Average growth-behavior and distribution properties of generalized weighted digit-block-counting functions

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Dedicated to Prof. Robert F. Tichy on the occasion of his 50th birthday.

Abstract

We introduce a generalized weighted digit-block-counting function on the non-negative integers, which is a generalization of many digit-depending functions as, for example, the well known sum-of-digits function. A formula for the first moment of the sum-of-digits function has been given by Delange in 1972. In the first part of this paper we provide a compact formula for the first moment of the generalized weighted digit-block-counting function and show that a (weak) Delange type formula holds if the sequence of weights converges. The question, whether the converse is true as well, can only be answered partially at the moment.

In the second part of this paper we study distribution properties of generalized weighted digit-block-counting sequences and their d-dimensional analogues. We give an *if and only if* condition under which such sequences are uniformly distributed modulo one.

Keywords: Digit-block-counting function, sum-of-digits function, first moment, uniform distribution modulo one.
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1 Introduction

Let $\boldsymbol{\gamma} = (\gamma_s)_{s \geq 0}$ be a sequence in $\mathbb{R}$. Let $t \geq 1$ and $q \geq 2$ be integers and $\Gamma = \{0, 1, \ldots, q-1\}^t$. Further let $g : \Gamma \to \mathbb{R}$ be a function.

For $k \in \mathbb{N}_0$ with base q representation $k = k_r q^r + \cdots + k_1 q + k_0$, where $k_r \neq 0$, we define the *generalized weighted digit-block-counting function*

$$s_q(k, \boldsymbol{\gamma}) := \sum_{s=0}^{r} \gamma_s g(k_s, \ldots, k_{s+t-1}), \tag{1}$$

where here $k_i = 0$ for all $i > r$. (This definition is a weighted version of the definition given by Cateland in [1].)

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In the first part of this paper (Section 2) we investigate the average growth-behavior of the generalized weighted digit-block-counting function. For $n \in \mathbb{N}$ with $n \geq 2$ we consider the first moment defined as

$$S_q(n, \boldsymbol{\gamma}) := \sum_{k=1}^{n-1} s_q(k, \boldsymbol{\gamma}).$$

The definition of the function in (1) covers many well-known and extensively studied sequences as, for example:

1. If $t = 1$ and $g : \Gamma \to \mathbb{R}$ is given by $g(a) = a$ for all $a \in \Gamma$, then $s_q(\cdot, \boldsymbol{\gamma})$ is the *weighted sum-of-digits function*. For the unweighted case, i.e., $\boldsymbol{\gamma} = \mathbf{1} = (1)_{s \geq 0}$, Delange [2] showed the following formula.

   Theorem 1 (Delange) *We have*

   $$S_q(n, \mathbf{1}) = \frac{q-1}{2} n \log_q(n) + n F_{\mathrm{Del}}(\log_q(n)),$$

   where $F_{\mathrm{Del}}(x)$ is a continuous, one-periodic, nowhere differentiable function.

   See also [4, 12, 14] and the references therein. A similar formula for the more general weighted case was proved in [7] and recently in [10] for $q = 2$.

2. If $t \geq 1$, $q \geq 2$ and the function $g : \Gamma \to \mathbb{R}$ is given by $g(\boldsymbol{a}) = 1$ for fixed $\boldsymbol{a} = (a_1, \ldots, a_t) \in \Gamma$ and zero otherwise, then $s_q(k, \boldsymbol{\gamma})$ counts (weighted) occurrences of the block $(a_1, \ldots, a_t)$ in the base q representation of k. In this case we will refer to $s_q(k, \boldsymbol{\gamma})$ as the *weighted single-block-occurrence function*. For the unweighted case a formula for the first moment was given by Kirschenhofer [6], see also [4].

3. If $t = 2$, $q = 2$, and the function $g : \Gamma \to \mathbb{R}$ is given by $g(0, 0) = g(1, 1) = 0$ and $g(1, 0) = g(0, 1) = 1$, then $s_q(k, \boldsymbol{\gamma})$ counts the (weighted) number of 1's in the Gray code representation of k. In this case we will refer to $s_q(k, \boldsymbol{\gamma})$ as the *weighted Gray code sum*. Here a formula for the first moment for the unweighted case was given by Flajolet & Ramshaw [5], see also [4].

