Distribution Properties of Generalized van der Corput-Halton Sequences and their Subsequences

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Abstract

We study the distribution properties of sequences which are a generalization of the well-known van der Corput-Halton sequences on the one hand, and digital $(\mathbf{T}, s)$-sequences on the other. In this paper we give precise results concerning the distribution properties of such sequences in the s-dimensional unit cube. Moreover, we consider subsequences of the above-mentioned sequences and study their distribution properties. Additionally, we give discrepancy estimates for some special cases, including subsequences of van der Corput and van der Corput-Halton sequences.

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1 Introduction

A sequence $(\boldsymbol{x}_n)_{n\geq 0}$ in the s-dimensional unit cube $[0,1)^s$ is said to be *uniformly distributed modulo one* if for all intervals $[\boldsymbol{a},\boldsymbol{b}) \subseteq [0,1)^s$ we have

$$\lim_{N\to\infty} \frac{\#\{n : 0 \leq n < N, \boldsymbol{x}_n \in [\boldsymbol{a},\boldsymbol{b})\}}{N} = \lambda([\boldsymbol{a},\boldsymbol{b})),$$

where λ denotes the s-dimensional Lebesgue measure.

In this paper, the distribution of a sequence modulo one will often be linked to the distribution properties of a sequence of integers. A sequence $(k_n)_{n\geq 0}$ of integers is said to be uniformly distributed modulo an integer $r \geq 2$, if we have

$$\lim_{N\to\infty} \frac{\#\{n : 0 \leq n < N, k_n \equiv j \pmod{r}\}}{N} = \frac{1}{r} \tag{1}$$

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for all integers $j \in \{0, \ldots, r-1\}$. Furthermore, $(k_n)_{n\geq 0}$ is said to be uniformly distributed in $\mathbb{Z}$, if (1) holds for all integers $r \geq 2$.

Excellent introductions to these and related topics can be found in the book of Kuipers & Niederreiter [12] or in the book of Drmota & Tichy [4].

It is an interesting question which subsequences of a given uniformly distributed sequence are uniformly distributed as well. This problem was studied, for example for the classical one-dimensional $(n\alpha)$-sequences, in detail. In this paper we consider other classical examples of uniformly distributed sequences, namely the van der Corput sequence and its multi-dimensional generalizations such as the van der Corput-Halton sequence or digital $(\mathbf{T}, s)$-sequences, or a hybrid of both. The van der Corput and the van der Corput-Halton sequence are defined as follows.

Definition 1 *Let $q \geq 2$ be an integer. For any nonnegative integer n with base q representation $n = \sum_{i\geq 0} n_i q^i$ (note that this expansion is finite) the radical inverse function to the base q is defined as $\varphi_q(n) = \sum_{i\geq 0} n_i q^{-i-1}$. Now the van der Corput sequence in base q is the sequence $\omega_{\mathrm{vdC}} = (x_n)_{n\geq 0}$ with $x_n = \varphi_q(n)$ for all $n \in \mathbb{N}_0$. For $s \geq 1$ and pairwise relatively prime bases $q_1, \ldots, q_s$ the van der Corput-Halton sequence is given by $(\boldsymbol{x}_n)_{n\geq 0}$ where $\boldsymbol{x}_n = (\varphi_{q_1}(n), \ldots, \varphi_{q_s}(n))$.*

The van der Corput sequence is also the prototype of other multi-dimensional uniformly distributed sequences as for example digital (t, s)-sequences over $\mathbb{Z}_q$ as introduced by Niederreiter (see [19, 20]) or, more generally, digital $(\mathbf{T}, s)$-sequences over $\mathbb{Z}_q$ as introduced by Larcher & Niederreiter [14]. Here, s is the dimension and $\mathbf{T} : \mathbb{N}_0 \to \mathbb{N}_0$ is a quality function for the uniformity of the sequence. The smaller the values of $\mathbf{T}$ are, the better the distribution properties of the sequence. The precise definition of a digital $(\mathbf{T}, s)$-sequence is as follows.

Definition 2 *Let s be a dimension and q be a prime. Let $C_1, \ldots, C_s$ be $\mathbb{N} \times \mathbb{N}$-matrices over the finite field $\mathbb{Z}_q$. We construct a sequence $(\boldsymbol{x}_n)_{n\geq 0}$, $\boldsymbol{x}_n = \left(x_n^{(1)}, \ldots, x_n^{(s)}\right)$, $n \in \mathbb{N}_0$, by generating the i-th coordinate of the n-th point, $x_n^{(i)}$, as follows. Represent $n = n_0 + n_1 q + n_2 q^2 + \cdots$ in base q. Then we set*

$$C_i \cdot (n_0, n_1, \ldots)^\top =: \left(y_0^{(i)}, y_1^{(i)}, \ldots\right)^\top \in \mathbb{Z}_q^\infty$$

and

$$x_n^{(i)} := \frac{y_0^{(i)}}{q} + \frac{y_1^{(i)}}{q^2} + \cdots .$$

For every $m \in \mathbb{N}$ let $\mathbf{T}(m)$, satisfying $0 \leq \mathbf{T}(m) \leq m$, be such that for all $d_1, d_2, \ldots, d_s \in \mathbb{N}_0$ with $d_1 + \cdots + d_s = m - \mathbf{T}(m)$ the $(m - \mathbf{T}(m)) \times m$-matrix consisting of the

left upper $d_1 \times m$-submatrix of C_1 together with the
left upper $d_2 \times m$-submatrix of C_2 together with the
$\vdots$
left upper $d_s \times m$-submatrix of C_s

has rank $m - \mathbf{T}(m)$. Then $(\boldsymbol{x}_n)_{n\geq 0}$ is called a digital $(\mathbf{T}, s)$-sequence over $\mathbb{Z}_q$. If $\mathbf{T}$ is minimal with this property, we speak of a strict digital $(\mathbf{T}, s)$-sequence. If $\mathbf{T}(m) \leq t$ for all m, then we speak of a digital (t, s)-sequence.

In this setting, the one-dimensional van der Corput sequence in (prime) base q can be considered as the digital $(0,1)$-sequence over $\mathbb{Z}_q$ (i.e., $\mathbf{T} \equiv 0$ and $s = 1$) generated by the $\mathbb{N} \times \mathbb{N}$-identity matrix.

Remark 1 *If the sequence $(\boldsymbol{x}_n)_{n \geq 0}$ in $[0,1)^s$ is a digital $(\mathbf{T}, s)$-sequence over $\mathbb{Z}_q$, then for each $m \in \mathbb{N}_0$ and $l \in \mathbb{N}_0$ we have that each interval of the form*

$$E = \prod_{j=1}^{s} \left[\frac{a_j}{q^{d_j}}, \frac{a_j + 1}{q^{d_j}} \right) \subseteq [0,1)^s$$

with volume $\lambda(E) = q^{\mathbf{T}(m)-m}$ contains exactly $q^{\mathbf{T}(m)}$ points of $\{\boldsymbol{x}_n : lq^m \leq n < (l+1)q^m\}$ (see [14]). In general, any sequence which satisfies this conditon is called a $(\mathbf{T}, s)$-sequence in base q. Thus, every digital $(\mathbf{T}, s)$-sequence over $\mathbb{Z}_q$ is also a $(\mathbf{T}, s)$-sequence in base q.

It was shown in [14] that a strict digital $(\mathbf{T}, s)$-sequence is uniformly distributed if and only if

$$\lim_{m \to \infty} (m - \mathbf{T}(m)) = +\infty.$$

Digital $(\mathbf{T}, s)$-sequences can be defined over general finite abelian groups but we restrict ourselves to the more important case of $\mathbb{Z}_q$ with prime q. For further information on this subject, we refer the interested reader to [14].

In the following, we will introduce a more general concept of digital sequences containing the van der Corput-Halton sequences as well as digital $(\mathbf{T}, s)$-sequences as special cases; we are going to discuss distribution properties of these new sequences and their subsequences.

The paper is organized as follows. In Section 2 we define the general class of sequences we are going to consider. These sequences are a mixture of the classical van der Corput-Halton sequence on the one hand and digital $(\mathbf{T}, s)$-sequences on the other. We also study some basic distribution properties of such "hybrid sequences" in Section 2. In Section 3, we give an upper bound on the star discrepancy of such "hybrid sequences".

In Sections 4, 5, and 6, we study the distribution properties of subsequences of "hybrid sequences". In particular, we study subsequences indexed by primes (Section 5), and give discrepancy estimates for certain subsequences of van der Corput- and van der Corput-Halton sequences (Section 6).

Finally, in the last section, we deal with a concrete example of a further generalization of our concept. We only deal with an example here, since this further generalization is more difficult to be handled than the case considered in the other sections. However, we mean this example as motivation for future research.

Througout the paper we denote the set of positive integers by $\mathbb{N}$ and we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for the set of nonnegative integers. The set of prime numbers is denoted by $\mathbb{P}$.

2 A generalization of van der Corput-Halton sequences and digital $(\mathbf{T}, s)$-sequences

Let us now introduce certain "hybrid sequences", a generalization of van der Corput-Halton sequences and digital $(\mathbf{T}, s)$-sequences.

Definition 3 *Let $q_1 < q_2 < \cdots < q_v$ be primes and let $v, w_1, \ldots, w_v$ be positive integers. For $i \in \{1, \ldots, v\}$ let $C_1^{(i)}, \ldots, C_{w_i}^{(i)}$ be $\mathbb{N} \times \mathbb{N}$-matrices over $\mathbb{Z}_{q_i}$. We now define a sequence $(\boldsymbol{x}_n)_{n \geq 0}$ in $[0,1)^s$, $s := w_1 + w_2 + \cdots + w_v$, with*

$$\boldsymbol{x}_n := \left(x_n^{(1,1)}, \ldots, x_n^{(1,w_1)}, x_n^{(2,1)}, \ldots, x_n^{(2,w_2)}, \ldots, x_n^{(v,1)}, \ldots, x_n^{(v,w_v)}\right).$$

The component $x_n^{(i,j)}$, for $j \in \{1, \ldots w_i\}$, $i \in \{1, \ldots, v\}$, is generated as follows.

Let $n = n_0^{(i)} + n_1^{(i)} q_i + n_2^{(i)} q_i^2 + \cdots$ be the base q_i-representation of n for $i \in \{1, \ldots, v\}$. Then we set

$$C_j^{(i)} \cdot \left(n_0^{(i)}, n_1^{(i)}, \ldots\right)^\top =: \left(y_0^{(i,j)}, y_1^{(i,j)}, \ldots\right) \in \mathbb{Z}_{q_i}$$

and

$$x_n^{(i,j)} := \frac{y_0^{(i,j)}}{q_i} + \frac{y_1^{(i,j)}}{q_i^2} + \cdots.$$

For the description of the distribution quality we define a slightly modified quality parameter (compared to Definition 2) for the sequences introduced in Definition 3. For each $i \in \{1, \ldots, v\}$, and for each choice of nonnegative integers $d_1^{(i)}, d_2^{(i)}, \ldots, d_{w_i}^{(i)}$ let $F^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)$ be minimal such that the $(d_1^{(i)} + \cdots + d_{w_i}^{(i)}) \times F^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)$-matrix $C^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}, F^{(i)}\right)$ formed by the

left upper $d_1^{(i)} \times F^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)$-submatrix of $C_1^{(i)}$ together with the
left upper $d_2^{(i)} \times F^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)$-submatrix of $C_2^{(i)}$ together with the
$\vdots$
left upper $d_s^{(i)} \times F^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)$-submatrix of $C_{w_i}^{(i)}$

has rank $d_1^{(i)} + \cdots + d_{w_i}^{(i)}$ (we set $F^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right) := +\infty$ if this is not satisfied for any finite $F^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)$).

