}{b^j}\right) = \left(\frac{b(b^2-1)}{12}\right)^2 b^{n-2}.$$

Hence

$$\begin{aligned}\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^2 &= \frac{2nb^{2(n+2)} - 2nb^{2n} + 5b^{2(n+1)} - 5b^2}{180b^{2+2n}}\\ &\quad + \left(\frac{b^2-1}{12b}\right)^2\sum_{\substack{i,j=1\\ i\neq j}}^{n}(-1)^{s_i+s_j}.\end{aligned}$$

We have

$$\sum_{\substack{i,j=1\\ i\neq j}}^{n}(-1)^{s_i+s_j} = \left(\sum_{i=1}^{n}(-1)^{s_i}\right)^2 - n = (n-2l)^2 - n,$$

and the result follows. $\square$

Now, we can give the proof of Theorem 4.

Proof. We have

$$\begin{aligned}
(L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}))^2 &= \int_0^1\int_0^1 (E(x,y,\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}))^2\,\mathrm{d}x\,\mathrm{d}y\\
&= \int_0^1\int_0^1 \left(E(x(n),y(n),\mathcal{H}_{b,n}^{\boldsymbol{\sigma}})+b^n(x(n)y(n)-xy)\right)^2\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\right)^2\\
&\quad +2b^n\sum_{\lambda,N=1}^{b^n}\int_{\frac{\lambda-1}{b^n}}^{\frac{\lambda}{b^n}}\int_{\frac{N-1}{b^n}}^{\frac{N}{b^n}} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\left(\frac{\lambda}{b^n}\frac{N}{b^n}-xy\right)\mathrm{d}x\,\mathrm{d}y\\
&\quad +b^{2n}\sum_{\lambda,N=1}^{b^n}\int_{\frac{\lambda-1}{b^n}}^{\frac{\lambda}{b^n}}\int_{\frac{N-1}{b^n}}^{\frac{N}{b^n}}\left(\frac{\lambda}{b^n}\frac{N}{b^n}-xy\right)^2\mathrm{d}x\,\mathrm{d}y\\
&=: \Sigma_1+\Sigma_2+\Sigma_3.
\end{aligned}$$

From Lemma 9 we find that

$$\Sigma_1=\frac{2nb^{2(n+2)}-2nb^{2n}+5b^{2(n+1)}-5b^2}{180b^{2+2n}}+\left(\frac{b^2-1}{12b}\right)^2((n-2l)^2-n).$$

Straightforward algebra shows that

$$\Sigma_3=\frac{1}{72b^{2n}}\left(1+18b^n+25b^{2n}\right).$$

So it remains to deal with Σ_2. We have

$$\begin{aligned}
\frac{1}{b^n}\Sigma_2 &= \frac{2}{b^{4n}}\sum_{\lambda,N=1}^{b^n}E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\lambda N\\
&\quad -\frac{1}{2b^{4n}}\sum_{\lambda,N=1}^{b^n}E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)(2\lambda-1)(2N-1)\\
&= \frac{1}{b^{4n}}\sum_{\lambda,N=1}^{b^n}(\lambda+N)E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)-\frac{1}{2b^{4n}}\sum_{\lambda,N=1}^{b^n}E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\\
&=: \Sigma_4-\Sigma_5.
\end{aligned}$$

From the first part of Lemma 9 we obtain

$$\Sigma_5=\frac{1}{b^{2n}}\frac{b^2-1}{12b}\left(l-\frac{n}{2}\right).$$

We finally have to consider Σ_4. We have

$$\begin{aligned}
\Sigma_4 &= \frac{1}{b^{4n}}\sum_{\lambda,N=1}^{b^n}\lambda E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)+\frac{1}{b^{4n}}\sum_{\lambda,N=1}^{b^n}NE\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right)\\
&=: \frac{1}{b^{4n}}(\Sigma_{4,1}+\Sigma_{4,2})
\end{aligned}$$

