L_2 discrepancy of generalized two-dimensional Hammersley point sets scrambled with arbitrary permutations

Henri Faure, Friedrich Pillichshammer, Gottlieb Pirsic and Wolfgang Ch. Schmid *

Abstract

The L_2 discrepancy is a quantitative measure for the irregularity of distribution of a finite point set. In this paper we consider the L_2 discrepancy of so-called *generalized* Hammersley point sets which can be obtained from the classical Hammersley point sets by introducing some permutations on the base b digits. While for the classical Hammersley point set it is *not* possible to achieve the optimal order of L_2 discrepancy with respect to a general lower bound due to Roth this disadvantage can be overcome with the generalized version thereof. For special permutations we obtain an exact formula for the L_2 discrepancy from which we obtain two-dimensional finite point sets with the lowest value of L_2 discrepancy known so far.

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Key words: L_2 discrepancy, generalized Hammersley point set.

1 Introduction

For a point set $\mathcal{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ of $N \geq 1$ points in the two-dimensional unit-square $[0,1)^2$ the L_2 *discrepancy* is defined by

$$L_2(\mathcal{P}) := \left(\int_0^1 \int_0^1 |E(x,y,\mathcal{P})|^2 \,\mathrm{d}x \,\mathrm{d}y\right)^{1/2},$$

where the so-called *discrepancy function* is given as $E(x,y,\mathcal{P}) = A([0,x) \times [0,y), \mathcal{P}) - Nxy$, where $A([0,x) \times [0,y), \mathcal{P})$ denotes the number of indices $1 \leq M \leq N$ for which $\boldsymbol{x}_M \in [0,x) \times [0,y)$. The L_2 discrepancy is a quantitative measure for the irregularity of distribution of $\mathcal{P}$, i.e., the deviation from perfect uniform distribution.

It was first shown by Roth [8] that for the L_2 discrepancy of any finite point set $\mathcal{P}$ consisting of N points in $[0,1)^2$ we have

$$L_2(\mathcal{P}) \geq c\sqrt{\log N} \tag{1}$$

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with a constant $c > 0$ independent of $\mathcal{P}$ and N. According to [6, Chapter 2, Proof of Lemma 2.5] one can choose $c = 1/(2^8\sqrt{\log 2}) = 0,0046918\ldots$.

In this paper we will consider the L_2 discrepancy of so-called generalized Hammersley point sets in base b with b^n points. These point sets, generalizations of the Hammersley point set in base b (which is also known as Roth net for $b = 2$), can be considered as finite two-dimensional versions of the generalized van der Corput sequences in base b as introduced by Faure [1].

Throughout the paper let $b \geq 2$ be an integer and let $\mathfrak{S}_b$ be the set of all permutations of $\{0, 1, \ldots, b-1\}$.

Definition 1 (generalized Hammersley point set) Let $b \geq 2$ and $n \geq 0$ be integers and let $\Sigma = (\sigma_0, \ldots, \sigma_{n-1}) \in \mathfrak{S}_b^n$. For an integer $1 \leq N \leq b^n$, write $N-1 = \sum_{r=0}^{n-1} a_r(N) b^r$ in the b-adic system and define $S_b^\Sigma(N) := \sum_{r=0}^{n-1} \frac{\sigma_r(a_r(N))}{b^{r+1}}$. Then the *generalized two-dimensional Hammersley point set in base* b consisting of b^n points associated to Σ is defined by

$$\mathcal{H}_{b,n}^\Sigma := \left\{ \left(S_b^\Sigma(N), \frac{N-1}{b^n} \right) \, : \, 1 \leq N \leq b^n \right\}.$$

In case of $\sigma_i = \sigma$ for all $0 \leq i < n$, we write also $\mathcal{H}_{b,n}^\sigma$ instead of $\mathcal{H}_{b,n}^\Sigma$. If in the above definition $\sigma_i = id$ for all $i \in \{0, \ldots, n-1\}$, then we obtain the classical Hammersley point set in base b which we simply denote by $\mathcal{H}_{b,n}$.

Let $\tau \in \mathfrak{S}_b$ be given by $\tau(k) = b - 1 - k$. Faure and Pillichshammer [3] investigated the (more general) L_p discrepancy of the generalized two-dimensional Hammersley point set in base b with $\Sigma \in \{id, \tau\}^n$. Especially, for the L_2 discrepancy they showed that, whenever l is the number of components of Σ which are equal to id, then

$$\left(L_2(\mathcal{H}_{b,n}^\Sigma)\right)^2 =$$
$$\left(\frac{b^2-1}{12b}\right)^2 ((n-2l)^2 - n) + \frac{b^2-1}{12b}\left(1 - \frac{1}{2b^n}\right)(2l - n) + n\frac{b^4-1}{90b^2} + \frac{3}{8} + \frac{1}{4b^n} - \frac{1}{72b^{2n}}.$$

This result generalizes older results due to Vilenkin [9], Halton and Zaremba [4], Pillichshammer [7] and Kritzer and Pillichshammer [5] in base $b = 2$ and White [10] in arbitrary bases $b \geq 2$.

Note that the L_2 discrepancy of $\mathcal{H}_{b,n}^\Sigma$ with $\Sigma \in \{id, \tau\}^n$ only depends on n, b and the number of permutations in Σ which are equal to id (and not on their distribution). Setting $l = n$ we get the formula for the L_2 discrepancy of the classical Hammersley point set.

The above result shows that generalized Hammersley point sets can achieve the best possible order of L_2 discrepancy in the sense of Roth's lower bound (1). More detailed we have

$$\lim_{n\to\infty} \min_{\Sigma \in \{id,\tau\}^n} \frac{L_2(\mathcal{H}_{b,n}^\Sigma)}{\sqrt{\log b^n}} = \frac{1}{b}\sqrt{\frac{(b^2-1)(3b^2+13)}{720 \log b}}. \tag{2}$$

This is not the case for the classical Hammersley point set $\mathcal{H}_{b,n}$ where

$$\lim_{n\to\infty} \frac{L_2(\mathcal{H}_{b,n})}{\log b^n} = \frac{b^2-1}{12b\log b}.$$

In this paper we intend to generalize the result mentioned above. Thereby we aim to minimize the constant in the leading term in the formula for the L_2 discrepancy, i.e., the quantity $\lim_{n\to\infty} L_2(\mathcal{H}_{b,n}^{\Sigma})/\sqrt{\log b^n}$. More detailed, for $\sigma \in \mathfrak{S}_b$ we define $\overline{\sigma} := \tau \circ \sigma$ and consider sequences of permutations $\Sigma \in \{\sigma, \overline{\sigma}\}^n$. We will show that for arbitrary $\sigma \in \mathfrak{S}_b$ one still can achieve the optimal order of L_2 discrepancy in the sense of (1). However, if we want to study the constant in the leading term, then we need some restrictions on σ, but only for technical reasons.

Let $\mathcal{A}(\tau) := \{\sigma \in \mathfrak{S}_b \,:\, \sigma \circ \tau = \tau \circ \sigma\}$. For permutations $\sigma \in \mathcal{A}(\tau)$ and $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ we provide an explicit formula for the L_2 discrepancy of $\mathcal{H}_{b,n}^{\Sigma}$. This also yields an explicit formula for the quantity

$$\lim_{n\to\infty} \min_{\substack{\sigma \in \mathcal{A}(\tau)\\ \Sigma \in \{\sigma,\overline{\sigma}\}^n}} L_2(\mathcal{H}_{b,n}^{\Sigma})/\sqrt{\log b^n}\ .$$

With this formula we can then search for the permutations in $\mathcal{A}(\tau)$ which yield the best result (see Section 5).

The results are presented in Section 2. In Section 3 we show some auxiliary results and the proofs are finally presented in Section 4.

We close this introduction with some definitions and notations that are used throughout this paper.

Basic Notations. Throughout the paper let $b \geq 2$ and $n \geq 1$ be integers. Let $\mathfrak{S}_b$ be the set of all permutations of $\{0, 1, \ldots, b-1\}$, let $\tau \in \mathfrak{S}_b$ be given by $\tau(k) = b - 1 - k$ and define $\mathcal{A}(\tau) := \{\sigma \in \mathfrak{S}_b \,:\, \sigma \circ \tau = \tau \circ \sigma\}$. The identity in $\mathfrak{S}_b$ is always denoted by *id*. In all examples and concrete results we will write down the permutations in the usual cycle notation, i.e. for $\sigma = \begin{pmatrix} 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7 \\ 0\ 4\ 2\ 6\ 1\ 5\ 3\ 7 \end{pmatrix}$ we will write $\sigma = (4\ 1)(6\ 3)$.