Note that, for $v = 1$, we have the case of a digital $(\mathbf{T}, s)$-sequence and the relation between the parameters $F^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)$ and $d_1^{(i)} + \cdots + d_{w_i}^{(i)}$ is similar to the relation between the parameters m and $m - \mathbf{T}(m)$. Indeed, it is easily checked that a digital $(\mathbf{T}, s)$-sequence, i.e., a sequence of the above form with $v = 1$ is uniformly distributed if and only if $F(d_1, \ldots, d_s) := F^{(1)}\left(d_1^{(1)}, \ldots, d_{w_1}^{(1)}\right) < \infty$ for all $d_1, \ldots, d_s$.

For general $v \geq 2$, it is by far not so easy to give necessary and sufficient conditions for the uniform distribution of our "hybrid sequences". The exact reason for this will be outlined later, but, to make things a little bit easier, we will restrict ourselves for the rest of the paper to considering sequences that are generated by matrices with "finite row length". I.e., for every $i \in \{1, \ldots, v\}$ in Definition 3, and arbitrary nonnegative integers $d_1^{(i)}, \ldots, d_{w_i}^{(i)}$, let $L^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)$ be minimal such that each of the first $d_j^{(i)}$ rows of $C_j^{(i)}$ has length less than or equal to $L^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)$ for $j \in \{1, \ldots, w_i\}$. By the "length" of a row $(c_1, c_2, c_3, \ldots) \neq (0, 0, 0 \ldots)$ we mean $\sup\{k : c_k \neq 0\}$ and for the zero row we define its length as 0. In the following we always assume $L^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right) < \infty$ for all

i and $d_1^{(i)},\ldots,d_{w_i}^{(i)}$. It seems to be very difficult to discuss distribution properties in more general cases, as we shall detail below.

Note that we trivially always have $F^{(i)}\left(d_1^{(i)},\ldots,d_{w_i}^{(i)}\right) \leq L^{(i)}\left(d_1^{(i)},\ldots,d_{w_i}^{(i)}\right)$ or $F^{(i)}\left(d_1^{(i)},\ldots,d_{w_i}^{(i)}\right) = +\infty$.

Definition 4 *We denote sequences as introduced in Definition 3 by digital* $\left((L^{(1)},\ldots,L^{(v)}),(F^{(1)},\ldots,F^{(v)}),s\right)$*-sequences in bases* $((q_1,w_1),\ldots,(q_v,w_v))$*. Often, we use the notation digital* $(\mathbf{L},\mathbf{F},s)$*-sequence, where* $\mathbf{L} := (L^{(1)},\ldots,L^{(v)})$ *and* $\mathbf{F} := (F^{(1)},\ldots,F^{(v)})$*.*

We mentioned above that we consider the case where the lengths of the rows of the generating matrices of a $(\mathbf{L},\mathbf{F},s)$-sequence are finite easier to be handled than the more general case in which also infinite row-lengths are permitted. Our first theorem gives a positive result for the case of finite $L^{(i)}$.

Theorem 1 *A digital* $\left((L^{(1)},\ldots,L^{(v)}),(F^{(1)},\ldots,F^{(v)}),s\right)$*-sequence in bases* $((q_1,w_1),\ldots,(q_v,w_v))$ *with finite* $L^{(i)}$ *is uniformly distributed in* $[0,1)^s$*,* $s = w_1+\cdots+w_v$*, if and only if the* $F^{(i)}$ *are finite.*

Proof. Assume that we are given a digital $\left((L^{(1)},\ldots,L^{(v)}),(F^{(1)},\ldots,F^{(v)}),s\right)$-sequence for which all the $F^{(i)}$ are finite.

We consider a so-called elementary interval of order

$$\left(d_1^{(1)},\ldots,d_{w_1}^{(1)},d_1^{(2)},\ldots,d_{w_2}^{(2)},\ldots,d_1^{(v)},\ldots,d_{w_v}^{(v)}\right),$$

i.e., an interval of the form

$$\prod_{i=1}^{v}\prod_{j=1}^{w_i}\left[\frac{a_{i,j}}{q_i^{d_j^{(i)}}},\frac{a_{i,j}+1}{q_i^{d_j^{(i)}}}\right) =: I,$$

where the $a_{i,j} < q_i^{d_j^{(i)}}$ are nonnegative integers.

In order to show that our sequence is uniformly distributed modulo one, it suffices to show that for each such interval I we have

$$\lim_{N\to\infty}\frac{1}{N}\left|\#\{n : 0\leq n<N, \boldsymbol{x}_n\in I\} - \frac{N}{\prod_{i=1}^{v}\prod_{j=1}^{w_i} q_i^{d_j^{(i)}}}\right| = 0.$$

For short, we write

$$E_N := \left|\#\{n : 0\leq n<N, \boldsymbol{x}_n\in I\} - \frac{N}{\prod_{i=1}^{v}\prod_{j=1}^{w_i} q_i^{d_j^{(i)}}}\right|.$$

Let us first consider those n satisfying

$$0\leq n<\prod_{i=1}^{v} q_i^{L^{(i)}\left(d_1^{(i)},\ldots,d_{w_i}^{(i)}\right)} =: Q.$$

Let $\widetilde{C}^{(i)} = \widetilde{C}^{(i)}\left(d_1^{(i)},\ldots,d_{w_i}^{(i)}\right)$ be the $\left(d_1^{(i)}+\cdots+d_{w_i}^{(i)}\right) \times L^{(i)}\left(d_1^{(i)},\ldots,d_{w_i}^{(i)}\right)$-matrix over $\mathbb{Z}_{q_i}$ formed by the

left upper $d_1^{(i)} \times L^{(i)}\left(d_1^{(i)},\ldots,d_{w_i}^{(i)}\right)$-submatrix of $C_1^{(i)}$, together with the
left upper $d_2^{(i)} \times L^{(i)}\left(d_1^{(i)},\ldots,d_{w_i}^{(i)}\right)$-submatrix of $C_2^{(i)}$, together with the
$\vdots$
left upper $d_{w_i}^{(i)} \times L^{(i)}\left(d_1^{(i)},\ldots,d_{w_i}^{(i)}\right)$-submatrix of $C_{w_i}^{(i)}$.

Let $a_{i,j} := a^{(i,j)}_{d_j^{(i)}-1} + a^{(i,j)}_{d_j^{(i)}-2} q_i + \cdots + a_0^{(i,j)} q_i^{d_j^{(i)}-1}$. Then $\boldsymbol{x}_n \in I$ if and only if

$$C_j^{(i)} \cdot \begin{pmatrix} n_0^{(i)} \\ \vdots \\ n_{r_i}^{(i)} \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_0^{(i,j)} \\ \vdots \\ a^{(i,j)}_{d_j^{(i)}-1} \\ b_1^{(i)} \\ b_2^{(i)} \\ \vdots \end{pmatrix}, \tag{2}$$

for all i and j (considered over $\mathbb{Z}_{q_i}$). Here, r_i is the maximal non-zero digit of n in its base q_i representation and the $b_k^{(i)}$ are arbitrary elements in $\mathbb{Z}_{q_i}$. Equation (2) holds if and only if

$$\widetilde{C}^{(i)}(d_1^{(i)},\ldots,d_{w_i}^{(i)}) \cdot \begin{pmatrix} n_0^{(i)} \\ n_1^{(i)} \\ \vdots \\ n_{r_i}^{(i)} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_0^{(i,1)} \\ \vdots \\ a^{(i,1)}_{d_1^{(i)}-1} \\ \vdots \\ \vdots \\ a_0^{(i,w_i)} \\ \vdots \\ a^{(i,w_i)}_{d_{w_i}^{(i)}-1} \end{pmatrix}, \tag{3}$$

for all $i \in \{1,\ldots,v\}$.

Note that $F^{(i)} \le L^{(i)}$ and hence $\widetilde{C}^{(i)}$ has rank $d_1^{(i)}+\cdots+d_{w_i}^{(i)}$. Thus, for every i, the system (3) has exactly $q_i^{L^{(i)}\left(d_1^{(i)},\ldots,d_{w_i}^{(i)}\right)}/q_i^{d_1^{(i)}+\cdots+d_{w_i}^{(i)}}$ solutions for $n \in \left\{0,1,\ldots,q_i^{L^{(i)}\left(d_1^{(i)},\ldots,d_{w_i}^{(i)}\right)}-1\right\}$ and hence has exactly $Q/q_i^{d_1^{(i)}+\cdots+d_{w_i}^{(i)}}$ solutions for $n \in \{0,1,\ldots,Q-1\}$.

Hence (since the q_i are pairwise different primes) by the Chinese Remainder Theorem the system (3) has exactly

$$S := \frac{Q}{\prod_{i=1}^{v}\prod_{j=1}^{w_i} q_i^{d_j^{(i)}}}$$

solutions, let us call them $n_1, n_2, \ldots, n_S$, in the range $\{0,1,\ldots,Q-1\}$ (note that $d_1^{(i)} + \cdots + d_{w_i}^{(i)} \le F^{(i)}(d_1^{(i)},\ldots,d_{w_i}^{(i)}) \le L^{(i)}\left(d_1^{(i)},\ldots,d_{w_i}^{(i)}\right)$ for all i, hence S is indeed a positive integer).

Therefore, it follows that $E_Q = 0$.

For n larger than Q, by the definition of $L^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)$ and of $\widetilde{C}^{(i)}(d_1^{(i)}, \ldots, d_{w_i}^{(i)})$, for n to satisfy $\boldsymbol{x}_n \in I$ we have again condition (3). Hence, $\boldsymbol{x}_n \in I$ if and only if

$$n \equiv n_k \pmod{Q} \text{ for some } k \in \{1, \ldots, S\}.$$

Consequently, $E_N = 0$ for N which are multiples of Q, hence $E_N < Q$ for all N, and hence $\lim_{N\to\infty} \frac{1}{N} E_N = 0$.

In order to show that the condition of finite $F^{(i)}$ is also a necessary condition for the uniform distribution of a digital $\left(\left(L^{(1)}, \ldots, L^{(v)}\right), \left(F^{(1)}, \ldots, F^{(v)}\right), s\right)$-sequence, assume that we are given such a sequence that is uniformly distributed in $[0,1)^s$.