It is the aim of the first part of this paper to calculate the first moment of the generalized weighted digit-block-counting function (1), see Subsection 2.1. Further we show in Subsection 2.2 that a formula like that in Theorem 1 holds if the sequence $\boldsymbol{\gamma}$ of weights converges. In this case we will say, the formula for the first moment is *(weak) Delange type* (see Subsection 2.3 for an exact definition). Of course our formula also contains the formulas from Kirschenhofer [6] and Flajolet & Ramshaw [5] as special cases. In many cases we can also prove the converse, i.e., if the first moment is (weak) Delange type, then the sequence of weights has to converge (see Subsection 2.3). However, a general answer to this question has to remain open for the moment.

In the second part of this paper (Section 3) we will study distribution properties of generalized weighted digit-block-counting sequences. Especially we will answer the question under which conditions the generalized weighted digit-block-counting sequence $(s_q(k, \boldsymbol{\gamma}))_{k=0,1,\ldots}$ is uniformly distributed modulo one.

We recall that a sequence $(\boldsymbol{x}_n)_{n\geq 0}$ in $\mathbb{R}^d$ is said to be *uniformly distributed modulo one* if for all intervals $[\boldsymbol{a},\boldsymbol{b})\subseteq[0,1)^d$ we have

$$\lim_{N\to\infty}\frac{\#\{n\,:\,0\leq n<N,\{\boldsymbol{x}_n\}\in[\boldsymbol{a},\boldsymbol{b})\}}{N}=\lambda_d([\boldsymbol{a},\boldsymbol{b})),$$

where λ_d denotes the d-dimensional Lebesgue measure. Here for a vector $\boldsymbol{x}$ the fractional part $\{\cdot\}$ is applied componentwise. An excellent introduction to this topic can be found in the book of Kuipers & Niederreiter [8] or in the book of Drmota & Tichy [3].

Even more generally we will ask under which conditions the following quite general d-dimensional generalized weighted digit-block-counting sequences are uniformly distributed modulo one:

For given dimension d we choose d functions $g^{(i)}:\Gamma\to\mathbb{R}$ for all $i\in\{1,\ldots,d\}$ where we assume $g^{(i)}(\mathbf{0})=0$ for all i. Further we choose d sequences of weights

$$\boldsymbol{\gamma}^{(i)}:=(\gamma_0^{(i)},\gamma_1^{(i)},\ldots)\quad\text{for all } i\in\{1,\ldots,d\}$$

and denote by $s_q^{(i)}(k,\boldsymbol{\gamma}^{(i)})$ the generalized weighted digit-block-counting function generated by $g^{(i)}$ and $\boldsymbol{\gamma}^{(i)}$.

Now we are interested in the distribution behavior modulo one of the sequence

$$(\boldsymbol{s}_q(k,\boldsymbol{\gamma}))_{k=0,1,\ldots}:=(s_q^{(1)}(k,\boldsymbol{\gamma}^{(1)}),\ldots,s_q^{(d)}(k,\boldsymbol{\gamma}^{(d)}))_{k=0,1,\ldots}\tag{2}$$

in the d-dimensional unit cube.

The question, under which conditions the sequence (2) is uniformly distributed modulo one, was fully answered for $t=1$ and $g^{(i)}$ the identity function for all $i\in\{1,\ldots,d\}$ by Pillichshammer [11]. For arbitrary t, but $d=1$ and constant weights $\boldsymbol{\gamma}$ it was answered by Larcher & Tichy [9].

In Section 3 of this paper we give an *if and only if* condition under which the sequence (2) in its full generality is uniformly distributed modulo one.