The analysis of the L_2 discrepancy is based on special functions which have been first introduced by Faure in [1] and which are defined as follows. For $\sigma \in \mathfrak{S}_b$ let $\mathcal{Z}_b^{\sigma} = (\sigma(0)/b, \sigma(1)/b, \ldots, \sigma(b-1)/b)$. For $h \in \{0, 1, \ldots, b-1\}$ and $x \in \left[\frac{k-1}{b}, \frac{k}{b}\right)$ where $k \in \{1, \ldots, b\}$ we define

$$\varphi_{b,h}^{\sigma}(x) := \begin{cases} A([0, h/b); k; \mathcal{Z}_b^{\sigma}) - hx & \text{if } 0 \leq h \leq \sigma(k-1), \\ (b-h)x - A([h/b, 1); k; \mathcal{Z}_b^{\sigma}) & \text{if } \sigma(k-1) < h < b, \end{cases}$$

where here for a sequence $X = (x_M)_{M \geq 1}$ we denote by $A(I; k; X)$ the number of indices $1 \leq M \leq k$ such that $x_M \in I$. Further, the function $\varphi_{b,h}^{\sigma}$ is extended to the reals by periodicity. Note that $\varphi_{b,0}^{\sigma} = 0$ for any σ and that $\varphi_{b,h}^{\sigma}(0) = 0$ for any $\sigma \in \mathfrak{S}_b$ and any $h \in \{0, \ldots, b-1\}$.

Let $\varphi_b^{\sigma,(r)} := \sum_{h=0}^{b-1} \left(\varphi_{b,h}^{\sigma}\right)^r$ where for $r = 1$ we omit the superscript, i.e., $\varphi_b^{\sigma,(1)} =: \varphi_b^{\sigma}$. Note that φ_b^{σ} is continuous, piecewise linear on the intervals $[k/b, (k+1)/b]$ and $\varphi_b^{\sigma}(0) = \varphi_b^{\sigma}(1)$. The function $\varphi_b^{\sigma,(2)}$ is continuous, piecewise quadratic on the intervals $[k/b, (k+1)/b]$ and $\varphi_b^{\sigma,(2)}(0) = \varphi_b^{\sigma,(2)}(1)$. For an example see Fig. 1.

2 The L_2 discrepancy of $\mathcal{H}_{b,n}^{\Sigma}$

We start with a general result for the L_2 discrepancy of generalized Hammersley point sets.

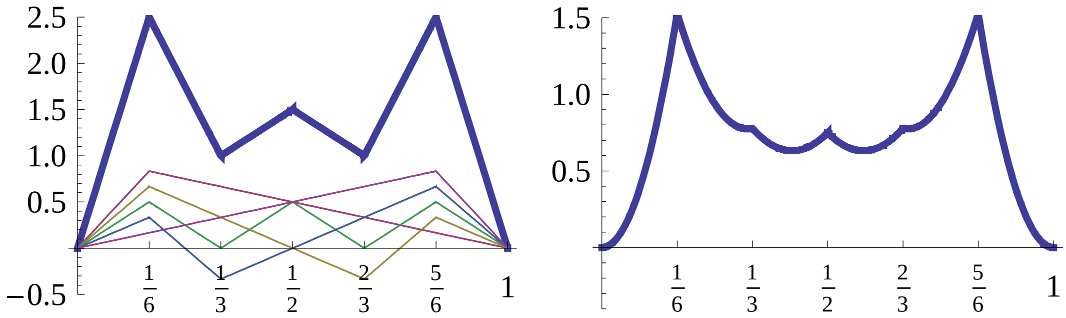

Figure 1: The functions $\varphi_{b,h}^{\sigma}$, $0 \leq h < b$ and φ_b^{σ} (left plot) and $\varphi_b^{\sigma,(2)}$ (right plot) for $b = 6$ and $\sigma = (4,1)$.

Theorem 1 *Let $\sigma \in \mathfrak{S}_b$ and let $\overline{\sigma} := \tau \circ \sigma$. Let $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ and let l denote the number of components of Σ which are equal to σ. Then we have*

$$\left(L_2(\mathcal{H}_{b,n}^{\Sigma})\right)^2 = (\Phi_b^{\sigma})^2((n-2l)^2 - n) + O(n),$$

where $\Phi_b^{\sigma} := \frac{1}{b}\int_0^1 \varphi_b^{\sigma}(x)\,\mathrm{d}x$ and where the constant in the O notation only depends on b.

The proof of this result will be given in Section 4.

Theorem 1 shows that one can always obtain $L_2(\mathcal{H}_{b,n}^{\Sigma}) = O(\sqrt{n})$ which is the best possible with respect to Roth's lower bound (1). Either one chooses a permutation $\sigma \in \mathfrak{S}_b$ for which $\Phi_b^{\sigma} = 0$ or, for arbitrary σ, one chooses l such that the term $(n-2l)^2 = O(n)$.

For permutations σ from the class $\mathcal{A}(\tau)$ we can even give an exact formula for the L_2 discrepancy of generalized two-dimensional Hammersley point sets. This result is a generalization of [3, Theorem 4] which can be obtained by choosing $\sigma = id$.

Theorem 2 *Let $\sigma \in \mathcal{A}(\tau)$ and let $\overline{\sigma} := \tau \circ \sigma$. Let $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ and let l denote the number of components of Σ which are equal to σ. Then we have*

$$\begin{aligned}&\left(L_2(\mathcal{H}_{b,n}^{\Sigma})\right)^2 = \\ &\quad (\Phi_b^{\sigma})^2((n-2l)^2 - n) + \Phi_b^{\sigma}\left(1 - \frac{1}{2b^n}\right)(2l-n) + n\Phi_b^{\sigma,(2)} + \frac{3}{8} + \frac{1}{4b^n} - \frac{1}{72b^{2n}},\end{aligned}$$

where $\Phi_b^{\sigma} := \frac{1}{b}\int_0^1 \varphi_b^{\sigma}(x)\,\mathrm{d}x$ and $\Phi_b^{\sigma,(2)} := \frac{1}{b}\int_0^1 \varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x$.

The proof of this result will be given in Section 4.

Remark 1 Note that the L_2 discrepancy of $\mathcal{H}_{b,n}^{\Sigma}$ with $\Sigma \in \{\sigma, \overline{\sigma}\}^n$, $\sigma \in \mathcal{A}(\tau)$, only depends on n, b, σ and the number l of permutations in Σ which are equal to σ. It does *not* depend on the distribution of σ and $\overline{\sigma}$ in Σ.

From Theorem 2 we find that among all sequences of permutations $\Sigma \in \{\sigma, \overline{\sigma}\}^n$, $\sigma \in \mathcal{A}(\tau)$, the one where all components are equal to σ gives the worst result for the L_2 discrepancy.

Corollary 1 *For any* $\Sigma \in \{\sigma, \overline{\sigma}\}^n$, $\sigma \in \mathcal{A}(\tau)$ *we have* $L_2(\mathcal{H}_{b,n}^{\Sigma}) \leq L_2(\mathcal{H}_{b,n}^{\sigma})$.

Again one has two possibilities to obtain the best possible order of L_2 discrepancy in the sense of Roth's lower bound (1). Either one chooses a permutation σ for which $\Phi_b^{\sigma} = 0$ (in which case the formula from Theorem 2 is *independent* of l) or, for arbitrary $\sigma \in \mathcal{A}(\tau)$, one chooses l such that $(n-2l)^2 = O(n)$.

Corollary 2 *Let* $\sigma \in \mathcal{A}(\tau)$ *and let* $\overline{\sigma} := \tau \circ \sigma$. *We have*

$$\min_{\Sigma \in \{\sigma, \overline{\sigma}\}^n} \left(L_2(\mathcal{H}_{b,n}^{\Sigma})\right)^2 = n\left(\Phi_b^{\sigma,(2)} - (\Phi_b^{\sigma})^2\right) + O(1).$$

Especially

$$\lim_{n\to\infty} \min_{\substack{\sigma\in\mathcal{A}(\tau)\\ \Sigma\in\{\sigma,\overline{\sigma}\}^n}} \frac{L_2(\mathcal{H}_{b,n}^{\Sigma})}{\sqrt{\log b^n}} = \min_{\sigma\in\mathcal{A}(\tau)} \sqrt{\frac{\Phi_b^{\sigma,(2)} - (\Phi_b^{\sigma})^2}{\log b}}.$$

Proof. The result follows from Theorem 2 together with the fact that the function $x \mapsto (\Phi_b^{\sigma})^2((n-2x)^2-n) + \Phi_b^{\sigma}\left(1-\frac{1}{2b^n}\right)(2x-n)$ attains it's minimum for $x = \frac{n}{2} - \frac{1}{4\Phi_b^{\sigma}}\left(1-\frac{1}{2b^n}\right)$. $\square$

Remark 2 Concerning the case $\Phi_b^{\sigma} = 0$ we can give explicit constructions for permutations in bases $b \equiv 0 \pmod 4$, $b \equiv 1 \pmod 4$ and $b \equiv 3 \pmod 4$, $b \notin \{3, 7, 11\}$. In bases $b = 3, 7$ and $b \equiv 2 \pmod 4$ there do not exist any permutations $\sigma \in \mathfrak{S}_b$ with $\Phi_b^{\sigma} = 0$, for $b = 11$ we will give an example in Table 2.