Then, for every fixed i, the digital $(\mathbf{T}, s)$-sequence over $\mathbb{Z}_{q_i}$ generated by $C_1^{(i)}, \ldots, C_{w_i}^{(i)}$ is uniformly distributed in $[0,1)^{w_i}$. As noted above, a necessary condition for this to hold is that $F^{(i)}$ is finite. $\square$

We remark that the more general case of $(\mathbf{L}, \mathbf{F}, s)$-sequences in bases $((q_1, w_1), \ldots, (q_v, w_v))$ with $v \geq 2$ and with some infinite values of $L^{(i)}$ seems to be much more difficult, as can be seen by the following very simple example.

Let $s = 2, q_1 = 2, q_2 = 3$, and $w_1 = w_2 = 1$. Furthermore, let

$$C_1^{(1)} := \begin{pmatrix} 1 & \gamma_1 & \gamma_2 & \gamma_3 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{Z}_2^{\infty\times\infty} \text{ and } C_1^{(2)} := \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{Z}_3^{\infty\times\infty},$$

with any given entries γ_j in $\mathbb{Z}_2$ such that $\gamma_j = 1$ for infinitely many j.

A necessary condition for this sequence to be uniformly distributed in $[0,1)^2$ is that

$$I := [0, 1/2) \times [0, 1/3)$$

contains the correct number of points in the limit, i.e.,

$$\lim_{N\to\infty} \frac{1}{N} \# \{n : 0 \leq n < N, n \equiv 0 \pmod{3} \text{ and } s_{\boldsymbol{\gamma},2}(n) \equiv 0 \pmod{2}\} = \frac{1}{6}, \quad (4)$$

where $s_{\boldsymbol{\gamma},2}(n)$ denotes the sum-of-digits function of n in base 2 weighted by the sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\geq 0}$, i.e., for $n = n_0 + n_1 2 + n_2 2^2 + \cdots$,

$$s_{\boldsymbol{\gamma},2}(n) := \gamma_0 n_0 + \gamma_1 n_1 + \gamma_2 n_2 + \cdots$$

(here, we deal with the special case $\gamma_0 = 1$). See, for example, [9, 15, 21] for more information on the weighted sum-of-digits function.

The fact that (4) holds in the special case $\boldsymbol{\gamma} = (1)_{j\geq 0}$ was shown, for example, by Newman [17], Coquet [3] or Solinas [22]. However, in the general case, the question for which weights $\boldsymbol{\gamma}$ the property (4) holds is not known until now and will be one of the topics of a forthcoming paper of the authors.

We will show later (see Section 7) that this sequence in the case $\boldsymbol{\gamma} = (1)_{j\geq 0}$ indeed is uniformly distributed, but has a relatively large discrepancy opposed to other, well-known sequences like the two-dimensional van der Corput-Halton sequence.

3 An upper bound on the discrepancy of digital $(\mathbf{L}, \mathbf{F}, s)$-sequences

The *star discrepancy* D_N^*, which is one of the most important measures for the quality of the uniformity of a finite point set $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}$ in $[0,1)^s$ is defined by

$$D_N^* = D_N^*(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}) := \sup_{B \subseteq [0,1)^s} \left| \frac{\#\{n : 0 \leq n < N, \boldsymbol{x}_n \in B\}}{N} - \lambda(B) \right|,$$

where the supremum is extended over all sub-boxes B of $[0,1)^s$ of the form $B = \prod_{i=1}^s [0, b_i)$ with $0 < b_i \leq 1$ for $i \in \{1, \ldots, s\}$.

For an infinite sequence $\omega = (\boldsymbol{x}_n)_{n \geq 0}$ in $[0,1)^s$, $D_N^*(\omega)$ denotes the star discrepancy of the first N elements of the sequence.

For the s-dimensional van der Corput-Halton sequence it is known that

$$D_N^*(\omega) = c_s \frac{(\log N)^s}{N}$$

for all $N \geq 2$ with a certain constant $c_s > 0$ depending only on the dimension s, see [1, 5, 8, 10, 16, 20]. The smalles value of c_s in this bound known so far is given in [1].

For a digital $(\mathbf{T}, s)$-sequence ω over $\mathbb{Z}_q$ it is known that

$$D_N^*(\omega) \leq \frac{c(q,s)}{N} \sum_{m=1}^{\left\lfloor \frac{\log N}{\log q} \right\rfloor} q^{\mathbf{T}(m)} m^{s-1}$$

and consequently for digital (t, s)-sequences over $\mathbb{Z}_q$ we have

$$D_N^*(\omega) \leq \widetilde{c}(q,s) q^t \frac{(\log N)^s}{N}$$

for all $N \geq 2$ with constants $c(q,s) > 0$ and $\widetilde{c}(q,s) > 0$ depending only on q and s, see [14] and [19, 20].

In Theorem 2 we will give an upper bound on the star discrepancy of digital $\left(\left(L^{(1)}, \ldots, L^{(v)}\right), \left(F^{(1)}, \ldots, F^{(v)}\right), s\right)$-sequences with finite $L^{(i)}$ and for $v \geq 2$, thereby generalizing the above discrepancy estimate for the van der Corput-Halton sequence.

Of course, the situation is again different for the cases $v = 1$ (which is the case of $(\mathbf{T}, s)$-sequences) and $v \geq 2$. Since $v = 1$ was already studied in detail in [14] and [19], we restrict ourselves to $v \geq 2$. What is more, we again restrict ourselves to uniformly distributed digital $(\mathbf{L}, \mathbf{F}, s)$-sequences with finite $\mathbf{L}$-parameters, i.e., with $F^{(i)} \leq L^{(i)}$ in all cases (according to Theorem 1).

The formulation of the upper discrepancy bound is going to be quite technical and we will need some notation in advance. After having formulated the result, we will, as an illustration, apply it to the van der Corput-Halton sequence. We will see that we obtain the order D_N^* in N known from previous results on the van der Corput-Halton sequence. In our discrepancy estimate, however, we will not take care of constants independent of N. Consequently, our results are weaker with respect to these constants than, for example, the discrepancy estimates given for the van der Corput-Halton sequence in [1].

For the statement and the proof of Theorem 2 we need some notation which will be given in the following.

We will consider intervals of the form

$$I=\prod_{i=1}^{v}\prod_{j=1}^{w_i}\left[0,\alpha_j^{(i)}\right)\subseteq[0,1)^s,\ \text{with}\ \alpha_j^{(i)}=\sum_{l=1}^{\infty}\frac{a_{j,l}^{(i)}}{q_i^l}.$$

We approximate an arbitrary interval I of the above form from the interior by J_N and from the exterior by $\widetilde{J}_N$ as defined below.

For strings $\mathbf{K}:=\left((k_{i,j})_{j=1}^{w_i}\right)_{i=1}^{v}$, we consider disjoint subintervals $I(\mathbf{K})$ of I of the form

$$I(\mathbf{K})=\prod_{i=1}^{v}\prod_{j=1}^{w_i}\left[\sum_{l=1}^{k_{i,j}-1}\frac{a_{j,l}^{(i)}}{q_i^l},\sum_{l=1}^{k_{i,j}}\frac{a_{j,l}^{(i)}}{q_i^l}\right).$$

For a string $\mathbf{K}$ as above, we define integers $Q(\mathbf{K})$ in the spirit of the quantity Q introduced in the proof of Theorem 1, namely

$$Q(\mathbf{K}):=\prod_{i=1}^{v}q_i^{L^{(i)}\left(k_{i,1},\ldots,k_{i,w_i}\right)}.$$

For a given positive integer N we will consider the following union of intervals $I(\mathbf{K})$:

$$J_N:=\bigcup_{\substack{\mathbf{K}\\ Q(\mathbf{K})\leq N}}I(\mathbf{K}).$$

Note that $J_N\subseteq I$ for all N.

In order to approximate I from the exterior we add to J_N some appropriate border area $\widetilde{R}_N$. For given $i_0\in\{1,\ldots,v\}$ and $j_0\in\{0,\ldots,w_{i_0}-1\}$ we consider strings $\zeta(i_0,j_0)$ of length $\sum_{i=1}^{v}w_i=s$ and of the form

$$\zeta(i_0,j_0)=\\ \left((k_{1,j})_{j=1}^{w_1},\ldots,(k_{i_0-1,j})_{j=1}^{w_{i_0-1}},\underbrace{(k_{i_0,1},\ldots,k_{i_0,j_0},0,0,\ldots,0)}_{w_{i_0}\text{ bits}},\underbrace{(0,\ldots,0)}_{w_{i_0+1}\text{ bits}},\ldots,\underbrace{(0,\ldots,0)}_{w_v\text{ bits}}\right).$$

For an arbitrary nonnegative integer θ let $\widetilde{\zeta}(i_0,j_0,\theta)$ be the same string as $\zeta(i_0,j_0)$ with the first zero following k_{i_0,j_0} replaced by θ.

For a given positive integer N let now $\zeta(i_0,j_0)$ be such that $Q(\zeta(i_0,j_0))\leq N$, then $\Theta:=\Theta(\zeta(i_0,j_0))$ is defined to be the maximal integer θ such that $Q(\widetilde{\zeta}(i_0,j_0,\theta))\leq N$.

Furthermore, for $\zeta(i_0,j_0)$ as defined above and for given N we will make use of the following intervals

$$\begin{aligned}I\left(\widetilde{\zeta}(i_0,j_0,\Theta)\right)\ :=\ &\prod_{i=1}^{i_0-1}\prod_{j=1}^{w_i}\left[\sum_{l=1}^{k_{i,j}-1}\frac{a_{j,l}^{(i)}}{q_i^l},\sum_{l=1}^{k_{i,j}}\frac{a_{j,l}^{(i)}}{q_i^l}\right)\times\prod_{j=1}^{j_0}\left[\sum_{l=1}^{k_{i_0,j}-1}\frac{a_{j,l}^{(i_0)}}{q_{i_0}^l},\sum_{l=1}^{k_{i_0,j}}\frac{a_{j,l}^{(i_0)}}{q_{i_0}^l}\right)\times\\ &\times\left[\sum_{l=1}^{\Theta}\frac{a_{j_0+1,l}^{(i_0)}}{q_{i_0}^l},\sum_{l=1}^{\Theta}\frac{a_{j_0+1,l}^{(i_0)}}{q_{i_0}^l}+\frac{1}{q_{i_0}^{\Theta}}\right)\times\prod_{j=w_1+\cdots+w_{i_0-1}+j_0+2}^{s}[0,1).\end{aligned}$$

We define

$$\widetilde{R}_N := \bigcup_{\substack{i_0,j_0\\ \zeta(i_0,j_0)\\ Q(\zeta(i_0,j_0))\le N}} I(\widetilde{\zeta}(i_0,j_0,\Theta))$$

and $\widetilde{J}_N := J_N \cup \widetilde{R}_N$. With this definition we have $I \subseteq \widetilde{J}_N$.