Throughout the paper let the integers $q\geq 2$ and $t\geq 1$ be fixed. Therefore also the set Γ is fixed. We define $\Gamma^*:=\Gamma\setminus\{\mathbf{0}\}$, where $\mathbf{0}=(0,\ldots,0)\in\Gamma$. If we write in the following $\boldsymbol{a}\in\Gamma$, then always $\boldsymbol{a}=(a_1,\ldots,a_t)$. Analogously, $\boldsymbol{x}=(x_1,\ldots,x_t)$, $\boldsymbol{j}=(j_1,\ldots,j_t)$ and so on. For a real number x, we denote by $\{x\}$ the fractional part of x, by $\lfloor x\rfloor$ the integer part of x and by $\|x\|$ the distance of x to the nearest integer, i.e., $\|x\|=\min\{x-\lfloor x\rfloor,1-(x-\lfloor x\rfloor)\}$. By $\log_q(x)$ we will denote the base q logarithm of x. For vectors $\boldsymbol{x}\in\mathbb{R}^d$ the functions $\lfloor\cdot\rfloor$ and $\{\cdot\}$ are applied componentwise. Further for $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^d$ we denote by $\boldsymbol{x}\cdot\boldsymbol{y}$ the usual inner product in $\mathbb{R}^d$. We will write $\mathbb{N}$ for the set of positive integers and $\mathbb{N}_0$ for the set of nonnegative integers.

2 The first moment of the generalized weighted digit-block-counting function

2.1 A formula for the first moment of the generalized weighted digit-block-counting function

Here we compute the first moment of the generalized weighted digit-block-counting function as defined in (1).

Theorem 2 *Let $\boldsymbol{\gamma} = (\gamma_s)_{s \geq 0}$ be a sequence in $\mathbb{R}$. For any positive integer $n = n_0 + n_1 q + n_2 q^2 + \cdots$ we have*

$$\begin{aligned} S_q(n, \boldsymbol{\gamma}) &= \frac{n}{q^t} \sum_{\boldsymbol{a} \in \Gamma} g(\boldsymbol{a}) \sum_{s=0}^{r(n)} \gamma_s - g(\mathbf{0}) \sum_{s=0}^{r(n)} \gamma_s q^s \\ &+ \frac{1}{q^t} \sum_{\boldsymbol{a} \in \Gamma} g(\boldsymbol{a}) \sum_{\boldsymbol{j} \in \Gamma^*} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{q} \sum_{l=1}^{t} j_l a_l} \sum_{s=0}^{r(n)} \gamma_s \\ &\times \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} \sum_{l=\bar{l}+1}^{t} j_l n_{s+l-1}} q^{\bar{l}+s-1} \left(\frac{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} j_{\bar{l}} n_{\bar{l}+s-1}} - 1}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} j_{\bar{l}}} - 1} + \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} j_{\bar{l}} n_{\bar{l}+s-1}} \left\{ \frac{n}{q^{\bar{l}+s-1}} \right\} \right), \end{aligned}$$

where $r(n) := \lfloor \log_q(n) \rfloor$ and $\bar{l} := \min\{l \in \{1, ..., t\} : j_l \neq 0\}$.

A more concise version of the above formula will be given in Subsection 2.2. Before we give the proof of this result let us consider some examples.

Example 1 For $t = 1$ and the function $g : \Gamma \to \mathbb{R}$ the identity function we get the first moment of the weighted sum-of-digits function in base q. It follows easily from Theorem 2 together with the formula

$$\frac{1}{q} \sum_{z=0}^{q-1} z \mathrm{e}^{-\frac{2\pi \mathrm{i}}{q} j z} = \frac{1}{\mathrm{e}^{-\frac{2\pi \mathrm{i}}{q} j} - 1},$$

that in this case for $n \geq 2$ we have

$$S_q(n, \boldsymbol{\gamma}) = \frac{n(q-1)}{2} \sum_{s=0}^{r(n)} \gamma_s + \sum_{j=1}^{q-1} \frac{1}{\mathrm{e}^{-\frac{2\pi \mathrm{i}}{q} j} - 1} \sum_{s=0}^{r(n)} \gamma_s q^s \left(\frac{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} j n_s} - 1}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} j} - 1} + \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} j n_s} \left\{ \frac{n}{q^s} \right\} \right).$$

For $q = 2$ this formula reduces to

$$S_2(n, \boldsymbol{\gamma}) = \frac{n}{2} \sum_{s=0}^{r(n)} \gamma_s - \frac{n}{2} \sum_{s=0}^{r(n)} \gamma_s \frac{2^{s+1}}{n} \left\| \frac{n}{2^{s+1}} \right\|$$

where in the last formula $r(n) = \lfloor \log_2(n) \rfloor$. This formula was proved in [10, Theorem 2]. See also [7].

