L_2 Discrepancy of Two-Dimensional Digitally Shifted Hammersley Point Sets in Base b

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Abstract We give an exact formula for the L_2 discrepancy of two-dimensional digitally shifted Hammersley point sets in base b. This formula shows that for certain bases b and certain shifts the L_2 discrepancy is of best possible order with respect to the general lower bound due to Roth. Hence, for the first time, it is proved that, for a thin, but infinite subsequence of bases b starting with $5, 19, 71, \ldots$, a single permutation only can achieve this best possible order, unlike previous results of White (1975) who needs b permutations and Faure & Pillichshammer (2008) who need 2 permutations.

1 Introduction and Statement of the Results

For a finite point set $\mathcal{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ of $N \geq 1$ (not necessarily distinct) points in the unit-square $[0,1)^2$ the L_2 *discrepancy* is defined by

$$L_2(\mathcal{P}) := \left(\int_0^1 \int_0^1 |E(x,y,\mathcal{P})|^2 \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2},$$

where the *discrepancy function* is given as $E(x,y,\mathcal{P}) = A([0,x) \times [0,y), \mathcal{P}) - Nxy$, where $A([0,x) \times [0,y), \mathcal{P})$ denotes the number of indices $1 \leq M \leq N$ for which $\boldsymbol{x}_M \in [0,x) \times [0,y)$. The L_2 discrepancy is a quantitative measure for the irregularity of distribution of $\mathcal{P}$, i.e., the deviation from perfect uniform

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distribution modulo one, which has a close relationship with the worst-case and average-case errors of quasi-Monte Carlo integration of functions from certain function classes. An introduction to the theory of uniform distribution modulo one and the discrepancy of sequences can be found in the books of Kuipers & Niederreiter [11] or of Drmota & Tichy [3]. Concerning the relationship between L_2 discrepancy and quasi-Monte Carlo integration we further refer to [16, 19, 20] for example.

It was first shown by Roth [15] (see also [11, Chapter 2, Section 2]) that there is a constant $c > 0$ with the property that for the L_2 discrepancy of any finite point set $\mathcal{P}$ consisting of N points in $[0,1)^2$ we have

$$L_2(\mathcal{P}) \geq c\sqrt{\log N}. \tag{1}$$

In this paper we will consider the L_2 discrepancy of so-called digitally shifted Hammersley point sets in base b with b^n points. These point sets form a sub-class of generalized Hammersley point sets in base b (the Hammersley point set is also known as Roth net for $b = 2$), which can be considered as finite two-dimensional versions of the generalized van der Corput sequences in base b as introduced by Faure [5].

Throughout the paper let $b \geq 2$ be an integer and let $\mathfrak{S}_b$ be the set of all permutations of $\{0, 1, \ldots, b-1\}$.

Definition 1 (generalized Hammersley point set). Let $b \geq 2$ and $n \geq 0$ be integers and let $\Sigma = (\sigma_0, \ldots, \sigma_{n-1}) \in \mathfrak{S}_b^n$. For an integer $1 \leq N \leq b^n$, write $N - 1 = \sum_{r=0}^{n-1} a_r(N) b^r$ in the b-adic system and define $S_b^\Sigma(N) := \sum_{r=0}^{n-1} \frac{\sigma_r(a_r(N))}{b^{r+1}}$. Then the *generalized two-dimensional Hammersley point set in base* b consisting of b^n points associated with Σ is defined by

$$\mathcal{H}_{b,n}^\Sigma := \left\{ \left(S_b^\Sigma(N), \frac{N-1}{b^n} \right) \,:\, 1 \leq N \leq b^n \right\}.$$

In case of $\sigma_i = \sigma$ for all $0 \leq i < n$, we also write $\mathcal{H}_{b,n}^\sigma$ instead of $\mathcal{H}_{b,n}^\Sigma$. If $\sigma = \mathrm{id}$, the identical permutation, then we obtain the classical two-dimensional Hammersley point set in base b.

Exact formulas for the L_2 discrepancy of the classical two-dimensional Hammersley point set $\mathcal{H}_{b,n}^{\mathrm{id}}$ in base b have been proved by Vilenkin [17], Halton & Zaremba [9] and Pillichshammer [13] in base $b = 2$ and by White [18] and Faure & Pillichshammer [8] for arbitrary bases. These results show that the classical Hammersley point set cannot achieve the best possible order of L_2 discrepancy with respect to Roth's general lower bound (1).

The first who obtained the best possible order of L_2 discrepancy for finite two-dimensional point sets was Davenport [2], with a modification of so-called $(N\alpha)$-sequences (α having a continued fraction expansion with bounded partial quotients), more precisely with the set consisting of the $2M$ points $\left(\{\pm N\alpha\}, \frac{N}{M}\right)$ for $1 \leq N \leq M$ where M is a positive integer and $\{x\}$ denotes the fractional part of x.

Next, observing that $\{-N\alpha\} = 1 - \{N\alpha\}$, Proinov [14] obtained the same result with the same set where generalized van der Corput sequences take the place of $(N\alpha)$-sequences and he named this process *symmetrization of a sequence*. Later on, the same process was used by Chaix & Faure [1] for infinite van der Corput sequences (improving at the same time the constants of Proinov) and by Larcher & Pillichshammer [12] for $(0, m, 2)$-nets and $(0, 1)$-sequences in base 2. It is important to note that all these results using the symmetrization process give the exact order with bounds only for the implied constants whereas in the following, with various cleverly generalized Hammersley point sets, different authors obtain exact formulas and hence exact values for the implied constants.