Finally, for any string $\mathbf{K}$, we use the following short notation.

$$P(\mathbf{K}) := \prod_{i=1}^{v}\prod_{j=1}^{w_i} q_i^{-k_{i,j}}.$$

In particular,

$$P(\zeta(i_0,j_0)) = \prod_{j=1}^{i_0-1}\prod_{j=1}^{w_i} q_i^{-k_{i,j}} \cdot \prod_{j=1}^{j_0} q_{i_0}^{-k_{i_0,j}}$$

and

$$P(\widetilde{\zeta}(i_0,j_0,\theta)) = P(\zeta(i_0,j_0))q_{i_0}^{-\theta}.$$

We can now formulate the following upper bound.

Theorem 2 *Let $(\boldsymbol{x}_n)_{n\ge 0}$ be a digital $(\mathbf{L},\mathbf{F},s)$-sequence in bases $((q_1,w_1),\dots,(q_v,w_v))$ with finite $\mathbf{L}$ and finite $\mathbf{F}$-parameters. Then for the star discrepancy D_N^* of the first N elements of the sequence we have*

$$D_N^* \le \frac{2c}{N}\sum_{\substack{\mathbf{K}\\ Q(\mathbf{K})\le N}} Q(\mathbf{K})P(\mathbf{K}) + c \sum_{\substack{i_0,j_0\\ \zeta(i_0,j_0)\\ Q(\zeta(i_0,j_0))\le N}} P\left(\widetilde{\zeta}(i_0,j_0,\Theta)\right),$$

where $c = \prod_{i=1}^{v} q_i^{w_i}$.

Proof. We use the notation from above and consider $A(I) - N\lambda(I)$, where

$$A(I) := \#\{n \,:\, 0 \le n < N, \boldsymbol{x}_n \in I\}.$$

As $J_N \subseteq I \subseteq \widetilde{J}_N$, we have

$$A(J_N) - N\lambda(J_N) - N\lambda\big(\widetilde{R}_N\big) \le A(I) - N\lambda(I) \le A\big(\widetilde{J}_N\big) - N\lambda(J_N)$$

and as for any string $\mathbf{K}$ the inequality $\lambda(I(\mathbf{K})) \le cP(\mathbf{K})$ holds we have

$$\lambda\big(\widetilde{R}_N\big) \le c \sum_{\substack{i_0,j_0\\ \zeta(i_0,j_0)\\ Q(\zeta(i_0,j_0))\le N}} P\left(\widetilde{\zeta}(i_0,j_0,\Theta)\right),$$

where, here and in the following, $c = \prod_{i=1}^{v} q_i^{w_i}$.

For any interval $I(\mathbf{K})$ with $Q(\mathbf{K}) \le N$ we have, like in the proof of Theorem 1, the following. If for any positive integer z we consider those n satisfying

$$0 \le n < zQ(\mathbf{K}),$$

then $I(\mathbf{K})$ contains $\boldsymbol{x}_n$ for exactly

$$z\frac{Q(\mathbf{K})}{\prod_{i=1}^{v}\prod_{j=1}^{w_i} q_i^{k_{i,j}}}\prod_{i=1}^{v}\prod_{j=1}^{w_i} a_{j,k_{i,j}}^{(i)} = zQ(\mathbf{K})\lambda(I(\mathbf{K}))$$

values of n.

Hence,

$$A(J_N) \geq \sum_{\substack{\mathbf{K}\\ Q(\mathbf{K})\leq N}} \left(\frac{N}{Q(\mathbf{K})}-1\right) Q(\mathbf{K})\lambda(I(\mathbf{K})) = N\lambda(J_N) - \sum_{\substack{\mathbf{K}\\ Q(\mathbf{K})\leq N}} Q(\mathbf{K})\lambda(I(\mathbf{K})),$$

and

$$A(I) - N\lambda(I) \geq - \sum_{\substack{\mathbf{K}\\ Q(\mathbf{K})\leq N}} Q(\mathbf{K})\lambda(I(\mathbf{K})) - Nc \sum_{\substack{i_0,j_0\\ \zeta(i_0,j_0)\\ Q(\zeta(i_0,j_0))\leq N}} P\left(\widetilde{\zeta}(i_0,j_0,\Theta)\right).$$

On the other hand,

$$\begin{aligned} A\big(\widetilde{J}_N\big) &= A(J_N) + A\big(\widetilde{R}_N\big) \\ &\leq \sum_{\substack{\mathbf{K}\\ Q(\mathbf{K})\leq N}} \left(\frac{N}{Q(\mathbf{K})}+1\right) Q(\mathbf{K})\lambda(I(\mathbf{K})) + A\big(\widetilde{R}_N\big) \\ &= N\lambda(J_N) + \sum_{\substack{\mathbf{K}\\ Q(\mathbf{K})\leq N}} Q(\mathbf{K})\lambda(I(\mathbf{K})) + A\big(\widetilde{R}_N\big). \end{aligned}$$

Furthermore,

$$\begin{aligned} A\big(\widetilde{R}_N\big) &\leq \sum_{\substack{i_0,j_0\\ \zeta(i_0,j_0)\\ Q(\zeta(i_0,j_0))\leq N}} \left(\frac{N}{Q\big(\widetilde{\zeta}(i_0,j_0,\Theta)\big)}+1\right) Q\big(\widetilde{\zeta}(i_0,j_0,\Theta)\big)\lambda\big(I(\widetilde{\zeta}(i_0,j_0,\Theta))\big) \\ &= N\lambda\big(\widetilde{R}_N\big) + \sum_{\substack{i_0,j_0\\ \zeta(i_0,j_0)\\ Q(\zeta(i_0,j_0))\leq N}} Q\big(\widetilde{\zeta}(i_0,j_0,\Theta)\big)\lambda\big(I(\widetilde{\zeta}(i_0,j_0,\Theta))\big), \end{aligned}$$

hence,

$$\begin{aligned} A(I) - N\lambda(I) &\leq \sum_{\substack{\mathbf{K}\\ Q(\mathbf{K})\leq N}} Q(\mathbf{K})\lambda(I(\mathbf{K})) + \sum_{\substack{i_0,j_0\\ \zeta(i_0,j_0)\\ Q(\zeta(i_0,j_0))\leq N}} Q\big(\widetilde{\zeta}(i_0,j_0,\Theta)\big)\lambda\big(I(\widetilde{\zeta}(i_0,j_0,\Theta))\big) \\ &\quad + Nc \sum_{\substack{i_0,j_0\\ \zeta(i_0,j_0)\\ Q(\zeta(i_0,j_0))\leq N}} P\left(\widetilde{\zeta}(i_0,j_0,\Theta)\right). \end{aligned}$$

We use the relation $\lambda(I(\mathbf{K})) \leq cP(\mathbf{K})$ for each string in the first and the second sum and obtain

$$A(I) - N\lambda(I) \leq c \sum_{\substack{\mathbf{K}\\ Q(\mathbf{K})\leq N}} Q(\mathbf{K})P(\mathbf{K}) + c \sum_{\substack{i_0,j_0\\ \zeta(i_0,j_0)\\ Q(\zeta(i_0,j_0))\leq N}} Q\big(\widetilde{\zeta}(i_0,j_0,\Theta)\big)P(\widetilde{\zeta}(i_0,j_0,\Theta))$$

$$+Nc \sum_{\substack{i_0,j_0 \\ \zeta(i_0,j_0) \\ Q(\zeta(i_0,j_0))\leq N}} P\left(\widetilde{\zeta}(i_0,j_0,\Theta)\right).$$

Now note that $\widetilde{\zeta}(i_0,j_0,\Theta)$ is also a string $\mathbf{K}$ with $Q(\mathbf{K}) \leq N$, hence the second summand on the right hand side of the above inequality is at most as large as the first summand and the result follows. $\square$

Example 1 To illustrate the result in Theorem 2, we apply it to the van der Corput-Halton sequence. In this case we have $v = s$ and $w_i = 1$ for all i. For simplicity we write here $\mathbf{K} = (k_1, \ldots, k_s)$. As $L^{(i)}(k_i) = k_i$ for all i we obtain $Q(\mathbf{K}) = q_1^{k_1} \cdots q_s^{k_s}$ and hence we always have $Q(\mathbf{K})P(\mathbf{K}) = 1$.

For all i_0 we have $j_0 = 0$ and

$$\zeta(i_0,j_0) = (k_1, \ldots, k_{i_0-1}, 0, 0, \ldots, 0).$$

We choose Θ maximal such that $q_1^{k_1} \cdots q_{i_0-1}^{k_{i_0-1}} q_{i_0}^{\Theta} \leq N$, i.e., $q_1^{k_1} \cdots q_{i_0-1}^{k_{i_0-1}} q_{i_0}^{\Theta+1} > N$, and hence

$$P\big(\widetilde{\zeta}(i_0,j_0,\Theta)\big) = \frac{1}{q_1^{k_1} \cdots q_{i_0-1}^{k_{i_0-1}} q_{i_0}^{\Theta}} < \frac{q_{i_0}}{N}.$$

Overall, for the discrepancy D_N^* of the first N elements of the van der Corput-Halton sequence we have

$$D_N^* \;\leq\; \frac{2c}{N} \sum_{\substack{k_1,\ldots,k_s=0 \\ q_1^{k_1}\cdots q_s^{k_s} \leq N}}^{\infty} 1 + \frac{c}{N}\left(\max_{1\leq i\leq s} q_i\right) \sum_{\substack{k_1,\ldots,k_s=0 \\ q_1^{k_1}\cdots q_s^{k_s} \leq N}}^{\infty} 1 = O\left(\frac{(\log N)^s}{N}\right),$$

which, concerning the order of magnitude in N, coincides with the best discrepancy estimates for the van der Corput-Halton sequence known until now.

4 Uniform distribution of subsequences of digital $(\mathbf{L}, \mathbf{F}, s)$-sequences

In the following, we try to classify which subsequences of van der Corput-Halton sequences or of digital $(\mathbf{T}, s)$-sequences are uniformly distributed.

First of all, we note that for digital $(\mathbf{T}, s)$-sequences with unbounded $\mathbf{L}$-parameters we cannot give an answer to this question, even in some of the most elementary cases, as we can see from the following example.

Example 2 Consider the digital $(0,1)$-sequence $(x_n)_{n\geq 0}$ over $\mathbb{Z}_2$ generated by a matrix of the form

$$C = \begin{pmatrix} 1 & \gamma_1 & \gamma_2 & \gamma_3 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\gamma_1, \gamma_2, \ldots$ are in $\mathbb{Z}_2$.

A necessary condition for the subsequence $(x_{3n})_{n\geq 0}$ to be uniformly distributed modulo one is that

$$\lim_{N\to\infty}\frac{1}{N}\#\{n\,:\,0\leq n<N, x_{3n}\in[0,1/2)\}=\frac{1}{2},$$

i.e.,

$$\lim_{M\to\infty}\frac{1}{M}\#\{n\,:\,0\leq n<M, n\equiv 0\pmod 3\text{ and } s_{\boldsymbol{\gamma},2}(n)\equiv 0\pmod 2\}=\frac{1}{6},$$

which, as already discussed in Section 2, is not solved until now for general sequences of weights $\boldsymbol{\gamma}$. The special case $\boldsymbol{\gamma}=(1)_{j\geq 0}$ is, as an example, discussed in Section 7.