Example 2 For $t = 2$, $q = 2$ and the function $g : \Gamma \to \mathbb{R}$ given by $g(0,0) = g(1,1) = 0$ and $g(0,1) = g(1,0) = 1$ we have

$$S_2(n, \boldsymbol{\gamma}) = \frac{n}{2} \sum_{s=0}^{r(n)} \gamma_s - \frac{n}{2} \sum_{s=0}^{r(n)} \gamma_s (-1)^{n_{s+1}} \frac{2^{s+1}}{n} \left\| \frac{n}{2^{s+1}} \right\|$$

for $n \geq 2$, where here $r(n) = \lfloor \log_2(n) \rfloor$ and $n_{r(n)+1} = 0$.

For the proof of Theorem 2 we need several lemmas.

Lemma 1 *For $\boldsymbol{y} \in \Gamma$ we have*

$$g(\boldsymbol{y}) = \frac{1}{q^t} \sum_{\boldsymbol{a} \in \Gamma} g(\boldsymbol{a}) \sum_{\boldsymbol{j} \in \Gamma} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} \sum_{l=1}^{t} j_l (y_l - a_l)}.$$

Proof. This follows easily from the fact that for an integer $q \geq 2$ we have

$$\frac{1}{q} \sum_{j=0}^{q-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} jz} = \begin{cases} 0 & \text{if } z \not\equiv 0 \pmod{q}, \\ 1 & \text{if } z \equiv 0 \pmod{q}. \end{cases}$$

$\square$

Lemma 2 *Let the nonnegative integer U have q-adic expansion $U = U_0 + \cdots + U_{m-1} q^{m-1}$. For any nonnegative integer $n \leq U - 1$ let $n = n_0 + \cdots + n_{m-1} q^{m-1}$ be the q-adic expansion of n. For $0 \leq p \leq m-1$ let $U(p) := U_0 + \cdots + U_p q^p$. Let $b_0, b_1, \ldots, b_{m-1}$ be arbitrary elements of $\mathbb{Z}_q$, not all zero. Then*

$$\sum_{n=0}^{U-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}(b_0 n_0 + \cdots + b_{m-1} n_{m-1})} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}(b_{w+1} U_{w+1} + \cdots + b_{m-1} U_{m-1})} q^w \left(\frac{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} b_w U_w} - 1}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} b_w} - 1} + \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} b_w U_w} \left\{ \frac{U}{q^w} \right\} \right),$$

where w is minimal such that $b_w \neq 0$.

Proof. The result easily follows from splitting up the sum.

$$\begin{aligned}
& \sum_{n=0}^{U-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}(b_0 n_0 + \cdots + b_{m-1} n_{m-1})} \\
&= \sum_{n=0}^{q^{w+1}(U_{w+1} + \cdots + U_{m-1} q^{m-w-2}) - 1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} n_w b_w} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}(b_{w+1} n_{w+1} + \cdots + b_{m-1} n_{m-1})} \\
&\quad + \sum_{n=0}^{U(w)-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} n_w b_w} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}(b_{w+1} U_{w+1} + \cdots + b_{m-1} U_{m-1})} \\
&= 0 + \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}(b_{w+1} U_{w+1} + \cdots + b_{m-1} U_{m-1})} \sum_{n=0}^{U(w)-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} n_w b_w}.
\end{aligned}$$

We study the last sum. We have

$$\begin{aligned}
& \sum_{n=0}^{U(w)-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} n_w b_w} \\
&= \sum_{n=0}^{q^w - 1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} 0 b_w} + \sum_{n=q^w}^{2q^w - 1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} b_w} + \cdots + \sum_{n=(U_w - 1) q^w}^{U_w q^w - 1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}(U_w - 1) b_w} + \sum_{n=U_w q^w}^{U(w)-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} U_w b_w} \\
&= q^w \sum_{k=0}^{U_w - 1} \left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} b_w} \right)^k + (U(w) - U_w q^w) \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} U_w b_w} \\
&= q^w \left(\frac{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} b_w U_w} - 1}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} b_w} - 1} + \left(\frac{U(w)}{q^w} - U_w \right) \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} U_w b_w} \right) \\
&= q^w \left(\frac{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} b_w U_w} - 1}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} b_w} - 1} + \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} b_w U_w} \left\{ \frac{U}{q^w} \right\} \right).
\end{aligned}$$