Below we first give a survey of results concerning generalized Hammersley point sets with best possible order of L_2 discrepancy together with some comparisons between the methods, showing the interest in considering only one permutation, i.e., a single sequence $\mathcal{H}_{b,n}^{\sigma}$.

First results were available in base $b = 2$: Let id be the identity and $\mathrm{id}_1(k) := k+1 \pmod 2$ be the *digital shift* in base 2; then Halton & Zaremba [9] and later, in a much more general form, Kritzer & Pillichshammer [10] gave sequences of permutations $\Sigma \in \{\mathrm{id}, \mathrm{id}_1\}^n$ (although they did not use this terminology), for which the generalized Hammersley point set $\mathcal{H}_{2,n}^{\Sigma}$ in base 2 achieves the best possible order of L_2 discrepancy in the sense of Roth (1). For more detailed results we refer to [10].

Results for arbitrary bases were first given by White [18] who generalized the result from [9] in a certain way. He considered sequences Σ of the form

$$\Sigma = (\mathrm{id}_0, \mathrm{id}_1, \ldots, \mathrm{id}_{b-1}, \mathrm{id}_0, \mathrm{id}_1, \ldots, \mathrm{id}_{b-1}, \ldots) \qquad (2)$$

of length n where $\mathrm{id}_l(k) := k + l \pmod b$ for $0 \leq l, k < b$ (White did not use this terminology). The permutations id_l are called *digital shifts* in base b; they are natural generalizations of the digital shift in base 2 used by Halton & Zaremba and Kritzer & Pillichshammer. For this specific Σ, White gave an exact formula for the L_2 discrepancy of the corresponding generalized Hammersley point set. Essentially this formula states that

$$\left(L_2(\mathcal{H}_{b,n}^{\Sigma})\right)^2 = n\frac{(b^2-1)(3b^2+13)}{720b^2} + O(1) \qquad (3)$$

whenever Σ is of the form (2).

Setting $b = 2$ in this formula gives the same sequence as in [9] and the simplest sequence in [10], that is $\Sigma = (\mathrm{id}_0, \mathrm{id}_1, \mathrm{id}_0, \mathrm{id}_1, \ldots)$, with the same constant 5/192. Note that we need only two permutations and therefore the formula for base 2 starts being valid for integers $n \geq 2$, that is, sets of $2^2 = 4$ points at least, which is very few.

The problem for arbitrary b is that we need $n \geq b$, i.e., sets of b^b points at least. Even for small bases like $b = 10$ the property requires sets consisting of more than 10^{10} points which is far away from usual numbers of points allowed in quasi-Monte Carlo simulation. If we want to use generalized Hammersley point sets in applications (image-processing, optimization of printers for instance), we must find a better way than White (in fact White used a trick due to Halton & Warnock, see [18, p. 221]) to improve the L_2 discrepancy of the original Hammersley point sets.

Another approach consists of using the so-called *swapping permutation* τ defined by $\tau(k) = b - k - 1$, for $0 \leq k < b$, instead of shifts (the term *swapping* is introduced and justified in [6] and [7, Section 2]). Applied to the L_2 discrepancy of Hammersley point sets, this generalization gives formula (3) with the simplest sequence $\Sigma = (\mathrm{id}_0, \tau, \mathrm{id}_0, \tau, \ldots)$ in arbitrary bases. We refer to [8] for detailed proofs together with extensions to the L_p discrepancy. Once again, we need only two permutations but our results are valid for arbitrary bases whereas Halton & Zaremba and Kritzer & Pillichshammer deal only with base 2. We also remark that in base 2, shift and swap is the same permutation, so that [8] fully generalizes the results of [9] (for L_2 discrepancy) and [10] from base 2 to base b.

Now, after White who needs b permutations and Faure & Pillichshammer who need two, the question arises if only one permutation is enough to get the same property, i.e., the best order of L_2 discrepancy.

In this paper, we consider this question for shifts in base b and we deal with sequences of permutations of the form $\Sigma_l := (\mathrm{id}_l, \ldots, \mathrm{id}_l)$ for arbitrary fixed integer $0 \leq l < b$, i.e., with our notation after Definition 1, we study generalized Hammersley point sets $\mathcal{H}_{b,n}^{\mathrm{id}_l}$. We call such sets *digitally shifted Hammersley point sets in base* b. We can prove an exact formula for the L_2 discrepancy of these sets which permits to answer the question above for the sub-class of digitally shifted Hammersley point sets. The proof relies on the approach of [8] and uses the fundamental Lemmas 1 and 2 from this paper. However here, for the first time, we have to manage with true permutations while in [8] we dealt with identity only (τ being simply a mirror of it); on the other hand, we obtained more results in this specific case.

Section 2 contains prerequisites and auxiliary results, and Section 3 contains the proof of the following result:

Theorem 1. *For the L_2 discrepancy of a digitally shifted Hammersley point set $\mathcal{H}_{b,n}^{\mathrm{id}_l}$, with integers $b \geq 2$, $0 \leq l < b$ and $n \geq 1$, we have*

$$\begin{aligned}
&\left(L_2\left(\mathcal{H}_{b,n}^{\mathrm{id}_l}\right)\right)^2 \\
&= \left(\frac{n}{b}\left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right)\right)^2 - \frac{1}{2b^n}\frac{n}{b}\left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right) \\
&\quad + \frac{n}{b}\left(\frac{b^2-1}{12} - \frac{l(b-l)}{2} + \frac{(b^2-1)(3b^2+13)}{720b}\right) + \frac{3}{8} + \frac{1}{4b^n} - \frac{1}{72b^{2n}}.
\end{aligned}$$

If we choose $l = 0$ then $\mathcal{H}_{b,n}^{\mathrm{id}_0}$ is the classical Hammersley point set and our formula recovers [8, Theorem 1] and [18, Eq. (15)].