As demonstrated by our example, we have to, quite naturally, restrict our investigations to the case of finite $\mathbf{L}$-parameters. We now show our main result concerning the distribution of subsequences of digital $(\mathbf{L},\mathbf{F},s)$-sequence with finite $\mathbf{L}$-and $\mathbf{F}$-parameters.

Theorem 3 *Let $(\boldsymbol{x}_n)_{n\geq 0}$ be a digital $(\mathbf{L},\mathbf{F},s)$-sequence in bases $((q_1,w_1),\ldots,(q_v,w_v))$ with finite $\mathbf{L}$- and $\mathbf{F}$-parameters.*

- *(a) Let $(k_n)_{n\geq 0}$ be a sequence of nonnegative integers. If for all positive integers d the sequence $(k_n)_{n\geq 0}$ is uniformly distributed modulo $(q_1\cdots q_v)^d$, then $(\boldsymbol{x}_{k_n})_{n\geq 0}$ is uniformly distributed in $[0,1)^s$.*
- *(b) The condition given in (a) for $(k_n)_{n\geq 0}$ is also a necessary condition for the uniform distribution of $(\boldsymbol{x}_{k_n})_{n\geq 0}$, if and only if $w_i=1$ for all i and $C_1^{(i)}$ is a lower triangular matrix for all i.*

Before we give the proof of Theorem 3, we state some remarks and exhibit a few examples.

Remark 2 *The condition on $(k_n)_{n\geq 0}$ for $(\boldsymbol{x}_{k_n})_{n\geq 0}$ to be uniformly distributed is not necessary in general. Consider, for example, the digital $(1,1)$-sequence $(x_n)_{n\geq 0}$ over $\mathbb{Z}_2$ generated by*

$$C=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots\\ 0 & 0 & 1 & 0 & 0 & \ldots\\ 0 & 0 & 0 & 1 & 0 & \ldots\\ 0 & 0 & 0 & 0 & 1 & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Here, the subsequence $(x_{2n})_{n\geq 0}$ is just the uniformly distributed van der Corput sequence, however, the integer sequence $(k_n)_{n\geq 0}=(2n)_{n\geq 0}$ is not uniformly distributed modulo 2.

Remark 3 *Note that, in the case of the van der Corput-Halton sequence, the conditions in Theorem 3 (b) are satisfied, i.e., a subsequence $(\boldsymbol{x}_{k_n})_{n\geq 0}$ of the van der Corput-Halton sequence $(\boldsymbol{x}_n)_{n\geq 0}$ is uniformly distributed modulo one if and only if $(k_n)_{n\geq 0}$ is uniformly distributed modulo $(q_1\cdots q_v)^d$ for all positive integers d.*

Remark 4 *It can be shown that, in the case of the van der Corput-Halton sequence, Theorem 3 holds true even if $q_1,\ldots,q_s$ are pairwise coprime but the q_i are not necessarily primes.*

Let us discuss some further examples.

Example 3 (a) Let $(\boldsymbol{x}_n)_{n\geq 0}$ be as above. If the sequence $K = (k_n)_{n\geq 0}$ of nonnegative integers is increasing and if K has density one, then it can be shown easily that $(k_n)_{n\geq 0}$ is uniformly distributed in $\mathbb{Z}$ and hence $(\boldsymbol{x}_{k_n})_{n\geq 0}$ is uniformly distributed modulo one by Theorem 3 (a).

(b) If $k_n = un + v$ and $(\boldsymbol{x}_n)_{n\geq 0}$ as above, then $(\boldsymbol{x}_{un+v})_{n\geq 0}$ is uniformly distributed if $\gcd(u, q_1 \cdots q_v) = 1$.

(c) If $(\boldsymbol{x}_n)_{n\geq 0}$ is as in Theorem 3 (b), then $(\boldsymbol{x}_{un+v})_{n\geq 0}$ is uniformly distributed if and only if $\gcd(u, q_1 \cdots q_v) = 1$. This result holds, for example, for the van der Corput-Halton sequence.

(d) If $(\boldsymbol{x}_n)_{n\geq 0}$ is as in Theorem 3 (b), then for any integer $\alpha \geq 2$ the sequence $(\boldsymbol{x}_{n^\alpha})_{n\geq 0}$ is not uniformly distributed modulo one.

Example 4 If $k_n = p_{n+1}$, the $(n+1)$-st prime, then in general, $(\boldsymbol{x}_{p_n})_{n\geq 1}$ is not uniformly distributed modulo one. For example, for all the sequences $(\boldsymbol{x}_n)_{n\geq 0}$ satisfying the conditions in Theorem 3 (b) (which includes the van der Corput-Halton sequence), the subsequence $(\boldsymbol{x}_{p_n})_{n\geq 1}$ is not uniformly distributed.

However, for the sequence $(\boldsymbol{x}_n)_{n\geq 0}$ discussed in Remark 2, we indeed have uniform distribution of $(\boldsymbol{x}_{p_n})_{n\geq 1}$ as will be seen in Section 5, where we will consider subsequences of the type $(\boldsymbol{x}_{p_n})_{n\geq 1}$ in greater detail.

Of course we can give many further examples of uniformly distributed subsequences $(\boldsymbol{x}_{k_n})_{n\geq 0}$. In the following example we focus on the one-dimensional van der Corput sequence ω_{vdC} in base q.

Example 5 Let $(x_n)_{n\geq 0}$ denote the van der Corput sequence in base q, q not necessarily a prime. Then the following statements hold true:

1. Let F_n, $n \in \mathbb{N}_0$, denote the n-th Fibonacci number, i.e., $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 3, \ldots$. Then the sequence $(x_{F_n})_{n\geq 0}$ in base q is uniformly distributed modulo one if and only if $q = 5^k$ for some $k \in \mathbb{N}$.

2. Let F_n be as above. Then, in any base q, the sequence $(x_{\lfloor \log F_n \rfloor})_{n\geq 1}$ is uniformly distributed modulo one.

3. Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ or $\xi = 1/d$ for some nonzero integer d. Then, in any base q, the sequence $(x_{\lfloor n\xi \rfloor})_{n\geq 0}$ is uniformly distributed modulo one.

4. For an integer $\widetilde{q} \geq 2$ and $n \in \mathbb{N}_0$ let $s_{\widetilde{q}}(n)$ denote the $\widetilde{q}$-ary sum-of-digits function. Then, in any base q, the sequence $(x_{s_{\widetilde{q}}(n)})_{n\geq 0}$ is uniformly distributed modulo one.

Proof. 1. It was shown by Kuipers & Shiue [13], that if the Fibonacci numbers are uniformly distributed modulo q, then q has to be a power of 5. Conversely, it has been shown by Niederreiter [18], that for any $k \in \mathbb{N}_0$ the Fibonacci sequence is uniformly distributed modulo 5^k. Now the result follows from these facts together with Theorem 3 (together with Remark 4).

2. ¿From [12, Chapter 1, Theorem 3.3] it follows that the sequence $\left(\left\{\frac{1}{q}\log F_n\right\}\right)_{n\geq 1}$ is uniformly distributed modulo one for every $q \in \mathbb{N}$. Now it follows from [12, Chapter 5, Theorem 1.4], that the sequence $(\lfloor \log F_n \rfloor)_{n\geq 1}$ is uniformly distributed in $\mathbb{Z}$. Hence the result follows from Theorem 3 (together with Remark 4).

3. Combine Theorem 3 with [12, Chapter 5, Theorem 1.5].

4. It was shown in [7, Théorème I] that for any integer $\widetilde{q} \geq 2$ the sequence $(s_{\widetilde{q}}(n))_{n\geq 0}$ is uniformly distributed in $\mathbb{Z}$. Hence the result follows by Theorem 3 (together with Remark 4).

□

For more examples of integer sequences which are uniformly distributed in $\mathbb{Z}$, and therefore for uniformly distributed subsequences of digital $(\mathbf{L}, \mathbf{F}, s)$-sequences with finite $\mathbf{L}$- and $\mathbf{F}$-parameters, we refer to Kuipers & Niederreiter [12, Chapter 5].

We now give the proof of Theorem 3.

Proof. We use the notation and facts given in the proof of Theorem 1.

(a) We have seen in the proof of Theorem 1 that $\boldsymbol{x}_{k_n} \in I$ if and only if

$$k_n \equiv n_i \pmod{Q}$$

for some $i \in \{1, \ldots, S\}$. Hence, $(\boldsymbol{x}_{k_n})_{n\geq 0}$ is uniformly distributed in $[0,1)^s$ if and only if for all $d_j^{(i)}$, $j \in \{1, \ldots, w_i\}$, $i \in \{1, \ldots, v\}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{l=1}^{S} \#\{n : 0 \leq n < N, k_n \equiv n_l \pmod{Q}\} = \frac{S}{Q}.$$

Note that we have $S = S\left(\left(d_j^{(i)}\right)\right)$, $Q = Q\left(\left(d_j^{(i)}\right)\right)$ and $n_l = n_l\left(\left(d_j^{(i)}\right)\right)$, here.

Recalling that $Q = \prod_{i=1}^{v} q_i^{L^{(i)}\left(d_1^{(i)},\ldots,d_{w_i}^{(i)}\right)}$, this condition is certainly satisfied if

$$\lim_{N\to\infty} \frac{1}{N} \{n : 0 \leq n < N, k_n \equiv l \pmod{q_1^{a_1} \cdots q_v^{a_v}}\} = \frac{1}{q_1^{a_1} \cdots q_v^{a_v}}$$

for all nonnegative integers $l, a_1, \ldots, a_v$ holds, and this is satisfied if and only if $(k_n)_{n\geq 0}$ is uniformly distributed modulo $(q_1 \cdots q_v)^d$ for all $d \in \mathbb{N}$.

(b) If in the proof of Part (a) the value of S equals 1 for all choices of $d_j^{(i)}$, $j \in \{1, \ldots, w_i\}$, $i \in \{1, \ldots, v\}$, then the uniform distribution of $(k_n)_{n\geq 0}$ modulo $(q_1 \cdots q_v)^d$ for all d is also a necessary condition.

The condition that S equals 1 in all cases is satisfied if and only if $Q = \prod_{i=1}^{v} \prod_{j=1}^{w_i} q_i^{d_i^{(j)}}$, i.e., if and only if

$$d_1^{(i)} + \cdots + d_{w_i}^{(i)} = L^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)$$

for all i, and all choices of $d_1^{(i)}, \ldots, d_{w_i}^{(i)}$. In particular, we have $L^{(i)}\left(0, \ldots, 0, d_j^{(i)}, 0, \ldots, 0\right) = d_j^{(i)}$, i.e., each of the $C_j^{(i)}$ is a lower triangular matrix. So, if for some i we had $w_i \geq 2$, then $F^{(i)}(1, 1, \ldots, 1) = +\infty$. Hence, $w_i = 1$ for all i and $C_1^{(i)}$ is a lower triangular matrix for all i.