The result follows. $\square$

Remark 1 For $q=2$ the formula from Lemma 2 can be further simplified with

$$q^w\left(\frac{\mathrm{e}^{\frac{2\pi\mathtt{i}}{q}b_wU_w}-1}{\mathrm{e}^{\frac{2\pi\mathtt{i}}{q}b_w}-1}+\mathrm{e}^{\frac{2\pi\mathtt{i}}{q}b_wU_w}\left\{\frac{U}{q^w}\right\}\right)=2^{w+1}\left\|\frac{U}{2^{w+1}}\right\|.$$

Now we give the proof of Theorem 2.

Proof. With the definition of the first moment we have

$$S_q(n,\boldsymbol{\gamma})=\sum_{k=1}^{n-1}s_q(k,\gamma)=\sum_{r=0}^{r(n)-1}\sum_{k=q^r}^{q^{r+1}-1}s_q(k,\gamma)+\sum_{k=q^{r(n)}}^{n-1}s_q(k,\gamma)=:\Sigma_1+\Sigma_2.$$

First we consider the sum Σ_1. We have

$$\Sigma_1=\sum_{r=0}^{r(n)-1}\sum_{k=q^r}^{q^{r+1}-1}s_q(k,\boldsymbol{\gamma})=\sum_{r=0}^{r(n)-1}\sum_{k=q^r}^{q^{r+1}-1}\sum_{s=0}^{r}\gamma_s g(k_s,\ldots,k_{s+t-1}).$$

Now we use Lemma 1 to replace $g(k_s,\ldots,k_{s+t-1})$ and get after changing the order of summation,

$$\Sigma_1=\frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma}g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma}\mathrm{e}^{-\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l a_l}\sum_{r=0}^{r(n)-1}\sum_{s=0}^{r}\gamma_s\sum_{k=q^r}^{q^{r+1}-1}\mathrm{e}^{\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l k_{s+l-1}}.$$

Now we take a closer look at the innermost sum in the above expression. Let

$$\Sigma_{1,1}:=\sum_{k=q^r}^{q^{r+1}-1}\mathrm{e}^{\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l k_{s+l-1}}$$

depending on $(j_1,\ldots,j_t)$ and s. We consider the following cases:

(a) If $j_1=\cdots=j_t=0$, then we get $\Sigma_{1,1}=q^r(q-1)$ and furthermore $\mathrm{e}^{-\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l a_l}=1$.

(b) If there exists at least one $l\in\{1,\ldots,t\}$ such that $j_l\neq 0$, then we define $\bar{l}:=\min\{l\in\{1,\ldots,t\}\,:\,j_l\neq 0\}$. We consider three cases for s:

(i) If $s+\bar{l}-1<r$, then we have

$$\Sigma_{1,1}=\sum_{k_0,\ldots,k_{r-1}=0}^{q-1}\sum_{k_r=1}^{q-1}\mathrm{e}^{\frac{2\pi\mathtt{i}}{q}\left(j_{\bar{l}}k_{s+\bar{l}-1}+\cdots+j_{\min\{t,r-s+1\}}k_{\min\{s+t-1,r\}}\right)}=0.$$

(ii) If $s+\bar{l}-1=r$, then we have

$$\Sigma_{1,1}=\sum_{k=q^r}^{q^{r+1}-1}\mathrm{e}^{\frac{2\pi\mathtt{i}}{q}j_{\bar{l}}k_r}=-q^r.$$

(iii) If $s+\bar{l}-1>r$, then we have

$$\Sigma_{1,1}=\sum_{k=q^r}^{q^{r+1}-1}1=q^r(q-1).$$

Altogether we find that

$$\begin{aligned}\Sigma_1 &= \frac{q-1}{q^t}\sum_{\boldsymbol{a}\in\Gamma}g(\boldsymbol{a})\sum_{r=0}^{r(n)-1}q^r\sum_{s=0}^{r}\gamma_s+\frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma}g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma^*}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{q}\sum_{l=1}^{t}j_la_l}\\ &\quad\times\sum_{r=0}^{r(n)-1}\left(-q^r\sum_{s=\max\{r-\bar{l}+1,0\}}^{r-\bar{l}+1}\gamma_s+(q-1)q^r\sum_{s=\max\{r-\bar{l}+2,0\}}^{r}\gamma_s\right).\end{aligned}$$