From Theorem 1 one can see that for certain values of b and l one can obtain the optimal order of L_2 discrepancy in the sense of Roth (1) with a single shift. In this case the implied leading constant is the same as in White's and Faure & Pillichshammer's result (3).

Corollary 1. *For integers $b \geq 2$, $0 \leq l < b$ and $n \geq 1$ we have*

$$\left(L_2\left(\mathcal{H}_{b,n}^{\mathrm{id}_l}\right)\right)^2 = n\frac{(b^2-1)(3b^2+13)}{720b^2} + \frac{3}{8} + \frac{1}{4b^n} - \frac{1}{72b^{2n}} \tag{4}$$

if and only if b satisfies the Pell-Fermat equation $b^2 - 3c^2 = -2$ with a suitable integer c and $l = \frac{1}{2}(b \pm c)$.

All solutions of this equation are given by $b + c\sqrt{3} = \pm(1+\sqrt{3})(2+\sqrt{3})^m$ with $m \in \mathbb{N}_0$.

Proof. Of course Eq. (4) holds if and only if $\frac{b^2-1}{12} = \frac{l(b-l)}{2}$ and this is equivalent to $l = \frac{1}{2}\left(b \pm \sqrt{\frac{b^2+2}{3}}\right)$. Since l is an integer this is equivalent to $\frac{b^2+2}{3} = c^2$ for some integer c or equivalently $b^2 - 3c^2 = -2$. Note that all solutions (b, c) have to consist of odd b and c only. This is in accordance with the fact that $l = \frac{1}{2}(b \pm c)$ is an integer.

For $z = x + y\sqrt{d}$ and its conjugate $\overline{z} = x - y\sqrt{d}$ we write $N(z) = z \cdot \overline{z} = x^2 - y^2 d$. It is known (see, for example [4]) that the general solution z (if it exists) of a Pell-Fermat equation $N(z) = a$ can be obtained as the product of the solution of the special Pell-Fermat equation $N(z) = 1$, which is given by $z = \pm(z_0)^m$, $m \in \mathbb{N}$, where $z_0 > 1$ is the minimal solution, with a special solution of $N(z) = a$ with $0 \leq z \leq z_0$.

In our case we have the minimal solution $z_0 = 2 + \sqrt{3}$ and the special solution $1 + \sqrt{3}$. Hence, all solutions are given by $z = \pm(1+\sqrt{3})(2+\sqrt{3})^m$, $m \in \mathbb{N}_0$. □

The first few of the infinitely many pairs (b, l) for which Eq. (4) holds are $(5, 1)$, $(5, 4)$, $(19, 4)$, $(19, 15)$, $(71, 15)$, $(71, 56)$, $(265, 56)$, $(265, 209)$, $(989, 209)$, $(989, 780)$, $(3691, 780)$, $(3691, 2911)$, ...

Hence, we have proved that for a thin (but infinite) subsequence of bases b a single shift only is sufficient to obtain the optimal order of L_2 discrepancy. Between the necessity of b shifts with White's method and the few bases we have found with a single shift, there are surely many other possibilities. Finding such alternatives will need more investigations and we plan to pursue this work in the near future.

2 Auxiliary Results

In this section we provide the main tools for the proof of Theorem 1. The analysis of the L_2 discrepancy is based on special functions which have been first introduced by Faure in [5] and which are defined as follows.

For $\sigma \in \mathfrak{S}_b$ let $\mathcal{Z}_b^\sigma = (\sigma(0)/b, \sigma(1)/b, \ldots, \sigma(b-1)/b)$. For $h \in \{0,1,\ldots,b-1\}$ and $x \in [(k-1)/b, k/b)$, where $k \in \{1,\ldots,b\}$, we define

$$\varphi_{b,h}^\sigma(x) = \begin{cases} A([0,h/b);k;\mathcal{Z}_b^\sigma) - hx & \text{if } 0 \le h \le \sigma(k-1), \\ (b-h)x - A([h/b,1);k;\mathcal{Z}_b^\sigma) & \text{if } \sigma(k-1) < h < b, \end{cases}$$

where here for a sequence $X = (x_M)_{M \ge 1}$ we denote by $A(I;k;X)$ the number of indices $1 \le M \le k$ such that $x_M \in I$. Further, the function $\varphi_{b,h}^\sigma$ is extended to the reals by periodicity. Note that $\varphi_{b,0}^\sigma = 0$ and $\varphi_{b,h}^\sigma(0) = 0$ for any $\sigma \in \mathfrak{S}_b$ and any $0 \le h < b$.

Furthermore, we define $\varphi_b^\sigma := \sum_{h=0}^{b-1} \varphi_{b,h}^\sigma$ and $\phi_b^\sigma := \sum_{h=0}^{b-1} (\varphi_{b,h}^\sigma)^2$. Note that φ_b^σ is continuous, piecewise linear on the intervals $[k/b,(k+1)/b]$ and $\varphi_b^\sigma(0) = \varphi_b^\sigma(1)$. For example for $\sigma = \mathrm{id}$ we have

$$\varphi_{b,h}^{\mathrm{id}}(x) = \begin{cases} (b-h)x & \text{if } x \in [0,h/b], \\ h(1-x) & \text{if } x \in [h/b,1], \end{cases} \tag{5}$$

from which one obtains (see [8, Lemma 3] for details) that for $x \in \left[\frac{k}{b}, \frac{k+1}{b}\right]$, $0 \le k < b$, we have

$$\varphi_b^{\mathrm{id}}(x) = \frac{b(b-2k-1)}{2}\left(x - \frac{k}{b}\right) + \frac{k(b-k)}{2} \tag{6}$$

and

$$\phi_b^{\mathrm{id}}(x) = (1-x)^2 \frac{k(k+1)(2k+1)}{6} + x^2 \frac{(b-k)(b-k-1)(2b-2k-1)}{6}. \tag{7}$$