We may choose $\sigma \in \mathcal{A}(\tau)$ such that for $b \equiv 0 \pmod 4$ and $b \equiv 1 \pmod 4$ we let

$$\sigma(k) = \begin{cases} k+1 \text{ for even } k \\ b-k \text{ for odd } k \end{cases} \quad \text{for } 0 \leq k < \left\lfloor \frac{b}{2} \right\rfloor,$$

and for $b = 4c+3$ with $c \geq 3$ we let

$$\sigma(k) = \begin{cases} 2c-k+1 & \text{for} & 1 \leq & k & \leq c-2 \\ 4c-k+1 & \text{for} & c-1 \leq & k & \leq c+1 \\ 2c+k+1 & \text{for} & c+2 \leq & k & \leq 2c-2 \\ 6c-k & \text{for} & 2c-1 \leq & k & \leq 2c\ . \end{cases}$$

Note that σ is completely determined since $\sigma \in \mathcal{A}(\tau)$, i.e. the other values are given by symmetry through $\sigma(b-1-k) = b-1-\sigma(k)$. However, the numerical values of $\Phi_b^{\sigma,(2)}$ are not optimal in these cases. We remark that we gave for fixed b only one example for a permutation σ with $\Phi_b^{\sigma} = 0$. Numerical experiments suggest that for any $b \not\equiv 2 \pmod 4$, $b \notin \{3, 7\}$ there exist many permutations with $\Phi_b^{\sigma} = 0$. We have tabulated those with the minimal L_2 discrepancy for bases $b \leq 17$ (see Section 5, Table 2).

We can also show that the L_2 discrepancy of the two-dimensional generalized Hammersley point set $\mathcal{H}_{b,n}^{\Sigma}$ with $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ and $\sigma \in \mathcal{A}(\tau)$ satisfies a central limit theorem. In particular, the following result states that the probability for $L_2\left(\mathcal{H}_{b,n}^{\Sigma}\right) \leq c\sqrt{n}$ with randomly chosen $\Sigma \in \{\sigma, \overline{\sigma}\}^n$, can be made arbitrarily close to 1 by choosing the constant c large enough.

Corollary 3 *Let $\sigma \in \mathcal{A}(\tau)$ and let $\overline{\sigma} := \tau \circ \sigma$. Then for any real $y \geq 0$ we have*

$$\lim_{n\to\infty} \frac{\#\left\{\Sigma \in \{\sigma, \overline{\sigma}\}^n \,:\, L_2(\mathcal{H}_{b,n}^{\Sigma}) \leq \sqrt{\Phi_b^{\sigma,(2)} - (\Phi_b^{\sigma})^2(1-y^2)}\sqrt{n}\right\}}{2^n} = 2\phi(y) - 1,$$

where $\phi(y) = \frac{1}{2\pi}\int_{-\infty}^{y} \mathrm{e}^{-\frac{t^2}{2}}\,\mathrm{d}t$ denotes the normal distribution function.

Proof. We denote the right hand side of the formula in Theorem 2 by $d_b(n,l)$. Then we have

$$\frac{\#\left\{\Sigma \in \{\sigma, \tau\}^n \,:\, L_2(\mathcal{H}_{b,n}^{\Sigma}) \leq x\sqrt{n}\right\}}{2^n} = \frac{1}{2^n} \sum_{\substack{l=0 \\ \sqrt{d_b(n,l)} \leq x\sqrt{n}}}^{n} \binom{n}{l}.$$

We have $\sqrt{d_b(n,l)} \leq x\sqrt{n}$ if and only if $a_n^-(x) \leq l \leq a_n^+(x)$, where

$$a_n^{\pm}(x) := \frac{n}{2} - \left(1 - \frac{1}{2b^n}\right)\frac{1}{4\Phi_b^{\sigma}} \pm \frac{\sqrt{4n((\Phi_b^{\sigma})^2 - \Phi_b^{\sigma,(2)} + x^2) + O(1)}}{4\Phi_b^{\sigma}}.$$

Therefore

$$\frac{\#\left\{\Sigma \in \{\sigma, \overline{\sigma}\}^n \,:\, L_2(\mathcal{H}_{b,n}^{\Sigma}) \leq x\sqrt{n}\right\}}{2^n} = \frac{1}{2^n} \sum_{a_n^-(x) \leq l \leq a_n^+(x)} \binom{n}{l}.$$

For $x \geq \sqrt{\Phi_b^{\sigma,(2)} - (\Phi_b^{\sigma})^2}$ we have

$$\lim_{n\to\infty} \frac{a_n^{\pm}(x) - \frac{n}{2}}{\sqrt{\frac{n}{4}}} = \pm\frac{\sqrt{(\Phi_b^{\sigma})^2 - \Phi_b^{\sigma,(2)} + x^2}}{\Phi_b^{\sigma}}$$

and the result follows from the central limit theorem together with the substitution $x = \sqrt{\Phi_b^{\sigma,(2)} - (\Phi_b^{\sigma})^2(1-y^2)}$. $\square$

3 Auxiliary results

In this section we prepare the basic tools which are used for the proof of Theorem 1 and Theorem 2. Some of the following results are interesting on their own.

Basic properties of φ_b^{σ}. We begin with some basic properties of the functions $\varphi_{b,h}^{\sigma}$ resp. φ_b^{σ}. It has been shown in [2, Propriété 3.4] that

$$(\varphi_{b,h}^{\sigma})'(k/b + 0) = (\varphi_{b,h}^{id})'(\sigma(k)/b + 0) \tag{3}$$

and from [2, Propriété 3.5] it is known that

$$\varphi_b^{\sigma}(k/b) = \frac{1}{b}\sum_{j=0}^{k-1} (\varphi_b^{\sigma})'(j/b + 0). \tag{4}$$

For $\sigma = id$ we have

$$\varphi_{b,h}^{id}(x) = \begin{cases} (b-h)x & \text{if } x \in [0, h/b], \\ h(1-x) & \text{if } x \in [h/b, 1]. \end{cases} \tag{5}$$

A formula for the discrepancy function. The following lemma provides a formula for the discrepancy function of generalized Hammersley point sets. This formula has been used already in [3].

Lemma 1 *For integers* $1 \leq \lambda, N \leq b^n$ *we have*

$$E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right) = \sum_{j=1}^{n} \varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right),$$

where the $\varepsilon_j = \varepsilon_j(\lambda, n, N)$ *can be given explicitly.*

As the exact definition of the ε_j's is not so important here and as this definition is of a very technical nature we omit it here. A proof of the above result together with explicit expressions for the ε_j's can be found in [3, Lemma 1].

Remark 3 Let $0 \leq x, y \leq 1$ be arbitrary. Since all points from $\mathcal{H}_{b,n}^{\Sigma}$ have coordinates of the form α/b^n for some $\alpha \in \{0, 1, \ldots, b^n - 1\}$, we have

$$E(x, y, \mathcal{H}_{b,n}^{\Sigma}) = E(x(n), y(n), \mathcal{H}_{b,n}^{\Sigma}) + b^n(x(n)y(n) - xy),$$

where for $0 \leq x \leq 1$ we define $x(n) := \min\{\alpha/b^n \geq x \, : \, \alpha \in \{0, \ldots, b^n\}\}$.

More involved properties of φ_b^{σ}. We give a series of lemmas which provide important properties of the functions $\varphi_{b,h}^{\sigma}$ resp. φ_b^{σ}. These results finally lead to the proof of Theorem 1 and Theorem 2.

A proof for the subsequent lemma can be found in [3, Lemma 2].

Lemma 2 *For* $1 \leq N \leq b^n$, $0 \leq j_1 < j_2 < \ldots < j_k < n$ *and* $r_1, \ldots, r_k \in \mathbb{N}$ *we have*

$$\sum_{\lambda=1}^{b^n}\left(\varphi_{b,\varepsilon_{j_1}}^{\sigma_{j_1}}\left(\frac{N}{b^{j_1}}\right)\right)^{r_1} \cdots \left(\varphi_{b,\varepsilon_{j_k}}^{\sigma_{j_k}}\left(\frac{N}{b^{j_k}}\right)\right)^{r_k} = b^{n-k}\varphi_b^{\sigma_{j_1},(r_1)}\left(\frac{N}{b^{j_1}}\right) \cdots \varphi_b^{\sigma_{j_k},(r_k)}\left(\frac{N}{b^{j_k}}\right),$$

where $\varphi_b^{\sigma,(r)} := \sum_{h=0}^{b-1}(\varphi_{b,h}^{\sigma})^r$.

Lemma 3 *Let* $\sigma \in \mathfrak{S}_b$ *and let* $\overline{\sigma} = \tau \circ \sigma$*. For any* $h \in \{0, \ldots, b-1\}$ *we have* $\varphi_{b,h}^{\overline{\sigma}} = -\varphi_{b,b-h}^{\sigma}$*. Furthermore, we have* $\varphi_b^{\sigma,(r)} = (-1)^r \varphi_b^{\overline{\sigma},(r)}$.