□

5 Digital $(\mathbf{L}, \mathbf{F}, s)$-sequences indexed by primes

As already pointed out in Example 4 of Section 4, the subsequence $(x_{p_n})_{n\geq 0}$ of the van der Corput sequence $(x_n)_{n\geq 0}$ in base $q \geq 2$ is not uniformly distributed. The main reason for this fact is that the sequence of primes is not uniformly distributed modulo q.

However, if we "eliminate", in every dimension, the first digit in the expansion of n when generating the n-th point, and then "blow up" the resulting point by a factor equal to the base of the digital sequence, then we again have uniform distribution.

The "elimination" of the first digit can be systematically done by inserting an all-zero-column as the first column in each of the generating matrices.

For the van der Corput-Halton sequence $(\boldsymbol{x}_n)_{n\geq 0}$ in bases $(q_1, \ldots, q_s)$ this means that we consider the sequence $\widetilde{\boldsymbol{x}}_n = \left(\widetilde{x}_n^{(1)}, \ldots, \widetilde{x}_n^{(s)}\right)$ with $\widetilde{x}_n^{(i)} = \left\{q_i x_n^{(i)}\right\}$. Here, $\{y\}$ denotes the fractional part of a real number y.

We have the following general result.

Theorem 4 *Let $(\boldsymbol{x}_n)_{n\geq 0}$ be a digital $(\mathbf{L}, \mathbf{F}, s)$-sequence in bases $((q_1, w_1), \ldots, (q_v, w_v))$ with finite $\mathbf{L}$- and $\mathbf{F}$-parameters and assume that the first column of each of the generating matrices of the sequence consists only of zeros.*

Let $(k_n)_{n\geq 0}$ be a subsequence of nonnegative integers, then the following statement holds.

If, for all nonnegative integers, $D_1, \ldots, D_v$, the sequence

$$\left(\left\lfloor \frac{k_n}{q_1} \right\rfloor, \ldots, \left\lfloor \frac{k_n}{q_v} \right\rfloor\right)_{n\geq 0}$$

is uniformly distributed modulo $\left(q_1^{D_1}, \ldots, q_v^{D_v}\right)$, then $(\boldsymbol{x}_{k_n})_{n\geq 0}$ is uniformly distributed modulo one.

Proof. The result follows almost immediately by following the proofs of Theorems 1 and 3. In fact, just note that if we consider, for a single i, the system (3) in the proof of Theorem 1, then since the first column of $\widetilde{C}^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)$ always is the all-zero-column, with any solution

$$\left(n_0^{(i)}, \ldots, n_{v_i}^{(i)}, 0, \ldots, 0\right)^\top$$

of (3) (for a certain i), also

$$\left(a, n_1^{(i)}, \ldots, n_{v_i}^{(i)}, 0, \ldots, 0\right)^\top,$$

where a is arbitrary in $\{0, 1, \ldots, q_i - 1\}$ is a solution of (3) for the same i.

Thus, the $q_i^{L^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)} / q_i^{d_1^{(i)} + \cdots + d_{w_i}^{(i)}}$ solutions n of (3) (for our fixed i) satisfying $0 \leq n < q_i^{L^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)}$ are uniquely determined by $q_i^{L^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right)} / q_i^{d_1^{(i)} + \cdots + d_{w_i}^{(i)} + 1}$ different vectors $(n_1^{(i)}, \ldots, n_{v_i}^{(i)})$. Moreover, a given n is a solution of the whole system (3) if and only if

$$\left\lfloor \frac{n}{q_i} \right\rfloor \equiv n_1^{(i)} + n_2^{(i)} q_i + \cdots + n_{v_i}^{(i)} q_i^{v_i - 1} \pmod{q_i^{L^{(i)}\left(d_1^{(i)}, \ldots, d_{w_i}^{(i)}\right) - 1}}$$

for all i for some admissible choice of $(n_1^{(i)}, \ldots, n_{v_i}^{(i)})$.

Hence, for any subsequence $(k_n)_{n \geq 0}$ of the nonnegative integers it suffices to request that

$$\left(\left\lfloor \frac{k_n}{q_1} \right\rfloor, \ldots, \left\lfloor \frac{k_n}{q_v} \right\rfloor \right)_{n \geq 0}$$

is uniformly distributed modulo $(q_1^{D_1}, \ldots, q_v^{D_v})$ for all $D_1, \ldots, D_v \in \mathbb{N}_0$. □

Corollary 1 *Let $(\boldsymbol{x}_n)_{n \geq 0}$ be a digital $(\mathbf{L}, \mathbf{F}, s)$-sequence in bases $((q_1, w_1), \ldots, (q_v, w_v))$ with finite $\mathbf{L}$- and $\mathbf{F}$-parameters and assume that the first column of each of the generating matrices consists only of zeros. Then $(\boldsymbol{x}_{p_n})_{n \geq 1}$, where p_n is the n-th prime number, is uniformly distributed modulo one.*

Let us give another example before we prove the corollary.

Example 6 Let $\boldsymbol{x}_n = \left(x_n^{(1)}, \ldots, x_n^{(s)} \right)$ be the van der Corput-Halton sequence, then the sequence of

$$\widetilde{\boldsymbol{x}}_{p_n} = \left(\widetilde{x}_{p_n}^{(1)}, \ldots, \widetilde{x}_{p_n}^{(s)} \right)$$

with $\widetilde{x}_l^{(i)} := \left\{ q_i x_l^{(i)} \right\}$, for $n \geq 1$, is uniformly distributed in $[0,1)^s$. A discrepancy estimate for this sequence in dimension $s = 1$ is given in the subsequent Theorem 7.

We now give the proof of the Corollary 1.

Proof. The result basically follows from Theorem 4 and from Dirichlet's theorem on primes in arithmetic progressions. Indeed, for given $\widetilde{D}_1, \ldots, \widetilde{D}_v$ and u with $\gcd(u, q_1 \cdots q_v) = 1$, by Dirichlet's theorem we have

$$\lim_{N \to \infty} \frac{1}{N} \# \left\{ n : 1 \leq n \leq N, p_n \equiv u \pmod{q_1^{\widetilde{D}_1} \cdots q_v^{\widetilde{D}_v}} \right\} = \frac{1}{\varphi\left(q_1^{\widetilde{D}_1}\right) \cdots \varphi\left(q_v^{\widetilde{D}_v}\right)}.$$

For given $u_1, \ldots, u_v$ with $\gcd(u_i, q_i) = 1$, the relation

$$\left(\left\lfloor \frac{p_n}{q_1} \right\rfloor, \ldots, \left\lfloor \frac{p_n}{q_v} \right\rfloor \right) \equiv (u_1, \ldots, u_v) \pmod{\left(q_1^{D_1}, \ldots, q_v^{D_v}\right)}$$

means

$$(p_n, \ldots, p_n) \equiv (q_1 u_1 + \tau_1, \ldots, q_v u_v + \tau_v) \pmod{\left(q_1^{D_1+1}, \ldots, q_v^{D_v+1}\right)}$$

for some $\tau_j \in \{1, \ldots, q_j - 1\}$, i.e.,

$$p_n \equiv j(\tau_1, \ldots, \tau_v) \pmod{q_1^{D_1+1} \cdots q_v^{D_v+1}}$$

for one of altogether $(q_1 - 1) \cdots (q_v - 1)$ admissible values $j(\tau_1, \ldots, \tau_v)$.

Hence,

$$\begin{aligned} &\lim_{N \to \infty} \frac{1}{N} \# \left\{ n : 1 \leq n \leq N, \left(\left\lfloor \frac{p_n}{q_1} \right\rfloor, \ldots, \left\lfloor \frac{p_n}{q_v} \right\rfloor \right) \equiv (u_1, \ldots, u_v) \pmod{\left(q_1^{D_1}, \ldots, q_v^{D_v}\right)} \right\} \\ &= \lim_{N \to \infty} \frac{1}{N} \sum_{j(\tau_1, \ldots, \tau_v)} \# \left\{ n : 1 \leq n \leq N, p_n \equiv j(\tau_1, \ldots, \tau_v) \pmod{q_1^{D_1+1} \cdots q_v^{D_v+1}} \right\} \end{aligned}$$

$$
\begin{aligned}
&= (q_1-1)\cdots(q_v-1)\frac{1}{\varphi\left(q_1^{D_1+1}\right)\cdots\varphi\left(q_v^{D_v+1}\right)}\\
&= \frac{1}{q_1^{D_1}\cdots q_v^{D_v}}.
\end{aligned}
$$

By Theorem 4, the result follows. □

6 Some discrepancy estimates for subsequences

In this section, we give some discrepancy estimates for special subsequences of van der Corput and van der Corput-Halton sequences. Note that only in this section, we do not necessarily assume that the bases of the sequences considered are prime. Our first theorem in this section gives a precise result on subsequences of van der Corput sequences indexed by arithmetic progressions.

Theorem 5 *Let ω_{vdC} be the van der Corput sequence in base q (not necessarily a prime). Let $u, v \in \mathbb{Z}$ with $u \neq 0$ and $\gcd(u,q)=1$. Further define $k_n = un+v$. Then the sequence $\omega = (x_{k_n})_{n\geq 0}$ is a $(0,1)$-sequence in base q and*

$$D_N^*(\omega) \leq D_N^*(\omega_{\mathrm{vdC}}) \leq c_q \frac{\log N}{N} + \frac{b_q}{N} \qquad \text{for all } N \in \mathbb{N},$$

where $c_q, b_q > 0$ are constants depending only on the base q and

$$c_q = \begin{cases} \frac{q^2}{4(q+1)\log q} & \text{if } q \text{ is even,} \\ \frac{q-1}{4\log q} & \text{if } q \text{ is odd.} \end{cases}$$

Proof. Assume that $\gcd(u,q)=1$. Then it follows that for $m \in \mathbb{N}_0$ and $A \in \{0,\ldots,q^m-1\}$ the congruence $un+v \equiv A \pmod{q^m}$ has a unique solution modulo q^m. Therefore, for each $l \geq 0$, $m > 0$ and $a \in \{0,\ldots,q^m-1\}$, we have that in the set $\{x_{k_n} : lq^m \leq n < (l+1)q^m\}$ exactly one point is contained in the interval $[a/q^m, (a+1)/q^m)$. Hence, the sequence $(x_{k_n})_{n\geq 0}$ is a $(0,1)$-sequence in base q (see Remark 1).