From (6) we immediately obtain for $y \in \left[0, \frac{1}{b}\right)$ the equation

$$\sum_{k=0}^{b-1} \varphi_b^{\mathrm{id}}\left(\frac{k}{b} + y\right) = \frac{b(b^2-1)}{12}. \tag{8}$$

Sometimes we will use the following property from [1, Propriété 3.4] stating that

$$(\varphi_{b,h}^\sigma)'(k/b+0) = (\varphi_{b,h}^{\mathrm{id}})'(\sigma(k)/b+0). \tag{9}$$

Here and later on by $f'(x+0)$ we mean the right-derivative of the function f at x.

The following lemma gives a relationship between the family of $\varphi_{b,h}^\sigma$ functions with respect to the permutations id and id_l.

Lemma 1. *For any $0 \le h, l < b$ and $x \in [0,1]$ we have*

$$\varphi_{b,h}^{\mathrm{id}_l}(x) = \varphi_{b,h}^{\mathrm{id}}\left(x + \frac{l}{b}\right) - \varphi_{b,h}^{\mathrm{id}}\left(\frac{l}{b}\right) \tag{10}$$

and in particular,

$$\varphi_{b}^{\mathrm{id}_l}(x) = \varphi_{b}^{\mathrm{id}}\left(x + \frac{l}{b}\right) - \varphi_{b}^{\mathrm{id}}\left(\frac{l}{b}\right).$$

Proof. It is enough to show that the equality holds for $x = k/b, k \in \{0, \dots, b-1\}$. Since the functions $\varphi_{b,h}^{\sigma}$ are continuous and linear on $\left[\frac{j}{b}, \frac{j+1}{b}\right)$, $0 \le j < b$, invoking Eq. (9) we have

$$\begin{aligned}\varphi_{b,h}^{\mathrm{id}_l}\left(\frac{k}{b}\right) &= \frac{1}{b}\sum_{j=0}^{k-1}\left(\varphi_{b,h}^{\mathrm{id}_l}\right)'\left(\frac{j}{b}+0\right) = \frac{1}{b}\sum_{j=0}^{k-1}\left(\varphi_{b,h}^{\mathrm{id}}\right)'\left(\frac{\mathrm{id}_l(j)}{b}+0\right) \\ &= \frac{1}{b}\sum_{j=l}^{k+l-1}\left(\varphi_{b,h}^{\mathrm{id}}\right)'\left(\frac{j}{b}+0\right) = \varphi_{b,h}^{\mathrm{id}}\left(\frac{k+l}{b}\right) - \varphi_{b,h}^{\mathrm{id}}\left(\frac{l}{b}\right)\end{aligned}$$

as desired. □

The following lemma provides a formula for the discrepancy function of generalized Hammersley point sets.

Lemma 2. *For integers $1 \le \lambda, N \le b^n$ and $\Sigma = (\sigma_0, \dots, \sigma_{n-1}) \in \mathfrak{S}_b^n$ we have*

$$E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right) = \sum_{j=1}^{n} \varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right),$$

where the $\varepsilon_j = \varepsilon_j(\lambda, n, N)$ can be given explicitly.

A proof of this result together with formulas for $\varepsilon_j = \varepsilon_j(\lambda, n, N)$ can be found in [8, Lemma 1].

Remark 1. Let $0 \le x, y \le 1$ be arbitrary. Since all points from $\mathcal{H}_{b,n}^{\Sigma}$ have coordinates of the form α/b^n for some $\alpha \in \{0, 1, \dots, b^n - 1\}$, we have

$$E(x, y, \mathcal{H}_{b,n}^{\Sigma}) = E(x(n), y(n), \mathcal{H}_{b,n}^{\Sigma}) + b^n(x(n)y(n) - xy), \tag{11}$$

where for $0 \le x \le 1$ we define $x(n) := \min\{\alpha/b^n \ge x \,:\, \alpha \in \{0, \dots, b^n\}\}$.

Now we will give a series of lemmas with further, more involved properties of the functions $\varphi_{b,h}^{\sigma}$, φ_b^{σ} and ϕ_b^{σ}. The first result is a special case of [8, Lemma 2] (see there for a proof).

Lemma 3. *For $1 \le N \le b^n$ and $0 \le j_1 < j_2 < \dots < j_k < n$ we have*

$$\sum_{\lambda=1}^{b^n} \prod_{i=1}^{k} \varphi_{b,\varepsilon_{j_i}}^{\sigma_{j_i}} \left(\frac{N}{b^{j_i}}\right) = b^{n-k} \prod_{i=1}^{k} \varphi_b^{\sigma_{j_i}} \left(\frac{N}{b^{j_i}}\right)$$

and

$$\sum_{\lambda=1}^{b^n} \prod_{i=1}^{k} \left(\varphi_{b,\varepsilon_{j_i}}^{\sigma_{j_i}} \left(\frac{N}{b^{j_i}}\right)\right)^2 = b^{n-k} \prod_{i=1}^{k} \phi_b^{\sigma_{j_i}} \left(\frac{N}{b^{j_i}}\right).$$