Proof. With Eq. (3) together with the fact that $\varphi_{b,h}^{\tau} = -\varphi_{b,b-h}^{id}$, as shown in [3, Lemma 4], we obtain

$$\begin{aligned} (\varphi_{b,h}^{\overline{\sigma}})'(k/b) &= (\varphi_{b,h}^{id})'(\overline{\sigma}(k)/b) = (\varphi_{b,h}^{id})'(\tau(\sigma(k))/b) \\ &= (\varphi_{b,h}^{\tau})'(\sigma(k)/b) = -(\varphi_{b,b-h}^{id})'(\sigma(k)/b) = -(\varphi_{b,b-h}^{\sigma})'(k/b). \end{aligned}$$

Since for any permutation σ the function $\varphi_{b,h}^{\sigma}$ is linear on any interval $[k/b, (k+1)/b]$ and since $\varphi_{b,h}^{\sigma}(0) = 0$ the first result follows. The second result follows easily from the first one. $\square$

Lemma 4 *Let $\sigma \in \mathfrak{S}_b$. For $1 \leq i, j \leq n$, $i \neq j$ we have*

$$\sum_{N=1}^{b^n} \varphi_b^\sigma\left(\frac{N}{b^j}\right) = b^n \int_0^1 \varphi_b^\sigma(x)\,\mathrm{d}x, \tag{6}$$

and

$$\sum_{N=1}^{b^n} \varphi_b^\sigma\left(\frac{N}{b^i}\right)\varphi_b^\sigma\left(\frac{N}{b^j}\right) = b^n \left(\int_0^1 \varphi_b^\sigma(x)\,\mathrm{d}x\right)^2, \tag{7}$$

and

$$\sum_{N=1}^{b^n} \varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right) = b^n \left(\int_0^1 \varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x + \frac{b(b^2-1)}{36 b^{2j}}\right). \tag{8}$$

Proof. We start with the proof of Eq. (6). Using the periodicity of φ_b^σ we have

$$\sum_{N=1}^{b^n} \varphi_b^\sigma\left(\frac{N}{b^j}\right) = \sum_{N=0}^{b^n-1} \varphi_b^\sigma\left(\frac{N}{b^j}\right) = b^{n-j} \sum_{N=0}^{b^j-1} \varphi_b^\sigma\left(\frac{N}{b^j}\right). \tag{9}$$

Since φ_b^σ is linear on the intervals $[k/b, (k+1)/b]$ we obtain from the trapezoidal rule for $0 \leq N < b^j$,

$$\int_{\frac{N}{b^j}}^{\frac{N+1}{b^j}} \varphi_b^\sigma(x)\,\mathrm{d}x = \frac{\varphi_b^\sigma\left(\frac{N}{b^j}\right) + \varphi_b^\sigma\left(\frac{N+1}{b^j}\right)}{2b^j}.$$

Hence

$$\int_0^1 \varphi_b^\sigma(x)\,\mathrm{d}x = \sum_{N=0}^{b^j-1} \int_{\frac{N}{b^j}}^{\frac{N+1}{b^j}} \varphi_b^\sigma(x)\,\mathrm{d}x = \sum_{N=0}^{b^j-1} \frac{\varphi_b^\sigma\left(\frac{N}{b^j}\right) + \varphi_b^\sigma\left(\frac{N+1}{b^j}\right)}{2b^j} = \frac{1}{b^j} \sum_{N=0}^{b^j-1} \varphi_b^\sigma\left(\frac{N}{b^j}\right), \tag{10}$$

since $\varphi_b^\sigma(0) = \varphi_b^\sigma(1) = 0$. Inserting (10) into (9) yields Eq. (6).

We turn to the proof of Eq. (7). Let $i =: i_1$ and $j =: i_2$. We may assume that $i_1 < i_2$. For $0 \leq N < b^n$ let $N = N_0 + N_1 b + \cdots + N_{n-1} b^{n-1}$ be it's b-adic representation. Then we have

$$\begin{aligned}
\sum_{N=1}^{b^n} \prod_{l=1}^{2} \varphi_b^\sigma\left(\frac{N}{b^{i_l}}\right) &= \sum_{N=0}^{b^n-1} \prod_{l=1}^{2} \varphi_b^\sigma\left(\frac{N}{b^{i_l}}\right) \\
&= \sum_{N_0,\ldots,N_{n-1}=0}^{b-1} \prod_{l=1}^{2} \varphi_b^\sigma\left(\frac{N_0 + N_1 b + \cdots + N_{n-1} b^{n-1}}{b^{i_l}}\right) \\
&= b^{n-i_2} \sum_{N_0,\ldots,N_{i_2-2}=0}^{b-1} \varphi_b^\sigma\left(\frac{N_0 + \cdots + N_{i_1-1} b^{i_1-1}}{b^{i_1}}\right) \\
&\qquad \times \sum_{k=0}^{b-1} \varphi_b^\sigma\left(\frac{k}{b} + \frac{N_0 + \cdots + N_{i_2-2} b^{i_2-2}}{b^{i_2}}\right).
\end{aligned}$$

Let $t := \frac{N_0+\cdots+N_{i_2-2}b^{i_2-2}}{b^{i_2}} \in [0,1/b)$. From the linearity of $\varphi_b^\sigma(x)$ for $x \in [k/b,(k+1)/b]$ it follows that $\varphi_b^\sigma\left(\frac{k}{b}+t\right) = \varphi_b^\sigma\left(\frac{k}{b}\right) + tb\left(\varphi_b^\sigma\left(\frac{k+1}{b}\right) - \varphi_b^\sigma\left(\frac{k}{b}\right)\right)$. Hence

$$\begin{aligned}\sum_{k=0}^{b-1}\varphi_b^\sigma\left(\frac{k}{b}+t\right) &= \sum_{k=0}^{b-1}\varphi_b^\sigma\left(\frac{k}{b}\right) + bt\sum_{k=0}^{b-1}\left(\varphi_b^\sigma\left(\frac{k+1}{b}\right) - \varphi_b^\sigma\left(\frac{k}{b}\right)\right) \\ &= \sum_{k=0}^{b-1}\varphi_b^\sigma\left(\frac{k}{b}\right) + bt(\varphi_b^\sigma(1)-\varphi_b^\sigma(0)) = b\int_0^1 \varphi_b^\sigma(x)\,\mathrm{d}x,\end{aligned}$$

where we used Eq. (10) with $j=1$ and the periodicity of φ_b^σ for the last equality. Therefore we obtain

$$\begin{aligned}\sum_{N=1}^{b^n}\prod_{l=1}^{2}\varphi_b^\sigma\left(\frac{N}{b^{i_l}}\right) &= b^{n-i_2+1}\left(\int_0^1 \varphi_b^\sigma(x)\,\mathrm{d}x\right)\sum_{N_0,\ldots,N_{i_2-2}=0}^{b-1}\varphi_b^\sigma\left(\frac{N_0+\cdots+N_{i_1-1}b^{i_1-1}}{b^{i_1}}\right) \\ &= \left(\int_0^1 \varphi_b^\sigma(x)\,\mathrm{d}x\right)\sum_{N=0}^{b^n-1}\varphi_b^\sigma\left(\frac{N}{b^{i_1}}\right) = b^n\left(\int_0^1 \varphi_b^\sigma(x)\,\mathrm{d}x\right)^2,\end{aligned}$$

where we used Eq. (6) for the last equality. This gives Eq. (7).

Finally, we prove Eq. (8). First let $j \geq 2$. The function $\varphi_b^{\sigma,(2)}(x)$ for $x \in [k/b,(k+1)/b]$ is a quadratic polynomial $a_kx^2+b_kx+c_k$. Hence from Simpson's rule we obtain

$$\int_{\frac{N}{b^j}}^{\frac{N+2}{b^j}} \varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x = \frac{1}{3b^j}\left(\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right) + 4\varphi_b^{\sigma,(2)}\left(\frac{N+1}{b^j}\right) + \varphi_b^{\sigma,(2)}\left(\frac{N+2}{b^j}\right)\right),$$

whenever $\frac{N}{b^j}, \frac{N+1}{b^j}, \frac{N+2}{b^j} \in \left[\frac{k}{b}, \frac{k+1}{b}\right]$. Hence for $0 \leq k < b$ we obtain

$$\begin{aligned}&\sum_{N=kb^{j-1}}^{(k+1)b^{j-1}-2}\frac{1}{3b^j}\left(\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right) + 4\varphi_b^{\sigma,(2)}\left(\frac{N+1}{b^j}\right) + \varphi_b^{\sigma,(2)}\left(\frac{N+2}{b^j}\right)\right) \\ &= \sum_{N=kb^{j-1}}^{(k+1)b^{j-1}-2}\left\{\int_{\frac{N}{b^j}}^{\frac{N+1}{b^j}} + \int_{\frac{N+1}{b^j}}^{\frac{N+2}{b^j}}\right\}\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x \\ &= \int_{\frac{k}{b}}^{\frac{k+1}{b}-\frac{1}{b^j}}\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x + \int_{\frac{k}{b}+\frac{1}{b^j}}^{\frac{k+1}{b}}\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x \\ &= 2\int_{\frac{k}{b}}^{\frac{k+1}{b}}\varphi_b^{\sigma,(2)}(x) - \int_{\frac{k}{b}}^{\frac{k}{b}+\frac{1}{b^j}}\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x - \int_{\frac{k+1}{b}-\frac{1}{b^j}}^{\frac{k+1}{b}}\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x.\end{aligned}$$