We have the following identities:

$$\sum_{r=0}^{r(n)-1}q^r\sum_{s=0}^{r}\gamma_s=\sum_{s=0}^{r(n)-1}\gamma_s\sum_{r=s}^{r(n)-1}q^r=\frac{q^{r(n)}}{q-1}\sum_{s=0}^{r(n)-1}\gamma_s-\frac{1}{q-1}\sum_{s=0}^{r(n)-1}\gamma_sq^s,$$

and

$$\sum_{r=0}^{r(n)-1}q^r\sum_{s=\max\{r-\bar{l}+1,0\}}^{r-\bar{l}+1}\gamma_s=\sum_{r=\bar{l}-1}^{r(n)-1}q^r\gamma_{r-\bar{l}+1}=\sum_{s=0}^{r(n)-\bar{l}}\gamma_sq^{s+\bar{l}-1}.$$

For the last term we have

$$\begin{aligned}\sum_{r=0}^{r(n)-1}(q-1)q^r\sum_{s=\max\{r-\bar{l}+2,0\}}^{r}\gamma_s &= (q-1)\sum_{s=0}^{r(n)-1}\gamma_s\sum_{r=s}^{\min\{s+\bar{l}-2,r(n)-1\}}q^r\\ &= \sum_{s=0}^{r(n)-1}\gamma_sq^{\min\{s+\bar{l}-1,r(n)\}}-\sum_{s=0}^{r(n)-1}\gamma_sq^s.\end{aligned}$$

Now we can express Σ_1 as

$$\begin{aligned}\Sigma_1 &= \frac{q^{r(n)}}{q^t}\sum_{\boldsymbol{a}\in\Gamma}g(\boldsymbol{a})\sum_{s=0}^{r(n)}\gamma_s-\frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma}g(\boldsymbol{a})\sum_{s=0}^{r(n)}\gamma_sq^s\\ &\quad-\frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma}g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma^*}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{q}\sum_{l=1}^{t}j_la_l}\sum_{s=0}^{r(n)-\bar{l}}\gamma_sq^{s+\bar{l}-1}\\ &\quad+\frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma}g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma^*}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{q}\sum_{l=1}^{t}j_la_l}\sum_{s=0}^{r(n)}\gamma_sq^{\min\{s+\bar{l}-1,r(n)\}}\\ &\quad-\frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma}g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma^*}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{q}\sum_{l=1}^{t}j_la_l}\sum_{s=0}^{r(n)}\gamma_sq^s.\end{aligned}$$

The second and the last sum together give

$$
\begin{aligned}
&-\frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{s=0}^{r(n)}\gamma_s q^s - \frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma^*} \mathrm{e}^{-\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l a_l}\sum_{s=0}^{r(n)}\gamma_s q^s \\
&= -\sum_{s=0}^{r(n)}\gamma_s q^s \frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma} \mathrm{e}^{-\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l a_l},
\end{aligned}
$$

and from Lemma 1 we know that

$$
g(\mathbf{0}) = \frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma} \mathrm{e}^{-\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l a_l}.
$$

Therefore we obtain

$$
\begin{aligned}
\Sigma_1 &= \frac{q^{r(n)}}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{s=0}^{r(n)}\gamma_s - g(\mathbf{0})\sum_{s=0}^{r(n)}\gamma_s q^s \\
&\quad -\frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma^*} \mathrm{e}^{-\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l a_l}\sum_{s=0}^{r(n)-\bar{l}}\gamma_s q^{s+\bar{l}-1} \\
&\quad +\frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma^*} \mathrm{e}^{-\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l a_l}\sum_{s=0}^{r(n)}\gamma_s q^{\min\{s+\bar{l}-1,r(n)\}}.
\end{aligned}
$$