It was shown in [11] that among all $(0,1)$-sequences in base q the van der Corput sequence in base q has the worst star discrepancy. The upper bound for the star discrepancy of the van der Corput sequence was shown by Faure in [6]. □

A natural question to ask when considering the result in Theorem 5 is: can these results be generalized to the case of the s-dimensional van der Corput-Halton sequence? The answer to this question is twofold. On the one hand, we will show in the subsequent theorem that the star discrepancy of the first N points of subsequences indexed by arithmetic progressions is of order $O\left((\log N)^s/N\right)$ which is one might expect. On the other hand, so far we have not been able to show more precise discrepancy estimates; this question remains open for future research.

Theorem 6 *Let $(\boldsymbol{x}_n)_{n\geq 0}$ be the van der Corput-Halton sequence in relatively prime bases $q_1,\ldots,q_s$. Furthermore, let $u \in \mathbb{N}$ with $\gcd(u,q_i)=1$ for all $i \in \{1,\ldots,s\}$ and let $k_n = un$. Then the sequence $\omega = (\boldsymbol{x}_{k_n})_{n\geq 0}$ satisfies*

$$D_N^*(\omega) \leq 2\prod_{i=1}^{s} \frac{q_i-1}{\log q_i}\frac{(\log N)^s}{N} + O\left(\frac{(\log N)^{s-1}}{N}\right),$$

where the implied factor in the O-notation is independent of N, but depends on $q_1, \dots, q_s$, u and s.

Proof. The result follows from an adaption of the proof of the well known discrepancy estimate for the classical van der Corput-Halton sequence as, for example, presented in [10]. We omit the tedious but not difficult details. $\square$

Finally we give some discrepancy estimates for subsequences of the modified one-dimensional van der Corput sequence (as defined in Example 6) indexed by primes.

Theorem 7 *Let $(x_n)_{n\geq 0}$ be the van der Corput sequence in prime base q. Moreover, let ω denote the sequence $(\{qx_p\})_{p\in\mathbb{P}}$, where $\mathbb{P}$ is the sequence of primes (in increasing order). Then for any $k > 0$ we have*

$$D_N^*(\omega) = O_k\left(\frac{1}{(\log N)^k}\right),$$

where the involved constant in the O-notation depends on k. If the Extended Riemann Hypothesis holds true, then we even have

$$D_N^*(\omega) = O\left(\frac{(\log N)^{3/8}}{N^{1/4}}\right).$$

For the proof of Theorem 7 we need the following results. For further information see [2] and the references therein.

Lemma 1 *Let $\mathrm{li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$ and $\lambda(x) = (\log x)^{3/5}(\log\log x)^{-1/5}$. For $x \to \infty$ we have*

$$\pi(x) = \mathrm{li}(x) + O\left(x\mathrm{e}^{-c\lambda(x)}\right). \tag{5}$$

and for each n such that $\gcd(a, n) = 1$ we have that for every $k > 0$ there exists a $c > 0$ for which

$$\#\{p : p \leq x, p \equiv a \pmod{n}\} = \frac{\mathrm{li}(x)}{\varphi(n)} + O\left(x\mathrm{e}^{-c\sqrt{\log x}}\right), \tag{6}$$

uniformly for $n \leq (\log x)^k$.

If the Riemann Hypothesis holds, then we have

$$\pi(x) = \mathrm{li}(x) + O\left(\sqrt{x}\log x\right), \tag{7}$$

and if the Extended Riemann Hypothesis holds, then we even have

$$\#\{p : p \leq x, p \equiv a \pmod{n}\} = \frac{\mathrm{li}(x)}{\varphi(x)} + O\left(\sqrt{x}(\log x + \log n)\right). \tag{8}$$

Now we give the proof of Theorem 7.

Proof. We will estimate the star discrepancy of the sequence $\omega^* := (\{qx_p\})_{\substack{p\in\mathbb{P}\\ p>q}}$. Prepending the finite sequence $(\{qx_p\})_{\substack{p\in\mathbb{P}\\ p\leq q}}$ changes the star discrepancy D_N^* by a term of order $1/N$ only.

For $p > q$ we have $\{qx_p\} = qx_p - \zeta_0(p)$ where $\zeta_0(p)$ is the first digit in the q-adic expansion of p. Hence we consider the sequence $(qx_p - \zeta_0(p))_{\substack{p\in\mathbb{P}\\ p>q}}$ and estimate its star discrepancy. We start with an elementary interval of the form $[a/q^m, (a+1)/q^m)$.

Let $p \in \mathbb{P}$, $p > q$ with q-adic expansion $p = \zeta_0(p) + \zeta_1(p)q + \zeta_2(p)q^2 + \cdots$. Hence, $qx_p - \zeta_0(p) = \frac{\zeta_1(p)}{q} + \frac{\zeta_2(p)}{q^2} + \cdots$. Choose $m \in \mathbb{N}_0$ and $a \in \{0, 1, \ldots, q^m - 1\}$. Then we have $qx_p - \zeta_0(p) \in [a/q^m, (a+1)/q^m)$ if and only if $p \equiv B_{q,\zeta_0(p)}(a) \pmod{q^{m+1}}$, where $B_{q,\zeta_0(p)}(a) = \zeta_0(p) + a_{m-1}q + \cdots + a_0 q^m$ for $a = a_0 + a_1 q + \cdots + a_{m-1}q^{m-1}$. Note that $\zeta_0(p) \in \{1, \ldots, q-1\}$ since $p \in \mathbb{P}$ and $p > q$. Hence we have $\gcd(B_{q,\zeta_0(p)}(a), q) = 1$ for all a and all $p > q$. Let p_n denote the n-th prime number and let n^* be the minimal positive integer with $p_{n^*} > q$. Then we have

$$
\begin{aligned}
&\#\{n : n^* \leq n < N + n^*, qx_{p_n} - \zeta_0(p_n) \in [a/q^m, (a+1)/q^m)\} \\
&= \sum_{\zeta_0=1}^{q-1} \#\left\{n : n^* \leq n < N + n^*, p_n \equiv \zeta_0 \pmod{q} \text{ and } p_n \equiv B_{q,\zeta_0}(a) \pmod{q^{m+1}}\right\} \\
&= \sum_{\zeta_0=1}^{q-1} \#\left\{n : n^* \leq n < N + n^*, p_n \equiv B_{q,\zeta_0}(a) \pmod{q^{m+1}}\right\}.
\end{aligned}
$$

For any integer B we have

$$
\begin{aligned}
&\#\left\{n : n^* \leq n < N + n^*, p_n \equiv B \pmod{q^{m+1}}\right\} = \\
&\quad = \#\left\{p : p_{n^*} \leq p \leq p_{N+n^*-1}, p \equiv B \pmod{q^{m+1}}\right\} \\
&\quad = \#\left\{p : p \leq p_{N+n^*-1}, p \equiv B \pmod{q^{m+1}}\right\} \\
&\qquad -\#\left\{p : p < p_{n^*}, p \equiv B \pmod{q^{m+1}}\right\}.
\end{aligned}
$$

Thus, the number of points $A([a/q^m, (a+1)/q^m), N, \omega^*)$ among the first N points from ω^* that fall into our given interval is

$$
A\left(\left[\frac{a}{q^m}, \frac{a+1}{q^m}\right), N, \omega^*\right) = \sum_{\zeta=1}^{q-1} \#\{p : p \leq p_{N+n^*-1}, p \equiv B_{q,\zeta}(a) \pmod{q^{m+1}}\} + O(1).
$$

Choose $k > 0$ and assume that $q^{m+1} \leq (\log p_{N+n^*-1})^k$. From Lemma 1, Eq. (5) and Eq. (6), we obtain

$$
\begin{aligned}
&A\left(\left[\frac{a}{q^m}, \frac{a+1}{q^m}\right), N, \omega^*\right)/N \\
&= \sum_{\zeta=1}^{q-1}\left[\frac{\mathrm{li}(p_{N+n^*-1})}{N}\frac{1}{q^m(q-1)} + O\left(\frac{p_{N+n^*-1}}{N}\,\mathrm{e}^{-c\sqrt{\log p_{N+n^*-1}}}\right)\right] \\
&= \frac{\mathrm{li}(p_{N+n^*-1})}{N\cdot q^m} + O\left(\frac{p_{N+n^*-1}}{N}\mathrm{e}^{-c\sqrt{\log p_{N+n^*-1}}}\right) \\
&= \frac{\pi(p_{N+n^*-1}) + O\left(p_{N+n^*-1}\mathrm{e}^{-\widetilde{c}\lambda(p_{N+n^*-1})}\right)}{N\cdot q^m} + O\left(\frac{p_{N+n^*-1}}{N}\mathrm{e}^{-c\sqrt{\log p_{N+n^*-1}}}\right)
\end{aligned}
$$

$$= \frac{1}{q^m} + O\left(\frac{p_{N+n^*-1}}{N} \mathrm{e}^{-c\sqrt{\log p_{N+n^*-1}}}\right).$$

Therefore, for $q^{m+1} \leq (\log p_{N+n^*-1})^k$ we have

$$\left|\frac{A\left(\left[a/q^m,(a+1)/q^m\right), N, \omega^*\right)}{N} - \frac{1}{q^m}\right| = O\left(\frac{p_{N+n^*-1}}{N} \mathrm{e}^{-c\sqrt{\log p_{N+n^*-1}}}\right)$$

(note that here $c = c(k)$).

Now let $[0,\alpha) \subset [0,1)$ and let $\alpha \in [a/q^m,(a+1)/q^m)$. Then we have

$$\frac{A([0,a/q^m), N, \omega^*)}{N} - \frac{a+1}{q^m} \leq \frac{A([0,\alpha), N, \omega^*)}{N} - \alpha \leq \frac{A([0,(a+1)/q^m), N, \omega^*)}{N} - \frac{a}{q^m}$$

and hence

$$\begin{aligned}
\left|\frac{A([0,\alpha), N, \omega^*)}{N} - \alpha\right| &\leq \frac{1}{q^m} + \max_{0\leq a<q^m} \left|\frac{A([0,a/q^m), N, \omega^*)}{N} - \frac{a}{q^m}\right| \\
&\leq \frac{1}{q^m} + \max_{0\leq a<q^m} \sum_{b=0}^{a-1} \left|\frac{A([b/q^m,(b+1)/q^m), N, \omega^*)}{N} - \frac{1}{q^m}\right| \\
&= \frac{1}{q^m} + q^m O\left(\frac{p_{N+n^*-1}}{N} \mathrm{e}^{-c\sqrt{\log p_{N+n^*-1}}}\right) \\
&= \frac{1}{q^m} + q^m O\left((\log N)\, \mathrm{e}^{-c\sqrt{\log N+\log\log N}}\right),
\end{aligned}$$

as for the n-th prime we have $p_n = (1+o(1))n\log n$ which is equivalent to the prime number theorem.