Summation over all $k=0,\ldots,b-1$ yields

$$\begin{aligned}&\sum_{N=0}^{b^j-1}\frac{1}{3b^j}\left(\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right) + 4\varphi_b^{\sigma,(2)}\left(\frac{N+1}{b^j}\right) + \varphi_b^{\sigma,(2)}\left(\frac{N+2}{b^j}\right)\right) \\ &= 2\int_0^1 \varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x - \sum_{k=0}^{b-1}\left\{\int_{\frac{k}{b}}^{\frac{k}{b}+\frac{1}{b^j}} + \int_{\frac{k+1}{b}-\frac{1}{b^j}}^{\frac{k+1}{b}}\right\}\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x \\ &\quad + \sum_{k=0}^{b-1}\frac{1}{3b^j}\left(\varphi_b^{\sigma,(2)}\left(\frac{k+1}{b}-\frac{1}{b^j}\right) + 4\varphi_b^{\sigma,(2)}\left(\frac{k+1}{b}\right) + \varphi_b^{\sigma,(2)}\left(\frac{k+1}{b}+\frac{1}{b^j}\right)\right).\end{aligned}$$

Now, using again the periodicity of $\varphi_b^{\sigma,(2)}$, we have

$$\begin{aligned}
&\sum_{N=0}^{b^j-1}\left(\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right)+4\varphi_b^{\sigma,(2)}\left(\frac{N+1}{b^j}\right)+\varphi_b^{\sigma,(2)}\left(\frac{N+2}{b^j}\right)\right)\\
&\quad=\sum_{N=0}^{b^j-1}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right)+4\sum_{N=1}^{b^j}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right)+\sum_{N=2}^{b^j+1}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right)=6\sum_{N=0}^{b^j-1}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{N=0}^{b^j-1}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right)\\
&\quad= b^j\int_0^1\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x-\frac{b^j}{2}\sum_{k=0}^{b-1}\left\{\int_{\frac{k}{b}}^{\frac{k}{b}+\frac{1}{b^j}}+\int_{\frac{k+1}{b}-\frac{1}{b^j}}^{\frac{k+1}{b}}\right\}\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x\\
&\qquad+\frac{1}{6}\sum_{k=1}^{b}\left(\varphi_b^{\sigma,(2)}\left(\frac{k}{b}-\frac{1}{b^j}\right)+4\varphi_b^{\sigma,(2)}\left(\frac{k}{b}\right)+\varphi_b^{\sigma,(2)}\left(\frac{k}{b}+\frac{1}{b^j}\right)\right)\\
&\quad= b^j\int_0^1\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x\\
&\qquad+\frac{b^j}{2}\sum_{k=1}^{b}\left(\frac{\varphi_b^{\sigma,(2)}\left(\frac{k}{b}-\frac{1}{b^j}\right)+4\varphi_b^{\sigma,(2)}\left(\frac{k}{b}\right)+\varphi_b^{\sigma,(2)}\left(\frac{k}{b}+\frac{1}{b^j}\right)}{3b^j}-\int_{\frac{k}{b}-\frac{1}{b^j}}^{\frac{k}{b}+\frac{1}{b^j}}\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x\right)\\
&\quad=: b^j\left(\int_0^1\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x+\frac{A(j,k,\sigma)}{2}\right),
\end{aligned}$$

and, using the periodicity of $\varphi_b^{\sigma,(2)}$, we get

$$\sum_{N=1}^{b^n}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right)=b^{n-j}\sum_{N=0}^{b^j}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^j}\right)=b^n\left(\int_0^1\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x+\frac{A(j,k,\sigma)}{2}\right)$$

for all $j\geq 2$. For $j=1$ this equation can be checked directly.

It remains to evaluate

$$A(j,k,\sigma)=\sum_{k=1}^{b}\left(\frac{\varphi_b^{\sigma,(2)}\left(\frac{k}{b}-\frac{1}{b^j}\right)+4\varphi_b^{\sigma,(2)}\left(\frac{k}{b}\right)+\varphi_b^{\sigma,(2)}\left(\frac{k}{b}+\frac{1}{b^j}\right)}{3b^j}-\int_{\frac{k}{b}-\frac{1}{b^j}}^{\frac{k}{b}+\frac{1}{b^j}}\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x\right).$$

For $1\leq k\leq b$ let $h_k(x)=\alpha_k x^2+\beta_k x+\gamma_k$ with $h_k\left(\frac{k}{b}-\frac{1}{b^j}\right)=\varphi_b^{\sigma,(2)}\left(\frac{k}{b}-\frac{1}{b^j}\right)$, $h_k\left(\frac{k}{b}\right)=\varphi_b^{\sigma,(2)}\left(\frac{k}{b}\right)$ and $h_k\left(\frac{k}{b}+\frac{1}{b^j}\right)=\varphi_b^{\sigma,(2)}\left(\frac{k}{b}+\frac{1}{b^j}\right)$. Then by Simpson's rule we have

$$\int_{\frac{k}{b}-\frac{1}{b^j}}^{\frac{k}{b}+\frac{1}{b^j}}h_k(x)\,\mathrm{d}x=\frac{\varphi_b^{\sigma,(2)}\left(\frac{k}{b}-\frac{1}{b^j}\right)+4\varphi_b^{\sigma,(2)}\left(\frac{k}{b}\right)+\varphi_b^{\sigma,(2)}\left(\frac{k}{b}+\frac{1}{b^j}\right)}{3b^j}.$$

By tedious but straightforward algebra it can be shown that

$$\int_{\frac{k}{b}-\frac{1}{b^j}}^{\frac{k}{b}+\frac{1}{b^j}}h_k(x)\,\mathrm{d}x-\int_{\frac{k}{b}-\frac{1}{b^j}}^{\frac{k}{b}+\frac{1}{b^j}}\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x=\frac{1}{6b^{2j}}\left(\left(\varphi_b^{\sigma,(2)}\right)'\left(\frac{k}{b}-0\right)-\left(\varphi_b^{\sigma,(2)}\right)'\left(\frac{k}{b}+0\right)\right).$$

By definition we have $\varphi_b^{\sigma,(2)} = \sum_{h=0}^{b-1} (\varphi_{b,h}^{\sigma})^2$ and hence

$$\begin{aligned}
&\left(\varphi_b^{\sigma,(2)}\right)' \left(\frac{k}{b} - 0\right) - \left(\varphi_b^{\sigma,(2)}\right)' \left(\frac{k}{b} + 0\right) \\
&= 2\sum_{h=0}^{b-1} \varphi_{b,h}^{\sigma}\left(\frac{k}{b}\right)\left(\left(\varphi_{b,h}^{\sigma}\right)'\left(\frac{k}{b}-0\right) - \left(\varphi_{b,h}^{\sigma}\right)'\left(\frac{k}{b}+0\right)\right) \\
&= 2\sum_{h=0}^{b-1} \varphi_{b,h}^{\sigma}\left(\frac{k}{b}\right)\left(\left(\varphi_{b,h}^{\sigma}\right)'\left(\frac{k-1}{b}+0\right) - \left(\varphi_{b,h}^{\sigma}\right)'\left(\frac{k}{b}+0\right)\right).
\end{aligned}$$

For short we define $f_{h,k} := \left(\varphi_{b,h}^{\sigma}\right)'\left(\frac{k}{b}+0\right)$. Hence we have

$$A(j,k,\sigma) = \frac{1}{3b^{2j}} \sum_{h=0}^{b-1} \sum_{k=1}^{b} \varphi_{b,h}^{\sigma}\left(\frac{k}{b}\right)(f_{h,k-1} - f_{h,k}). \tag{11}$$

Since $\varphi_{b,h}^{\sigma}$ is linear on every interval $[k/b,(k+1)/b]$ we have $\varphi_{b,h}^{\sigma}(k/b) = \int_0^{k/b} \left(\varphi_{b,h}^{\sigma}\right)'(x)\,\mathrm{d}x = \frac{1}{b}\sum_{l=0}^{k-1} f_{h,l}$ and especially $\sum_{l=0}^{b-1} f_{h,l} = 0$. Hence for every fixed h we obtain

$$\sum_{k=1}^{b} \varphi_{b,h}^{\sigma}\left(\frac{k}{b}\right)(f_{h,k-1} - f_{h,k}) = \frac{1}{b}\sum_{l=0}^{b-1} f_{h,l} \sum_{k=l+1}^{b} (f_{h,k-1} - f_{h,k}) = \frac{1}{b}\sum_{l=0}^{b-1} f_{h,l}^2. \tag{12}$$

Inserting (12) into (11) and using Eq. (3) gives

$$\begin{aligned}
A(j,k,\sigma) &= \frac{1}{3b^{2j+1}} \sum_{h=0}^{b-1}\sum_{l=0}^{b-1} \left(\left(\varphi_{b,h}^{\sigma}\right)'\left(\frac{l}{b}+0\right)\right)^2 = \frac{1}{3b^{2j+1}} \sum_{h=0}^{b-1}\sum_{l=0}^{b-1} \left(\left(\varphi_{b,h}^{id}\right)'\left(\frac{\sigma(l)}{b}+0\right)\right)^2 \\
&= \frac{1}{3b^{2j+1}} \sum_{h=0}^{b-1}\sum_{l=0}^{b-1} \left(\left(\varphi_{b,h}^{id}\right)'\left(\frac{l}{b}+0\right)\right)^2 = A(j,k,id).
\end{aligned}$$

This means that $A(j,k,\sigma)$ does not depend on the choice of the permutation σ. Now we may use known results for the case $\sigma = id$. It has been shown in [3, Lemma 5] that

$$\sum_{N=1}^{b^n} \varphi_b^{id,(2)}\left(\frac{N}{b^j}\right) = b^n\left(\int_0^1 \varphi_b^{id,(2)}(x)\,\mathrm{d}x + \frac{b(b^2-1)}{36b^{2j}}\right)$$

(we remark that $\int_0^1 \varphi_b^{id,(2)}(x)\,\mathrm{d}x = \frac{b^4-1}{90b}$ which follows from [3, Lemma 3]). Hence

$$\frac{A(j,k,\sigma)}{2} = \frac{A(j,k,id)}{2} = \frac{b(b^2-1)}{36b^{2j}}$$

and this finishes the proof. □

Lemma 5 *For any $\sigma \in \mathfrak{S}_b$ we have*

$$\int_0^1 \varphi_b^{\sigma}(x)\,\mathrm{d}x = \frac{1}{b}\sum_{k=0}^{b-1} \sigma(k)k - \left(\frac{b-1}{2}\right)^2.$$

In particular $\int_0^1 \varphi_b^{\sigma}(x)\,\mathrm{d}x = \int_0^1 \varphi_b^{\sigma^{-1}}(x)\,\mathrm{d}x$.