If we finally split the last sum,

$$
\sum_{s=0}^{r(n)}\gamma_s q^{\min\{s+\bar{l}-1,r(n)\}} = \sum_{s=0}^{r(n)-\bar{l}}\gamma_s q^{s+\bar{l}-1} + \sum_{s=r(n)-\bar{l}+1}^{r(n)}\gamma_s q^{r(n)},
$$

then we end at

$$
\begin{aligned}
\Sigma_1 &= \frac{q^{r(n)}}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{s=0}^{r(n)}\gamma_s - g(\mathbf{0})\sum_{s=0}^{r(n)}\gamma_s q^s \\
&\quad +\frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma^*} \mathrm{e}^{-\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l a_l}\sum_{s=r(n)-\bar{l}+1}^{r(n)}\gamma_s q^{r(n)}.
\end{aligned}
$$

Now we turn to the sum Σ_2. Again we use Lemma 1 to replace $g(k_s,\ldots,k_{s+t-1})$ and get

$$
\Sigma_2 = \frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma} \mathrm{e}^{-\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l a_l}\sum_{s=0}^{r(n)}\gamma_s\sum_{k=q^{r(n)}}^{n-1} \mathrm{e}^{\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l k_{s+l-1}}.
$$

To compute $\Sigma_{2,1} := \sum_{k=q^{r(n)}}^{n-1} \mathrm{e}^{\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l k_{s+l-1}}$ depending on $j_1,\ldots,j_t$ and s we have to consider the following cases:

(a) If $j_1=\cdots=j_t=0$, then we have $\Sigma_{2,1}=n-q^{r(n)}$ and furthermore $\mathrm{e}^{-\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l a_l}=1$.

(b) If there exists $j_l \neq 0$, then let again $\overline{l} := \min\{l \in \{1, ..., t\} : j_l \neq 0\}$. We consider three cases for s.

(i) If $s < r(n) - \overline{l} + 1$, then $s + \overline{l} - 1 \neq r(n)$ and we get after replacing the summation index k by $k' = k - q^{r(n)}$ and using Lemma 2,

$$\Sigma_{2,1} = \mathrm{e}^{\frac{2\pi\mathrm{i}}{q}\sum_{l=\overline{l}+1}^{t} j_l n_{s+l-1}} q^{\overline{l}+s-1} \left(\frac{\mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}} n_{\overline{l}+s-1}} - 1}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}}} - 1} + \mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}} n_{\overline{l}+s-1}} \left\{ \frac{n}{q^{\overline{l}+s-1}} \right\} \right).$$

(ii) If $s = r(n) - \overline{l} + 1$, then $s + \overline{l} - 1 = r(n)$ and we get again with Lemma 2,

$$\begin{aligned} \Sigma_{2,1} &= \mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}}} \sum_{k=0}^{n-q^{r(n)}-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{q}\sum_{l=1}^{t} j_l k_{s+l-1}} \\ &= \mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}}} \mathrm{e}^{\frac{2\pi\mathrm{i}}{q}\sum_{l=\overline{l}+1}^{t} j_l n_{s+l-1}} q^{\overline{l}+s-1} \left(\frac{\mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}}(n_{\overline{l}+s-1}-1)} - 1}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}}} - 1} + \mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}}(n_{\overline{l}+s-1}-1)} \left\{ \frac{n}{q^{\overline{l}+s-1}} \right\} \right) \\ &= q^{\overline{l}+s-1} \left(\frac{\mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}} n_{\overline{l}+s-1}} - 1}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}}} - 1} + \mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}} n_{\overline{l}+s-1}} \left\{ \frac{n}{q^{\overline{l}+s-1}} \right\} \right) - q^{s+\overline{l}-1}, \end{aligned}$$

as $\mathrm{e}^{\frac{2\pi\mathrm{i}}{q}\sum_{l=\overline{l}+1}^{t} j_l n_{s+l-1}} = 1$.