Now choose m such that $q^{m+1} \leq (\log p_{N+n^*-1})^k < q^{m+2}$. Then we obtain

$$\begin{aligned}
&\left|\frac{A([0,\alpha), N, \omega^*)}{N} - \alpha\right| \\
&\leq \frac{q^2}{(\log p_{N+n^*-1})^k} + (\log p_{N+n^*-1})^k O\left((\log N)\, \mathrm{e}^{-c\sqrt{\log N+\log\log N}}\right) \\
&= O\left(\frac{1}{(\log N)^k}\right),
\end{aligned}$$

uniformly for all $\alpha \in [0,1)$, where the involved constant in the O-notation depends on k. This concludes the first part of the proof.

Now assume that the Extended Riemann Hypothesis holds. We proceed as above but now we may use Eq. (7) and Eq. (8) from Lemma 1. Then we obtain

$$\left|\frac{A\left(\left[a/q^m,(a+1)/q^m\right), N, \omega^*\right)}{N} - \frac{1}{q^m}\right| = O\left(\sqrt{\frac{\log N}{N}}\,(\log N + \log q^m)\right). \tag{9}$$

Let $[0,\alpha) \subset [0,1)$. As above, but now using inequality (9), we obtain

$$\left|\frac{A([0,\alpha), N, \omega^*)}{N} - \alpha\right| \leq \frac{1}{q^m} + q^m O\left(\sqrt{\frac{\log N}{N}}\,(\log N + \log q^m)\right).$$

Choose m such that $q^m \leq N^{1/4}(\log N)^{-3/8} < q^{m+1}$. Then we obtain

$$\left|\frac{A([0,\alpha), N, \omega^*)}{N} - \alpha\right| = O\left(\frac{(\log N)^{3/8}}{N^{1/4}}\right)$$

uniformly for all $\alpha \in [0,1)$ and the result follows. $\square$

7 On sequences generated by matrices with infinite row length: an example

In the preceding sections we have given a detailed analysis of the sequences generated by matrices with finite row length and their subsequences. We already have pointed out several times before that the problem becomes essentially harder if the generating matrices also have rows with infinitely many entries different from zero.

To give at least some results for this kind of sequences, and to motivate further research in this direction, we consider the simplest examples of this type in this section.

First, for primes $q_1 \neq q_2$ we discuss uniform distribution of the two-dimensional sequence ν generated by the matrices

$$D_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{Z}_{q_1}^{\infty \times \infty} \text{ and } D_2 := \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{Z}_{q_2}^{\infty \times \infty}.$$

Further, let $\boldsymbol{\gamma} = (\gamma_i)_{i \geq 1}$ in $\mathbb{Z}_q$. Then we will study the circumstances under which a subsequence of the one-dimensional sequence generated by a matrix

$$E(\boldsymbol{\gamma}) := \begin{pmatrix} 1 & \gamma_1 & \gamma_2 & \gamma_3 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{Z}_q^{\infty \times \infty}$$

is uniformly distributed in $[0, 1)$. We shall refer to this sequence as $\sigma(\boldsymbol{\gamma})$ in the following.

Theorem 8 *The sequence $\nu = (\boldsymbol{x}_n)_{n \geq 0}$ defined above is uniformly distributed in $[0, 1)^2$.*

Proof. Let a, b be integers with $0 \leq a < q_1^{d_1}$ and $0 \leq b < q_2^{d_2}$ and let

$$I = \left[\frac{a}{q_1^{d_1}}, \frac{a+1}{q_1^{d_1}}\right) \times \left[\frac{b}{q_2^{d_2}}, \frac{b+1}{q_2^{d_2}}\right) \subseteq [0, 1)^2.$$

We need to show that

$$\lim_{N \to \infty} \frac{1}{N} \#\{n : n \leq N, \boldsymbol{x}_n \in I\} = \frac{1}{q_1^{d_1} q_2^{d_2}}.$$

Let $a = a_{d_1 - 1} + a_{d_1 - 2} q_1 + \cdots + a_0 q_1^{d_1 - 1}$ and $b = b_{d_2 - 1} + b_{d_2 - 2} q_2 + \cdots + b_0 q_2^{d_2 - 1}$. Then, for $n = n_0^{(i)} + n_1^{(i)} q_i + n_2^{(i)} q_i^2 + \cdots$, $i \in \{1, 2\}$, we have $\boldsymbol{x}_n \in I$ if and only if

$$D_1 \cdot \begin{pmatrix} n_0^{(1)} \\ n_1^{(1)} \\ n_2^{(1)} \\ \vdots \end{pmatrix} = \begin{pmatrix} a_0 \\ \vdots \\ a_{d_1 - 1} \\ \vdots \end{pmatrix} \text{ and } D_2 \cdot \begin{pmatrix} n_0^{(2)} \\ n_1^{(2)} \\ n_2^{(2)} \\ \vdots \end{pmatrix} = \begin{pmatrix} b_0 \\ \vdots \\ b_{d_2 - 1} \\ \vdots \end{pmatrix},$$

i.e., if and only if $n \equiv \widetilde{a} \pmod{q_1^{d_1}}$ and $s_{q_2}(n) \equiv b_0 \pmod{q_2}$ and $n \equiv c + b_1 q_2 + b_2 q_2^2 + \cdots + b_{d_2-1} q_2^{d_2-1} \pmod{q_2^{d_2}}$ for some $c \in \{0, \ldots, q_2 - 1\}$, where $\widetilde{a} = a_0 + a_1 q_1 + \cdots + a_{d_1-1} q_1^{d_1-1}$ and where $s_{q_2}(n) := n_0^{(2)} + n_1^{(2)} + \cdots$ denotes the (unweighted) sum-of-digits function of n in base q_2.

This means that for a certain $\alpha(c) \in \left\{0, 1, \ldots, q_1^{d_1} q_2^{d_2} - 1\right\}$,

$$n \equiv \alpha(c) \pmod{q_1^{d_1} q_2^{d_2}} \text{ and } s_{q_2}(n) \equiv b_0 \pmod{q_2}.$$

By a result of Gel'fond [7], see also Solinas [22] and its generalization given on page 150 of [22], we have

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{n : n \le N, n \equiv \alpha(c) \pmod{q_1^{d_1} q_2^{d_2}} \text{ and } s_{q_2}(n) \equiv b_0 \pmod{q_2}\right\} = \frac{1}{q_1^{d_1} q_2^{d_2+1}}$$

for each $c \in \{0, \ldots, q_2 - 1\}$. Hence,

$$\lim_{N\to\infty} \frac{1}{N} \#\{n : n \le N, \boldsymbol{x}_n \in I\} = \frac{1}{q_1^{d_1} q_2^{d_2}}$$

and the result follows. □

Remark 5 *Gel'fond [7] even gave an error term in the above limit. Using this more exact result we can easily find that* $D_N^*(\nu) = O\left(N^{-\frac{1}{2}+\lambda}\right)$ *for some* $\lambda < 1/2$*.*

Finally, we consider subsequences of the sequence $\sigma(\boldsymbol{\gamma})$.

Theorem 9 *Let* $\boldsymbol{\gamma} = (\gamma_i)_{i\ge1}$ *be a sequence in* $\mathbb{Z}_q$ *and let the sequence* $\sigma(\boldsymbol{\gamma}) = (x_n)_{n\ge0}$ *be defined as above. Furthermore, let* $(k_n)_{n\ge0}$ *be a subsequence of* $\mathbb{N}_0$*.*

Then the sequence $(x_{k_n})_{n\ge0}$ *is uniformly distributed modulo one if and only if for every* $m \in \mathbb{N}$*, every* $\alpha \in \{0, \ldots, q^{m-1} - 1\}$*, and every* $\beta \in \{0, 1, \ldots, q - 1\}$ *we have*

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{n : n \le N, \lfloor k_n/q \rfloor \equiv \alpha \pmod{q^{m-1}} \text{ and } s_{\boldsymbol{\gamma},q}(k_n) \equiv \beta \pmod{q}\right\} = \frac{1}{q^m}. \tag{10}$$

Here, for $n = n_0 + n_1 q + n_2 q^2 + \cdots$*,* $s_{\boldsymbol{\gamma},q}(n)$ *denotes the* $\boldsymbol{\gamma}$*-weighted sum-of-digits of* n *in base* q*, i.e.* $s_{\boldsymbol{\gamma},q}(n) = n_0 + \gamma_1 n_1 + \gamma_2 n_2 + \cdots$*.*

Proof. Let $I = \left[\frac{a}{q^m}, \frac{a+1}{q^m}\right)$ with $a = a_{m-1} + a_{m-2} q + \cdots + a_0 q^{m-1}$. We need to show that

$$\lim_{N\to\infty} \frac{1}{N} \#\{n : n \le N, x_{k_n} \in I\} = \frac{1}{q^m}$$

for all $m \in \mathbb{N}$ and $0 \le a < q^m$ if and only if (10) holds.

We have $x_n \in I$ for $n = n_0 + n_1 q + \cdots$ if and only if

$$s_{\boldsymbol{\gamma},q}(n) \equiv a_0 \pmod{q} \text{ and } n_i \equiv a_i \pmod{q} \text{ for all } i \in \{1, \ldots, m-1\}.$$

This is equivalent to $s_{\boldsymbol{\gamma},q}(n) \equiv a_0 \pmod{q}$ and

$$n \equiv a_{m-1} q^{m-1} + \cdots + a_1 q + c \pmod{q^m} \text{ for some } c \in \{0, 1, \ldots, q-1\}$$

which in turn is equivalent to

$$s_{\boldsymbol{\gamma},q}(n) \equiv a_0 \pmod{q} \text{ and } \left\lfloor \frac{n}{q} \right\rfloor \equiv a_{m-1} q^{m-2} + \cdots + a_1 \pmod{q^{m-1}}.$$

The result follows. □

Example 7 Let $k_n = un + v$ with $u \geq 2$ and $v \in \{0, 1, \ldots, u-1\}$ and $\gcd(u, q) = 1$. Then $\lfloor k_n/q \rfloor \equiv \alpha \pmod{q^{m-1}}$ means that

$$un + v \equiv q\alpha + \delta \pmod{q^m}$$

for some $\delta \in \{0, 1, \ldots, q-1\}$, i.e.,

$$un + v = u\delta' + v + wuq^m$$

for some w, and $\delta' = (q\alpha + \delta - v)u^{-1} \pmod{q^m}$. Here, u^{-1} denotes the inverse of u modulo q^m. Hence, if for every $d \in \{0, 1, \ldots, q^m - 1\}$ and every $\beta \in \{0, 1, \ldots, q-1\}$ we have

$$\lim_{W \to \infty} \frac{1}{W} \# \{w : w \leq W, s_{\boldsymbol{\gamma},q}(ud + v + wuq^m) \equiv \beta \pmod{q}\} = \frac{1}{q},$$

then the subsequence $(x_{un+v})_{n \geq 0}$ of $\sigma(\boldsymbol{\gamma})$ is uniformly distributed modulo one.

For the special case $\boldsymbol{\gamma} = (1)_{j \geq 1}$, again by Theorem 1 in [22] and its generalization on page 150 of [22], we obtain uniform distribution of $(x_{un+v})_{n \geq 0}$ for all u with $\gcd(u, q) = 1$ and all v.

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