We have

$$\Sigma_{4,2} = \sum_{i=1}^{n}\sum_{N=1}^{b^n} N \sum_{\lambda=1}^{b^n} \varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right) = b^{n-1}\sum_{i=1}^{n}(-1)^{s_i}\sum_{N=1}^{b^n} N\varphi_b\left(\frac{N}{b^i}\right),$$

where we used Lemma 2 and the definition of the s_i from the proof of Lemma 8. Now we have

$$\sum_{N=1}^{b^n} N\varphi_b\left(\frac{N}{b^j}\right) = \sum_{k=0}^{b^{n-j+1}-1}\sum_{N=kb^{j-1}+1}^{(k+1)b^{j-1}} N\varphi_b\left(\frac{N}{b^j}\right).$$

For $0 \le k < b^{n-j+1}$ let $k = qb + r$ with integers $0 \le r < b-1$ and $0 \le q < b^{n-j}$. Then for $kb^{j-1}+1 \le N \le (k+1)b^{j-1}$ we have $r/b \le N/b^j - q \le (r+1)/b$. Using the periodicity of φ_b we therefore obtain

$$\sum_{N=1}^{b^n} N\varphi_b\left(\frac{N}{b^j}\right) = \sum_{r=0}^{b-1}\sum_{q=0}^{b^{n-j}-1}\sum_{N=qb^j+rb^{j-1}+1}^{qb^j+(r+1)b^{j-1}} N\varphi_b\left(\frac{N}{b^j}-q\right) = b^{2n}\frac{b^2-1}{24},$$

where we used Lemma 3 for the last equality. Hence

$$\Sigma_{4,2} = b^{n-1}\sum_{i=1}^{n}(-1)^{s_i} b^{2n}\frac{b^2-1}{24} = b^{3n-1}\frac{b^2-1}{12}\left(l-\frac{n}{2}\right).$$

It remains to compute $\Sigma_{4,1}$. We have

$$\begin{aligned}
&\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\\
&= \left\{\left(\frac{\sigma_0(a_0)}{b}+\cdots+\frac{\sigma_{n-1}(a_{n-1})}{b^n}, \frac{a_{n-1}}{b}+\cdots+\frac{a_0}{b^n}\right) : a_0,\ldots,a_{n-1}\in\{0,\ldots,b-1\}\right\}\\
&= \left\{\left(\frac{x_0}{b}+\cdots+\frac{x_{n-1}}{b^{n-1}}, \frac{\sigma_{n-1}^{-1}(x_{n-1})}{b}+\cdots+\frac{\sigma_0^{-1}(x_0)}{b^n}\right) : x_0,\ldots,x_{n-1}\in\{0,\ldots,b-1\}\right\},
\end{aligned}$$

with $(\sigma_0,\ldots,\sigma_{n-1}) \in \{id,\tau\}^n$. Note that for $\sigma\in\{id,\tau\}$ we have $\sigma=\sigma^{-1}$. Let $g:[0,1]^2\to[0,1]^2$ be defined by $g(x,y)=(y,x)$ and for $\boldsymbol{\sigma}=(\sigma_0,\ldots,\sigma_{n-1})$ define $\boldsymbol{\sigma}^*=(\sigma_{n-1},\ldots,\sigma_0)$. Then we have found that

$$\mathcal{H}_{b,n}^{\boldsymbol{\sigma}} = g\left(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}^*}\right).$$

Therefore we obtain

$$\begin{aligned}
\Sigma_{4,1} &= \sum_{\lambda,N=1}^{b^n}\lambda E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}\right) = \sum_{\lambda,N=1}^{b^n}\lambda E\left(\frac{\lambda}{b^n},\frac{N}{b^n},g\left(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}^*}\right)\right)\\
&= \sum_{\lambda,N=1}^{b^n}\lambda E\left(\frac{N}{b^n},\frac{\lambda}{b^n},\mathcal{H}_{b,n}^{\boldsymbol{\sigma}^*}\right) = b^{3n-1}\frac{b^2-1}{12}\left(l-\frac{n}{2}\right),
\end{aligned}$$

where for the last equality we could use the formula for $\Sigma_{4,2}$ since both $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^*$ have the same number l of components equal to id.

Now we obtain the desired formula from

$$(L_2(\mathcal{H}_{b,n}^{\boldsymbol{\sigma}}))^2 = \Sigma_1 + \frac{1}{b^{3n}}(\Sigma_{4,1}+\Sigma_{4,2}) - b^n\Sigma_5 + \Sigma_3.$$

The evaluation of this sum is a matter of straight forward calculations and hence we omit the details. $\square$

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