Proof. Using integration by parts and (3) we have

$$\begin{aligned}\int_0^1 \varphi_b^\sigma(x)\,\mathrm{d}x &= x\varphi_b^\sigma(x)\Big|_0^1 - \int_0^1 x(\varphi_b^\sigma)'(x)\,\mathrm{d}x = -\sum_{k=0}^{b-1}\int_{\frac{k}{b}}^{\frac{k+1}{b}} x(\varphi_b^\sigma)'\left(\frac{k}{b}+0\right)\,\mathrm{d}x \\ &= -\sum_{k=0}^{b-1}(\varphi_b^{id})'\left(\frac{\sigma(k)}{b}+0\right)\frac{2k+1}{2b^2}.\end{aligned}$$

From Eq. (5) we obtain

$$\left(\varphi_{b,h}^{id}\right)'(x+0) = \begin{cases} b-h & \text{if } x\in[0,h/b], \\ -h & \text{if } x\in[h/b,1],\end{cases}$$

and therefore for any $0\le l<b$ we have

$$\begin{aligned}\left(\varphi_b^{id}\right)'\left(\frac{l}{b}+0\right) &= \sum_{h=0}^{b-1}(\varphi_{b,h}^{id})'\left(\frac{l}{b}+0\right) \\ &= \sum_{h=0}^{l}(-h) + \sum_{h=l+1}^{b-1}(b-h) = \frac{b(b-1-2l)}{2}.\end{aligned} \tag{13}$$

Therefore we have

$$\int_0^1 \varphi_b^\sigma(x)\,\mathrm{d}x = -\sum_{k=0}^{b-1}\frac{(b-1-2\sigma(k))(2k+1)}{4b} = \frac{1}{b}\sum_{k=0}^{b-1}\sigma(k)k - \left(\frac{b-1}{2}\right)^2.$$

□

Lemma 6 *We have $\sigma\in\mathcal{A}(\tau)$ if and only if for all $x\in[0,1]$ we have $\varphi_b^\sigma(x)=\varphi_b^\sigma(1-x)$.*

Proof. Since φ_b^σ is continuous, piecewise linear and $\varphi_b^\sigma(0)=\varphi_b^\sigma(1)=0$, we have $\varphi_b^\sigma(x)=\varphi_b^\sigma(1-x)$ if and only if $(\varphi_b^\sigma)'(x)=-(\varphi_b^\sigma)'(1-x)$ for all $x\in[0,1]$. Now if $\sigma\in\mathcal{A}(\tau)$, i.e., $\sigma(k)+\sigma(b-k-1)=b-1$, we have with Eq. (13),

$$\begin{aligned}(\varphi_b^\sigma)'\left(\frac{k}{b}+0\right) &= \frac{b(b-1)}{2} - b\sigma(k) = \frac{b(b-1)}{2} - b(b-1-\sigma(b-k-1)) \\ &= -\left(\frac{b(b-1)}{2} - b\sigma(b-k-1)\right) = -(\varphi_b^\sigma)'\left(1-\frac{k+1}{b}+0\right).\end{aligned}$$

This gives the desired property on the interval $\left[\frac{k}{b},\frac{k+1}{b}\right]$ for $(\varphi_b^\sigma)'$ and vice versa. □

4 The proof of Theorem 1 and Theorem 2

First we give a discrete version of Theorem Theorem 2.

Lemma 7 *Let $\sigma\in\mathfrak{S}_b$ and let $\overline{\sigma}:=\tau\circ\sigma$. Let $\Sigma\in\{\sigma,\overline{\sigma}\}^n$ and let l to denote the number of components of Σ which are equal to σ. Then we have*

$$\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right) = (2l-n)\Phi_b^\sigma \tag{14}$$

and

$$\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\right)^2 = n\Phi_b^{\sigma,(2)} + ((n-2l)^2-n)(\Phi_b^{\sigma})^2 + \frac{1}{36}\left(1-\frac{1}{b^{2n}}\right). \quad (15)$$

Here $\Phi_b^{\sigma} := \frac{1}{b}\int_0^1 \varphi_b^{\sigma}(x)\,\mathrm{d}x$ *and* $\Phi_b^{\sigma,(2)} := \frac{1}{b}\int_0^1 \varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x$.

Proof. Let $\Sigma = (\sigma_0,\ldots,\sigma_{n-1}) \in \{\sigma,\overline{\sigma}\}^n$ and define for $1 \le i \le n$,

$$s_i := \begin{cases} 1 & \text{if } \sigma_{i-1} = \overline{\sigma}, \\ 0 & \text{if } \sigma_{i-1} = \sigma. \end{cases}$$

For Eq. (14) we use Lemma 1, Lemma 2, Lemma 3 with the definition of the s_i and Eq. (6) from Lemma 4 (in that order) to obtain

$$\begin{aligned}
\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right) &= \frac{1}{b^{2n}}\sum_{j=1}^{n}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) \\
&= \frac{1}{b^{n+1}}\sum_{j=1}^{n}\sum_{N=1}^{b^n}\varphi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) \\
&= \frac{1}{b^{n+1}}\sum_{j=1}^{n}(-1)^{s_j}\sum_{N=1}^{b^n}\varphi_b^{\sigma}\left(\frac{N}{b^j}\right) \\
&= \Phi_b^{\sigma}\sum_{j=1}^{n}(-1)^{s_j} = (2l-n)\Phi_b^{\sigma}.
\end{aligned}$$

Now we prove Eq. (15). Using Lemma 1 we have

$$\begin{aligned}
\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\right)^2 &= \frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\sum_{i,j=1}^{n}\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) \\
&= \frac{1}{b^{2n}}\sum_{i=1}^{n}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\left(\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\right)^2 \\
&\quad + \frac{1}{b^{2n}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right).
\end{aligned}$$

By Lemma 2 we have

$$\sum_{\lambda=1}^{b^n}\left(\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\right)^2 = b^{n-1}\varphi_b^{\sigma_{i-1},(2)}\left(\frac{N}{b^i}\right)$$

and for $i \neq j$,

$$\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right) = b^{n-2}\varphi_b^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right).$$

Therefore we obtain

$$\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\right)^2 = \frac{1}{b^{2n}}\sum_{i=1}^{n}\sum_{N=1}^{b^n} b^{n-1}\varphi_b^{\sigma_{i-1},(2)}\left(\frac{N}{b^i}\right) + \frac{1}{b^{2n}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{N=1}^{b^n} b^{n-2}\varphi_b^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right).$$

From Lemma 3 we find that $\varphi_b^{\sigma,(2)}=\varphi_b^{\overline{\sigma},(2)}$ and $\varphi_b^{\sigma}=-\varphi_b^{\overline{\sigma}}$. Now we obtain

$$\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\right)^2 = \frac{1}{b^{2n}}\sum_{i=1}^{n}\sum_{N=1}^{b^n} b^{n-1}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^i}\right) + \frac{1}{b^{2n}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}(-1)^{s_i+s_j}\sum_{N=1}^{b^n} b^{n-2}\varphi_b^{\sigma}\left(\frac{N}{b^i}\right)\varphi_b^{\sigma}\left(\frac{N}{b^j}\right).$$

Using Eq. (8) from Lemma 4 we obtain

$$\begin{aligned}\sum_{i=1}^{n}\sum_{N=1}^{b^n} b^{n-1}\varphi_b^{\sigma,(2)}\left(\frac{N}{b^i}\right) &= b^{n-1}\sum_{i=1}^{n} b^n\left(\int_0^1\varphi_b^{\sigma,(2)}(x)\,\mathrm{d}x+\frac{b(b^2-1)}{36b^{2j}}\right)\\ &= b^{2n}n\Phi_b^{\sigma,(2)}+b^{2n}\sum_{i=1}^{n}\frac{b^2-1}{36b^{2j}} = b^{2n}n\Phi_b^{\sigma,(2)}+\frac{b^{2n}}{36}\left(1-\frac{1}{b^{2n}}\right),\end{aligned}$$

and, by Eq. (7) from Lemma 4 for $i\neq j$,

$$\sum_{N=1}^{b^n}\varphi_b^{\sigma}\left(\frac{N}{b^i}\right)\varphi_b^{\sigma}\left(\frac{N}{b^j}\right) = b^n\left(\int_0^1\varphi_b^{\sigma}(x)\,\mathrm{d}x\right)^2 = b^{n+2}(\Phi_b^{\sigma})^2.$$

Hence

$$\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\right)^2 = n\Phi_b^{\sigma,(2)}+\frac{1}{36}\left(1-\frac{1}{b^{2n}}\right)+\sum_{\substack{i,j=1\\ i\neq j}}^{n}(-1)^{s_i+s_j}(\Phi_b^{\sigma})^2.$$

Finally we note that $\sum_{\substack{i,j=1\\ i\neq j}}^{n}(-1)^{s_i+s_j}=\left(\sum_{i=1}^{n}(-1)^{s_i}\right)^2-n=(n-2l)^2-n$, from which the result follows. □

Now, we give the proof of Theorem 2. For the proof of Theorem 1 we add some remarks subsequent this proof.