Lemma 4. *For $0 \le h < k < n$ and $0 \le l < b$ we have*

$$\sum_{N=1}^{b^n} \varphi_b^{\mathrm{id}_l} \left(\frac{N}{b^h}\right) \varphi_b^{\mathrm{id}_l} \left(\frac{N}{b^k}\right) = b^n \left(\frac{b^2-1}{12} - \varphi_b^{\mathrm{id}} \left(\frac{l}{b}\right)\right)^2.$$

Proof. Using Lemma 1 we have

$$\begin{aligned}
&\sum_{N=1}^{b^n} \varphi_b^{\mathrm{id}_l} \left(\frac{N}{b^h}\right) \varphi_b^{\mathrm{id}_l} \left(\frac{N}{b^k}\right) \\
&= \sum_{N=1}^{b^n} \varphi_b^{\mathrm{id}} \left(\frac{N}{b^h} + \frac{l}{b}\right) \varphi_b^{\mathrm{id}} \left(\frac{N}{b^k} + \frac{l}{b}\right) + b^n \left(\varphi_b^{\mathrm{id}} \left(\frac{l}{b}\right)\right)^2 \\
&\quad -\varphi_b^{\mathrm{id}} \left(\frac{l}{b}\right) \sum_{N=1}^{b^n} \varphi_b^{\mathrm{id}} \left(\frac{N}{b^h} + \frac{l}{b}\right) - \varphi_b^{\mathrm{id}} \left(\frac{l}{b}\right) \sum_{N=1}^{b^n} \varphi_b^{\mathrm{id}} \left(\frac{N}{b^k} + \frac{l}{b}\right). \qquad (12)
\end{aligned}$$

Let $N = N_0 + N_1 b + \cdots + N_{n-1} b^{n-1}$ be the b-adic expansion of $N \in \{0, \ldots, b^n - 1\}$. From the periodicity of φ_b^{id} and using Eq. (8) we obtain

$$\begin{aligned}
\sum_{N=1}^{b^n} \varphi_b^{\mathrm{id}} \left(\frac{N}{b^h} + \frac{l}{b}\right) &= \sum_{N_0,\ldots,N_{n-1}=0}^{b-1} \varphi_b^{\mathrm{id}} \left(\frac{N_0 + \cdots + N_{n-1} b^{n-1}}{b^h} + \frac{l}{b}\right) \\
&= b^{n-h} \sum_{N_0,\ldots,N_{h-1}=0}^{b-1} \varphi_b^{\mathrm{id}} \left(\frac{N_0 + \cdots + N_{h-1} b^{h-1}}{b^h} + \frac{l}{b}\right) \\
&= b^{n-h} \sum_{N=0}^{b^{h-1}-1} \sum_{N_{h-1}=0}^{b-1} \varphi_b^{\mathrm{id}} \left(\frac{N}{b^h} + \frac{N_{h-1}+l}{b}\right) \\
&= b^{n-h} \sum_{N=0}^{b^{h-1}-1} \sum_{z=0}^{b-1} \varphi_b^{\mathrm{id}} \left(\frac{N}{b^h} + \frac{z}{b}\right) = b^n \frac{b^2-1}{12}. \qquad (13)
\end{aligned}$$

Similar reasoning as above and noting that $h < k$ gives

$$\sum_{N=1}^{b^n} \varphi_b^{\mathrm{id}} \left(\frac{N}{b^h} + \frac{l}{b}\right) \varphi_b^{\mathrm{id}} \left(\frac{N}{b^k} + \frac{l}{b}\right)$$

$$= b^{n-k}\sum_{N=0}^{b^k-1}\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right)\varphi_b^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{l}{b}\right)$$
$$= b^{n-k}\sum_{N=0}^{b^{k-1}-1}\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right)\sum_{z=0}^{b-1}\varphi_b^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{z}{b}\right)$$
$$= \frac{b^2-1}{12}\sum_{N=0}^{b^n-1}\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right) = b^n\left(\frac{b^2-1}{12}\right)^2. \tag{14}$$

Now the result follows from inserting (13) and (14) into (12). □

Lemma 5. *For $0 \le k < n$ and $0 \le l < b$ we have*

$$\sum_{N=1}^{b^n}\phi_b^{\mathrm{id}_l}\left(\frac{N}{b^k}\right) = b^n\left(\frac{b^4-1}{90b}+\frac{b(b^2-1)}{36b^{2k}}\right)+b^n\phi_b^{\mathrm{id}}\left(\frac{l}{b}\right) - \frac{b^{n-1}}{12}l(b-l)(1+b^2+lb-l^2).$$

Proof. We have

$$\phi_b^{\mathrm{id}_l}\left(\frac{N}{b^k}\right) = \sum_{h=0}^{b-1}\left(\varphi_{b,h}^{\mathrm{id}_l}\left(\frac{N}{b^k}\right)\right)^2 = \sum_{h=0}^{b-1}\left(\varphi_{b,h}^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{l}{b}\right)-\varphi_{b,h}^{\mathrm{id}}\left(\frac{l}{b}\right)\right)^2$$
$$= \phi_b^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{l}{b}\right)+\phi_b^{\mathrm{id}}\left(\frac{l}{b}\right)-2\sum_{h=0}^{b-1}\varphi_{b,h}^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{l}{b}\right)\varphi_{b,h}^{\mathrm{id}}\left(\frac{l}{b}\right).$$

By using the periodicity of ϕ_b^{id} we obtain

$$\sum_{N=1}^{b^n}\phi_b^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{l}{b}\right) = b^{n-k}\sum_{N=1}^{b^k}\phi_b^{\mathrm{id}}\left(\frac{N}{b^k}\right) = b^{n-k}\sum_{j=0}^{b-1}\sum_{N=jb^{k-1}+1}^{(j+1)b^{k-1}}\phi_b^{\mathrm{id}}\left(\frac{N}{b^k}\right).$$