Proof. We have

$$\begin{aligned}
(L_2(\mathcal{H}_{b,n}^{\Sigma}))^2 &= \int_0^1\int_0^1 (E(x,y,\mathcal{H}_{b,n}^{\Sigma}))^2\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1\int_0^1 \left(E(x(n),y(n),\mathcal{H}_{b,n}^{\Sigma}) + b^n(x(n)y(n)-xy)\right)^2\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\right)^2 \\
&\quad +2b^n\sum_{\lambda,N=1}^{b^n}\int_{\frac{\lambda-1}{b^n}}^{\frac{\lambda}{b^n}}\int_{\frac{N-1}{b^n}}^{\frac{N}{b^n}} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\left(\frac{\lambda}{b^n}\frac{N}{b^n}-xy\right)\mathrm{d}x\,\mathrm{d}y \\
&\quad +b^{2n}\sum_{\lambda,N=1}^{b^n}\int_{\frac{\lambda-1}{b^n}}^{\frac{\lambda}{b^n}}\int_{\frac{N-1}{b^n}}^{\frac{N}{b^n}}\left(\frac{\lambda}{b^n}\frac{N}{b^n}-xy\right)^2\mathrm{d}x\,\mathrm{d}y \\
&=: \Sigma_1+\Sigma_2+\Sigma_3.
\end{aligned}$$

From Eq. (15) of Lemma 7 we find that

$$\Sigma_1 = n\Phi_b^{\sigma,(2)} + ((n-2l)^2-n)(\Phi_b^{\sigma})^2 + \frac{1}{36}\left(1-\frac{1}{b^{2n}}\right)$$

and straightforward algebra shows that $\Sigma_3=(1+18b^n+25b^{2n})/(72b^{2n})$. So it remains to deal with Σ_2. We have

$$\begin{aligned}
\Sigma_2 &= \frac{2}{b^{3n}}\sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\lambda N \\
&\quad -\frac{1}{2b^{3n}}\sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)(2\lambda-1)(2N-1) \\
&= \frac{1}{b^{3n}}\sum_{\lambda,N=1}^{b^n}(\lambda+N)E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right) - \frac{1}{2b^{3n}}\sum_{\lambda,N=1}^{b^n} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right) \\
&=: \Sigma_4-\Sigma_5.
\end{aligned}$$

From Eq. (14) of Lemma 7 we obtain $\Sigma_5=(2l-n)\,\Phi_b^{\sigma}/(2b^n)$ and for Σ_4 we have

$$\Sigma_4 = \frac{1}{b^{3n}}\sum_{\lambda,N=1}^{b^n}\lambda E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right) + \frac{1}{b^{3n}}\sum_{\lambda,N=1}^{b^n} NE\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right) =: \frac{1}{b^{3n}}(\Sigma_{4,1}+\Sigma_{4,2}).$$

Again let $\Sigma=(\sigma_0,\ldots,\sigma_{n-1})\in\{\sigma,\overline{\sigma}\}^n$ and, for $1\le i\le n$,

$$s_i := \begin{cases} 1 & \text{if } \sigma_{i-1}=\overline{\sigma} \\ 0 & \text{if } \sigma_{i-1}=\sigma. \end{cases}$$

Then we have

$$\Sigma_{4,2} = \sum_{i=1}^{n}\sum_{N=1}^{b^n} N\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right) = b^{n-1}\sum_{i=1}^{n}(-1)^{s_i}\sum_{N=1}^{b^n}N\varphi_b^{\sigma}\left(\frac{N}{b^i}\right),$$

where we used Lemma 2. We have

$$\sum_{N=1}^{b^n} N\varphi_b^\sigma\left(\frac{N}{b^j}\right) =$$

$$\begin{array}{rcccccccccc}
\varphi_b^\sigma\left(\frac{1}{b^j}\right) & + & \varphi_b^\sigma\left(\frac{2}{b^j}\right) & + & \cdots & + & \varphi_b^\sigma\left(\frac{b^n-2}{b^j}\right) & + & \varphi_b^\sigma\left(\frac{b^n-1}{b^j}\right) & + & \varphi_b^\sigma\left(\frac{b^n}{b^j}\right)\\
 & + & \varphi_b^\sigma\left(\frac{2}{b^j}\right) & + & \cdots & + & \varphi_b^\sigma\left(\frac{b^n-2}{b^j}\right) & + & \varphi_b^\sigma\left(\frac{b^n-1}{b^j}\right) & + & \varphi_b^\sigma\left(\frac{b^n}{b^j}\right)\\
 & & & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
 & & & & & + & \varphi_b^\sigma\left(\frac{b^n-2}{b^j}\right) & + & \varphi_b^\sigma\left(\frac{b^n-1}{b^j}\right) & + & \varphi_b^\sigma\left(\frac{b^n}{b^j}\right)\\
 & & & & & & & + & \varphi_b^\sigma\left(\frac{b^n-1}{b^j}\right) & + & \varphi_b^\sigma\left(\frac{b^n}{b^j}\right)\\
 & & & & & & & & & + & \varphi_b^\sigma\left(\frac{b^n}{b^j}\right).
\end{array}$$

Since φ_b^σ is 1-periodic and since $\sigma \in \mathcal{A}(\tau)$ and hence, by Lemma 6, $\varphi_b^\sigma(x) = \varphi_b^\sigma(1-x)$ for $x \in [0,1]$, it follows that

$$\sum_{N=1}^{b^n} N\varphi_b^\sigma\left(\frac{N}{b^j}\right) = \frac{b^n}{2}\sum_{N=1}^{b^n} \varphi_b^\sigma\left(\frac{N}{b^j}\right) = \frac{b^{2n}}{2}\int_0^1 \varphi_b^\sigma(x)\,\mathrm{d}x = \frac{b^{2n+1}}{2}\Phi_b^\sigma.$$

This leads to

$$\Sigma_{4,2} = b^{n-1}\sum_{i=1}^{n}(-1)^{s_i}\frac{b^{2n+1}}{2}\Phi_b^\sigma = \frac{b^{3n}}{2}\Phi_b^\sigma\sum_{i=1}^{n}(-1)^{s_i} = \frac{b^{3n}}{2}(2l-n)\Phi_b^\sigma.$$

It remains to compute $\Sigma_{4,1}$. We have

$$\begin{aligned}
&\mathcal{H}_{b,n}^{\Sigma}\\
&= \left\{\left(\frac{\sigma_0(a_0)}{b}+\cdots+\frac{\sigma_{n-1}(a_{n-1})}{b^n}, \frac{a_{n-1}}{b}+\cdots+\frac{a_0}{b^n}\right) : a_0,\ldots,a_{n-1}\in\{0,\ldots,b-1\}\right\}\\
&= \left\{\left(\frac{x_0}{b}+\cdots+\frac{x_{n-1}}{b^{n-1}}, \frac{\sigma_{n-1}^{-1}(x_{n-1})}{b}+\cdots+\frac{\sigma_0^{-1}(x_0)}{b^n}\right) : x_0,\ldots,x_{n-1}\in\{0,\ldots,b-1\}\right\},
\end{aligned}$$

with $(\sigma_0,\ldots,\sigma_{n-1}) \in \{\sigma,\overline{\sigma}\}^n$. Note that for $\sigma\in\mathcal{A}(\tau)$ we also have $\sigma^{-1}\in\mathcal{A}(\tau)$. Let $g:[0,1]^2\to[0,1]^2$ be defined by $g(x,y)=(y,x)$ and for $\Sigma=(\sigma_0,\ldots,\sigma_{n-1})$ define $\Sigma^*=(\sigma_{n-1}^{-1},\ldots,\sigma_0^{-1})\in\{\sigma^{-1},\overline{\sigma^{-1}}\}^n$. Then we have found that $\mathcal{H}_{b,n}^{\Sigma} = g\left(\mathcal{H}_{b,n}^{\Sigma^*}\right)$ and therefore we obtain

$$\begin{aligned}
\Sigma_{4,1} &= \sum_{\lambda,N=1}^{b^n}\lambda E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right) = \sum_{\lambda,N=1}^{b^n}\lambda E\left(\frac{\lambda}{b^n},\frac{N}{b^n},g\left(\mathcal{H}_{b,n}^{\Sigma^*}\right)\right)\\
&= \sum_{\lambda,N=1}^{b^n}\lambda E\left(\frac{N}{b^n},\frac{\lambda}{b^n},\mathcal{H}_{b,n}^{\Sigma^*}\right) = \frac{b^{3n}}{2}(2l-n)\Phi_b^{\sigma^{-1}},
\end{aligned}$$

where for the last equality we used the formula for $\Sigma_{4,2}$ since the number of components of Σ which are equal to σ is the same as the number of components of Σ^* which are equal to σ^{-1}. By Lemma 5 we have $\Phi_b^{\sigma^{-1}}=\Phi_b^\sigma$ and hence $\Sigma_{4,1}=\frac{b^{3n}}{2}(2l-n)\Phi_b^\sigma$. Together we obtain $\Sigma_4=(2l-n)\Phi_b^\sigma$.