For $jb^{k-1}+1 \le N \le (j+1)b^{k-1}$ we have $j/b < N/b^k \le (j+1)/b$ and hence we can use Eq. (7) to obtain

$$\sum_{N=1}^{b^n}\phi_b^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{l}{b}\right) = b^{n-k}\sum_{j=0}^{b-1}\sum_{N=jb^{k-1}+1}^{(j+1)b^{k-1}}\left[\left(1-\frac{N}{b^k}\right)^2\frac{j(j+1)(2j+1)}{6} + \left(\frac{N}{b^k}\right)^2\frac{(b-j)(b-j-1)(2b-2j-1)}{6}\right]$$
$$= b^n\left(\frac{b^4-1}{90b}+\frac{b(b^2-1)}{36b^{2k}}\right).$$

Furthermore we have

$$\sum_{N=1}^{b^n}\sum_{h=0}^{b-1}\varphi_{b,h}^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{l}{b}\right)\varphi_{b,h}^{\mathrm{id}}\left(\frac{l}{b}\right)=\sum_{h=0}^{b-1}\varphi_{b,h}^{\mathrm{id}}\left(\frac{l}{b}\right)\sum_{N=1}^{b^n}\varphi_{b,h}^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{l}{b}\right).$$

Using the periodicity of $\varphi_{b,h}^{\mathrm{id}}$ and Eq. (5) for the innermost sum we obtain

$$\begin{aligned}\sum_{N=1}^{b^n}\varphi_{b,h}^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{l}{b}\right)&=b^{n-k}\sum_{N=0}^{b^k-1}\varphi_{b,h}^{\mathrm{id}}\left(\frac{N}{b^k}\right)\\&=b^{n-k}\left(\sum_{N=0}^{hb^{k-1}}(b-h)\frac{N}{b^k}+\sum_{N=hb^{k-1}+1}^{b^k-1}h\left(1-\frac{N}{b^k}\right)\right)\\&=b^{n-1}\frac{(b-h)h}{2}.\end{aligned}$$

Hence, using again Eq. (5),

$$\begin{aligned}&\sum_{N=1}^{b^n}\sum_{h=0}^{b-1}\varphi_{b,h}^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{l}{b}\right)\varphi_{b,h}^{\mathrm{id}}\left(\frac{l}{b}\right)=\frac{b^{n-1}}{2}\sum_{h=0}^{b-1}\varphi_{b,h}^{\mathrm{id}}\left(\frac{l}{b}\right)(b-h)h\\&=\frac{b^{n-1}}{2}\left(\sum_{h=0}^{l-1}(b-h)h^2\left(1-\frac{l}{b}\right)+\sum_{h=l}^{b-1}(b-h)^2h\frac{l}{b}\right)\\&=\frac{b^{n-1}}{24}l(b-l)(1+b^2+lb-l^2).\end{aligned}$$

The result follows. □

Lemma 6. *For $0\le h<n$ and $0\le l<b$ we have*

$$\sum_{N=1}^{b^n}N\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right)=b^{2n}\frac{b^2-1}{24}+\frac{b^nl(b-l)}{12b}(3b-b^h(b-2l)).$$

Proof. Splitting up the range of summation we have

$$\sum_{N=1}^{b^n}N\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right)=\sum_{k=0}^{b^{n-h+1}-1}\sum_{N=kb^{h-1}+1}^{(k+1)b^{h-1}}N\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right).$$

For $0\le k<b^{n-h+1}$ let $k=qb+r$ with integers $0\le r<b$ and $0\le q<b^{n-h}$. Then for $kb^{h-1}+1\le N\le(k+1)b^{h-1}$ we have $r/b\le N/b^h-q\le(r+1)/b$. Hence, if $0\le r<b-l$, then $0\le(r+l)/b\le N/b^h-q+l/b\le(r+l+1)/b\le 1$ and if $b-l\le r<b$, then $0\le(r+l-b)/b\le N/b^h-q+l/b-1\le(r+l-b+1)/b<1$. Using the periodicity of φ_b^{id} and Eq. (6) we therefore obtain

$$\sum_{N=1}^{b^n} N\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right) = \sum_{r=0}^{b-1}\sum_{q=0}^{b^{n-h}-1}\sum_{N=qb^h+rb^{h-1}+1}^{qb^h+(r+1)b^{h-1}} N\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}-q+\frac{l}{b}\right)$$

$$= \sum_{r=0}^{b-l-1}\sum_{q=0}^{b^{n-h}-1}\sum_{N=qb^h+rb^{h-1}+1}^{qb^h+(r+1)b^{h-1}} N\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}-q+\frac{l}{b}\right)$$

$$+ \sum_{r=b-l}^{b-1}\sum_{q=0}^{b^{n-h}-1}\sum_{N=qb^h+rb^{h-1}+1}^{qb^h+(r+1)b^{h-1}} N\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}-q+\frac{l}{b}-1\right)$$

$$= \sum_{r=0}^{b-l-1}\sum_{q=0}^{b^{n-h}-1}\sum_{N=qb^h+rb^{h-1}+1}^{qb^h+(r+1)b^{h-1}} N\left(\frac{b(b-2(r+l)-1)}{2}\left(\frac{N}{b^h}-q-\frac{r}{b}\right)\right. \\ \left. +\frac{(r+l)(b-r-l)}{2}\right)$$

$$+ \sum_{r=b-l}^{b-1}\sum_{q=0}^{b^{n-h}-1}\sum_{N=qb^h+rb^{h-1}+1}^{qb^h+(r+1)b^{h-1}} N\left(\frac{b(b-2(r+l-b)-1)}{2}\left(\frac{N}{b^h}-q-\frac{r}{b}\right)\right. \\ \left. +\frac{(r+l-b)(2b-r-l)}{2}\right)$$

$$= b^{2n}\frac{b^2-1}{24}+\frac{b^n l(b-l)}{12b}(3b-b^h(b-2l)).$$

This is the desired result. □

3 The Proof of Theorem 1

First we show a discrete version of Theorem 1. The following result is a generalization of [8, Lemma 6]. The original is obtained when putting $l=0$ below.

Lemma 7. *For $0 \le l < b$ we have*

$$\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\mathrm{id}_l}\right) = \frac{n}{b}\left(\frac{b^2-1}{12}-\frac{l(b-l)}{2}\right) \tag{15}$$

and

$$\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\mathrm{id}_l}\right)\right)^2 \tag{16}$$

$$= \left(\frac{n}{b}\left(\frac{b^2-1}{12}-\frac{l(b-l)}{2}\right)\right)^2 + n\frac{(b^2-1)(3b^2+13)}{720b^2}+\frac{1}{36}\left(1-\frac{1}{b^{2n}}\right).$$

Proof. We just give the (much more involved) proof of Eq. (16). Using Lemmas 2, 3, 4 and 5 we have

$$\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\mathrm{id}_l}\right)\right)^2=\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\sum_{i,j=1}^{n}\varphi_{b,\varepsilon_i}^{\mathrm{id}_l}\left(\frac{N}{b^i}\right)\varphi_{b,\varepsilon_j}^{\mathrm{id}_l}\left(\frac{N}{b^j}\right)$$
$$=\frac{1}{b^{2n}}\sum_{i=1}^{n}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\left(\varphi_{b,\varepsilon_i}^{\mathrm{id}_l}\left(\frac{N}{b^i}\right)\right)^2+\frac{1}{b^{2n}}\sum_{\substack{i,j=1\\i\neq j}}^{n}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_i}^{\mathrm{id}_l}\left(\frac{N}{b^i}\right)\varphi_{b,\varepsilon_j}^{\mathrm{id}_l}\left(\frac{N}{b^j}\right)$$
$$=\frac{1}{b^{2n}}\sum_{i=1}^{n}\sum_{N=1}^{b^n}b^{n-1}\phi_b^{\mathrm{id}_l}\left(\frac{N}{b^i}\right)+\frac{1}{b^{2n}}\sum_{\substack{i,j=1\\i\neq j}}^{n}\sum_{N=1}^{b^n}b^{n-2}\varphi_b^{\mathrm{id}_l}\left(\frac{N}{b^i}\right)\varphi_b^{\mathrm{id}_l}\left(\frac{N}{b^j}\right)$$
$$=\frac{1}{b}\sum_{i=1}^{n}\left(\left(\frac{b^4-1}{90b}+\frac{b(b^2-1)}{36b^{2i}}\right)+\phi_b^{\mathrm{id}}\left(\frac{l}{b}\right)-\frac{l(b-l)(1+b^2+lb-l^2)}{12b}\right)$$
$$+\frac{n^2-n}{b^2}\left(\frac{b^2-1}{12}-\varphi_b^{\mathrm{id}}\left(\frac{l}{b}\right)\right)^2$$
$$=\left(\frac{n}{b}\left(\frac{b^2-1}{12}-\varphi_b^{\mathrm{id}}\left(\frac{l}{b}\right)\right)\right)^2-\frac{n}{b^2}\left(\frac{b^2-1}{12}-\varphi_b^{\mathrm{id}}\left(\frac{l}{b}\right)\right)^2$$
$$+n\frac{b^4-1}{90b^2}+\frac{1}{36}\left(1-\frac{1}{b^{2n}}\right)+\frac{n}{b}\left(\phi_b^{\mathrm{id}}\left(\frac{l}{b}\right)-\frac{l(b-l)(1+b^2+lb-l^2)}{12b}\right)$$
$$=\left(\frac{n}{b}\left(\frac{b^2-1}{12}-\frac{l(b-l)}{2}\right)\right)^2+n\frac{(b^2-1)(3b^2+13)}{720b^2}+\frac{1}{36}\left(1-\frac{1}{b^{2n}}\right),$$

where for the last equality we used that $\varphi_b^{\mathrm{id}}(l/b)=l(b-l)/2$ according to Eq. (6) and $\phi_b^{\mathrm{id}}(l/b)=(1-l/b)^2l(l+1)(2l+1)/6+(b-l)(b-l-1)(2b-2l-1)l^2/(6b^2)$ according to Eq. (7). □

Now we give the proof of Theorem 1.

Proof. Using Eq. (11) we obtain

$$\left(L_2\left(\mathcal{H}_{b,n}^{\mathrm{id}_l}\right)\right)^2=\int_0^1\int_0^1\left(E\left(x(n),y(n),\mathcal{H}_{b,n}^{\mathrm{id}_l}\right)+b^n(x(n)y(n)-xy)\right)^2\,\mathrm{d}x\,\mathrm{d}y$$
$$=\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\mathrm{id}_l}\right)\right)^2$$
$$+2b^n\sum_{\lambda,N=1}^{b^n}\int_{\frac{\lambda-1}{b^n}}^{\frac{\lambda}{b^n}}\int_{\frac{N-1}{b^n}}^{\frac{N}{b^n}}E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\mathrm{id}_l}\right)\left(\frac{\lambda}{b^n}\frac{N}{b^n}-xy\right)\,\mathrm{d}x\,\mathrm{d}y$$
$$+b^{2n}\sum_{\lambda,N=1}^{b^n}\int_{\frac{\lambda-1}{b^n}}^{\frac{\lambda}{b^n}}\int_{\frac{N-1}{b^n}}^{\frac{N}{b^n}}\left(\frac{\lambda}{b^n}\frac{N}{b^n}-xy\right)^2\,\mathrm{d}x\,\mathrm{d}y$$
$$=:S_1+S_2+S_3.$$

The term S_1 has been evaluated in Lemma 7 and straightforward algebra shows that $S_3 = (1 + 18b^n + 25b^{2n})/(72b^{2n})$. So it remains to deal with S_2.