Now we obtain the desired formula from $\left(L_2(\mathcal{H}_{b,n}^{\Sigma})\right)^2 = \Sigma_1+\Sigma_4-\Sigma_5+\Sigma_3$. The evaluation of this sum is a matter of straight forward calculations and hence we omit the details. $\square$

For the *Proof of Theorem 1* we just remark that the only place in the proof of Theorem 2 where we used that $\sigma \in \mathcal{A}(\tau)$ was in the exact evaluation of Σ_4. However, it is easy to see that for arbitrary permutations $\sigma \in \mathfrak{S}_b$ we always have $\Sigma_4 = O(n)$ and hence the result of Theorem 1 follows as well from the proof above.

5 Numerical Results

In view of Corollary 2 we search for permutations $\sigma \in \mathcal{A}(\tau)$ giving the minimal L_2 discrepancy for a fixed base b. In fact, we want to minimize the expression $\Phi_b^{\sigma,(2)} - (\Phi_b^\sigma)^2$. To this aim we use an alternative formula for $\Phi_b^{\sigma,(2)}$ that can be derived similarly as the formula for Φ_b^σ given in Lemma 5.

Lemma 8 *For any $\sigma \in \mathfrak{S}_b$ we have*

$$\begin{aligned}\Phi_b^{\sigma,(2)} &= \frac{1-6b^2+9b^3-4b^4}{18b^2} + \\ &\sum_{k_1,k_2=0}^{b-1} \frac{\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}}{b^3}\left(b\max\{k_1,k_2\} - \frac{k_1^2+k_1+k_2^2+k_2}{2}\right).\end{aligned}$$

If in addition $\sigma \in \mathcal{A}(\tau)$ then

$$\Phi_b^{\sigma,(2)} = \frac{1}{2b^3}\left(2bS_3(\sigma) - S_2(\sigma) - (2b-1)S_1(\sigma) - \frac{b(6-11b+6b^2+3b^3-12b^4+8b^5)}{18}\right),$$

where

$$\begin{aligned}S_1(\sigma) &= \sum_{k=0}^{b-1} k\sigma(k), \qquad S_2(\sigma) = \sum_{k=0}^{b-1} k^2\sigma(k)^2, \\ S_3(\sigma) &= \sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}\max\{\sigma(k_1)\sigma(k_2)\}.\end{aligned}$$

From the second formula we have that σ and σ^{-1} can be interchanged. Therefore for $\sigma \in \mathcal{A}(\tau)$ we can replace σ^{-1} by σ in the first formula for $\Phi_b^{\sigma,(2)}$.

Using the alternative formulas from Lemma 5 and 8 we have performed a full search over all permutations $\sigma \in \mathcal{A}(\tau)$ for bases $4 \le b \le 23$. Note that we improved the best results known until now in all of these bases which were obtained for the identical permutation (see (2) — the best value 0.03757 appeared in base 2). In particular the minimal value occurs in base 22 (see Table 1).

Additionally we have performed a full search over all permutations $\sigma \in \mathcal{A}(\tau)$ where $\Phi_b^\sigma = 0$ for bases $b \le 17$, $b \notin \{2,3,6,7,10,14\}$, and tabulated those with the minimal L_2 discrepancy (see Table 2).

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b	$\frac{\Phi_b^{\sigma,(2)}-\left(\Phi_b^{\sigma}\right)^2}{\log b}$	num. value	σ
2	$\frac{5}{192\log(2)}$	0.037570	id
3	$\frac{4}{81\log(3)}$	0.044950	id
4	$\frac{5}{96\log(4)}$	0.037570	(2,1)
5	$\frac{112}{1875\log(5)}$	0.037114	(3,1)
6	$\frac{343}{5184\log(6)}$	0.036927	(4,1)
7	$\frac{512}{7203\log(7)}$	0.036529	(2,0)(5,1)(6,4)
8	$\frac{5}{64\log(8)}$	0.037570	(4,1)(6,3)
9	$\frac{512}{6561\log(9)}$	0.035516	(5,1)(7,3)
10	$\frac{3391}{40000\log(10)}$	0.036817	(2,8,4,6,9,7,1,5,3,0)
11	$\frac{3680}{43923\log(11)}$	0.034940	(7,1)(4,2)(9,3)(8,6)
12	$\frac{1759}{20736\log(12)}$	0.034137	(5,4,10,6,7,1)(8,9,3,2)
13	$\frac{574}{6591\log(13)}$	0.033953	(5,12,7,0)(10,11,2,1)(8,9,4,3)
14	$\frac{41581}{460992\log(14)}$	0.034178	(2,5,7,3,12,4,0)(9,13,11,8,6,10,1)
15	$\frac{4714}{50625\log(15)}$	0.034385	(8,10,12,9,1)(5,13,6,4,2)(11,3)
16	$\frac{17573}{196608\log(16)}$	0.032237	(7,6,14,8,9,1)(12,11,5,2)(4,10,13,3)
17	$\frac{8040}{83521\log(17)}$	0.033977	(9,1)(4,6,2)(13,3)(11,5)(15,7)(14,12,10)
18	$\frac{40631}{419904\log(18)}$	0.033478	(10,15,11,9,5,16,7,2,6,8,12,1)(13,14,4,3)
19	$\frac{12970}{130321\log(19)}$	0.033800	(7,12,13,2,14,15,8,1)(10,17,11,6,5,16,4,3)
20	$\frac{46733}{480000\log(20)}$	0.032500	(11,1)(7,2)(16,3)(14,5)(9,6)(18,8)(13,10)(17,12)
21	$\frac{19402}{194481\log(21)}$	0.032768	(12,1)(7,2)(17,3)(15,5)(9,6)(19,8)(14,11)(18,13)
22	$\frac{278629}{2811072\log(22)}$	**0.032066**	(10,5,7,2,15,8,20,11,16,14,19,6,13,1)(4,18,17,3)
23	$\frac{87112}{839523\log(23)}$	0.033093	(9,21,13,1)(16,20,6,2)(5,12,18,3)(19,17,10,4)(14,15,8,7)

Table 1: Numerical results for the full search in $\mathcal{A}(\tau)$.

b	$\frac{\Phi_b^{\sigma,(2)}}{\log b}$	num. value	σ
4	$\frac{5}{96\log(4)}$	0.037570	(1,3,2,0)
5	$\frac{26}{375\log(5)}$	0.043079	(1,4,3,0)
8	$\frac{5}{64\log(8)}$	0.037570	(2,4,7,5,3,0)(6,1)
9	$\frac{20}{243\log(9)}$	0.037458	(3,8,5,0)(6,7,2,1)
11	$\frac{38}{363\log(11)}$	0.043656	(3,1,6,0)(8,2)(10,7,9,4)
12	$\frac{235}{2592\log(12)}$	0.036486	(3,1,9,5,4,11,8,10,2,6,7,0)
13	$\frac{574}{6591\log(13)}$	0.033953	(5,12,7,0)(10,11,2,1)(8,9,4,3)
15	$\frac{964}{10125\log(15)}$	0.035158	(5,3,10,6,14,9,11,4,8,0)(12,13,2,1)
16	$\frac{37}{384\log(16)}$	0.034752	(4,15,11,0)(12,14,3,1)(8,13,7,2)(6,10,9,5)
17	$\frac{28}{289\log(17)}$	0.034196	(6,16,10,0)(14,15,2,1)(11,13,5,3)(9,12,7,4)

Table 2: Numerical results for the full search in $\mathcal{A}(\tau)$ where $\Phi_b^{\sigma}=0$ (see Remark 2).

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Henri Faure, Institut de Mathématiques de Luminy, U.M.R. 6206 CNRS, 163 avenue de Luminy, case 907, 13288 Marseille Cedex 09, France and Université Paul Cézanne, Aix-en-Provence-Marseille III, France. Email: faure(AT)iml.univ-mrs.fr

Friedrich Pillichshammer and Gottlieb Pirsic, Institut für Finanzmathematik, Universität Linz, Altenbergstraße 69, A-4040 Linz, Austria. Email: friedrich.pillichshammer(AT)jku.at, gpirsic(AT)gmail.com

Wolfgang Ch. Schmid, Fachbereich Mathematik, Universität Salzburg, Hellbrunnerstraße 34, A-5020 Salzburg, Austria. Email: wolfgang.schmid(AT)sbg.ac.at