Evaluating the integral appearing in S_2 we obtain

$$S_2 = \frac{1}{b^{3n}} \sum_{\lambda,N=1}^{b^n} (\lambda + N) E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\mathrm{id}_l}\right) - \frac{1}{2b^{3n}} \sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\mathrm{id}_l}\right)$$
$$=: S_4 - S_5.$$

The term S_5 can be obtained from Lemma 7, Eq. (15). For S_4 we have

$$S_4 = \frac{1}{b^{3n}} \sum_{\lambda,N=1}^{b^n} \lambda E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\mathrm{id}_l}\right) + \frac{1}{b^{3n}} \sum_{\lambda,N=1}^{b^n} N E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\mathrm{id}_l}\right)$$
$$=: \frac{1}{b^{3n}}(S_{4,1} + S_{4,2}).$$

With Lemma 2, Lemma 3, Lemma 1 and Lemma 6 we obtain

$$\begin{aligned}
S_{4,2} &= b^{n-1} \sum_{i=1}^{n} \sum_{N=1}^{b^n} N \left(\varphi_b^{\mathrm{id}}\left(\frac{N}{b^i} + \frac{l}{b}\right) - \varphi_b^{\mathrm{id}}\left(\frac{l}{b}\right)\right) \\
&= b^{2n-1} \sum_{i=1}^{n} \left(b^n \frac{b^2-1}{24} + l(b-l)\left(\frac{3b - b^{i+1} + 2lb^i}{12b} - \frac{b^n+1}{4}\right)\right) \\
&= b^{3n} \frac{b^2-1}{24b} n - \frac{b^{2n}}{12b^2} \sum_{i=1}^{n} (b-l)l\left(b^i(b-2l) + 3b^{n+1}\right) \\
&= b^{3n} \frac{b^2-1}{24b} n - \frac{b^{2n}}{12b}(b-l)l\left((b-2l)\frac{b^n-1}{b-1} + 3b^n n\right).
\end{aligned}$$

We turn to $S_{4,1}$. We have

$$\begin{aligned}
\mathcal{H}_{b,n}^{\mathrm{id}_l} &= \left\{\left(\frac{\mathrm{id}_l(a_0)}{b} + \cdots + \frac{\mathrm{id}_l(a_{n-1})}{b^n}, \frac{a_{n-1}}{b} + \cdots + \frac{a_0}{b^n}\right) : 0 \le a_i < b\right\} \\
&= \left\{\left(\frac{x_0}{b} + \cdots + \frac{x_{n-1}}{b^{n-1}}, \frac{\mathrm{id}_l^{-1}(x_{n-1})}{b} + \cdots + \frac{\mathrm{id}_l^{-1}(x_0)}{b^n}\right) : 0 \le x_i < b\right\}.
\end{aligned}$$

Let $g : [0,1]^2 \to [0,1]^2$ be defined by $g(x,y) = (y,x)$. For $l = 0$ we have $\mathrm{id}_0^{-1} = \mathrm{id}_0$ and for $0 < l < b$ we have $\mathrm{id}_l^{-1} = \mathrm{id}_{b-l}$. Hence we have $\mathcal{H}_{b,n}^{\mathrm{id}_l} = g\left(\mathcal{H}_{b,n}^{\mathrm{id}_{b-l}}\right)$ for $0 < l < b$ and $\mathcal{H}_{b,n}^{\mathrm{id}_0} = g\left(\mathcal{H}_{b,n}^{\mathrm{id}_0}\right)$. Therefore, for $0 < l < b$ we obtain

$$S_{4,1} = \sum_{\lambda,N=1}^{b^n} \lambda E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\mathrm{id}_l}\right) = \sum_{\lambda,N=1}^{b^n} \lambda E\left(\frac{N}{b^n}, \frac{\lambda}{b^n}, \mathcal{H}_{b,n}^{\mathrm{id}_{b-l}}\right)$$

$$= b^{3n}\frac{b^2-1}{24b}n - \frac{b^{2n}}{12b}(b-l)l\left((2l-b)\frac{b^n-1}{b-1}+3b^n n\right)$$

where we used the formula for $S_{4,2}$ in the last equation. The same formula holds true for $l = 0$.

Hence we have

$$S_4 = \frac{n}{b}\left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right).$$

Now we obtain

$$\begin{aligned}\left(L_2\left(\mathcal{H}_{b,n}^{\mathrm{id}_l}\right)\right)^2 &= \left(\frac{n}{b}\left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right)\right)^2 + n\frac{(b^2-1)(3b^2+13)}{720b^2} \\ &\quad + \frac{1}{36}\left(1-\frac{1}{b^{2n}}\right) + \frac{n}{b}\left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right) \\ &\quad - \frac{n}{2b^{n+1}}\left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right) + \frac{1+18b^n+25b^{2n}}{72b^{2n}}\end{aligned}$$

which yields the desired result. □

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