(iii) If $s > r(n) - \overline{l} + 1$, then we have $k_{s+l-1} = 0$ for $l \geq \overline{l}$. Thus we get

$$\Sigma_{2,1} = n - q^{r(n)}.$$

Altogether we find

$$\begin{aligned} \Sigma_2 = \; & \frac{n - q^{r(n)}}{q^t} \sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a}) \sum_{s=0}^{r(n)} \gamma_s \\ & + \frac{1}{q^t} \sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a}) \sum_{\boldsymbol{j}\in\Gamma^*} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{q}\sum_{l=1}^{t} j_l a_l} \sum_{s=0}^{r(n)-\overline{l}+1} \gamma_s \\ & \times \mathrm{e}^{\frac{2\pi\mathrm{i}}{q}\sum_{l=\overline{l}+1}^{t} j_l n_{s+l-1}} q^{\overline{l}+s-1} \left(\frac{\mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}} n_{\overline{l}+s-1}} - 1}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}}} - 1} + \mathrm{e}^{\frac{2\pi\mathrm{i}}{q} j_{\overline{l}} n_{\overline{l}+s-1}} \left\{ \frac{n}{q^{\overline{l}+s-1}} \right\} \right) \\ & - \frac{1}{q^t} \sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a}) \sum_{\boldsymbol{j}\in\Gamma^*} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{q}\sum_{l=1}^{t} j_l a_l} \gamma_{r(n)-\overline{l}+1} q^{r(n)} \\ & + \frac{n - q^{r(n)}}{q^t} \sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a}) \sum_{\boldsymbol{j}\in\Gamma^*} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{q}\sum_{l=1}^{t} j_l a_l} \sum_{s=r(n)-\overline{l}+2}^{r(n)} \gamma_s. \end{aligned}$$

We can split the last sum in terms with factor n and factor $q^{r(n)}$. The terms with factor n go with the second sum and the terms with factor $q^{r(n)}$ go with the third sum.

Hence

$$
\begin{aligned}
\Sigma_2 \;=\; & \frac{n-q^{r(n)}}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{s=0}^{r(n)}\gamma_s \\
& +\frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma^*}\mathrm{e}^{-\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l a_l}\sum_{s=0}^{r(n)}\gamma_s \\
& \times \mathrm{e}^{\frac{2\pi\mathtt{i}}{q}\sum_{l=\bar{l}+1}^t j_l n_{s+l-1}} q^{\bar{l}+s-1}\left(\frac{\mathrm{e}^{\frac{2\pi\mathtt{i}}{q}j_{\bar{l}}n_{\bar{l}+s-1}}-1}{\mathrm{e}^{\frac{2\pi\mathtt{i}}{q}j_{\bar{l}}}-1}+\mathrm{e}^{\frac{2\pi\mathtt{i}}{q}j_{\bar{l}}n_{\bar{l}+s-1}}\left\{\frac{n}{q^{\bar{l}+s-1}}\right\}\right) \\
& -\frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma^*}\mathrm{e}^{-\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l a_l}\sum_{s=r(n)-\bar{l}+1}^{r(n)}\gamma_s q^{r(n)}.
\end{aligned}
$$

The result follows by adding Σ_1 and Σ_2. □

2.2 Delange type results

In this section we show for the generalized weighted digit-block-counting function a (weak) Delange type result in the case of convergent weights $\boldsymbol{\gamma}$. (For an exact definition of what we mean by (weak) Delange type see Section 2.3.) Thereby we once again prove the formula of Delange [2] for the first moment of the sum-of-digits function, the formula of Flajolet & Ramshaw [5] for the first moment of the Gray code sum and the formula of Kirschenhofer [6] for the first moment of the single-block-occurrence function. Furthermore, this generalizes a result of Larcher & Pillichshammer [10] for the weighted sum-of-digits function in base 2.

For a further investigation of the formula from Theorem 2 we introduce the notation

$$
f_q(a_1,\ldots,a_l,x_{t-l+1},\ldots,x_t) := \sum_{a_1,\ldots,a_l=0}^{q-1} g(a_1,\ldots,a_l,x_{t-l+1},\ldots,x_t)
$$

where $l\in\{1,\ldots,t\}$ and $x_{t-l+1},\ldots,x_t$ are arbitrary integers from $\{0,\ldots,q-1\}$. Especially, $f_q(a_1,\ldots,a_t)=\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})$ and $f_q(x_1,\ldots,x_t)=g(x_1,\ldots,x_t)$.

In the formula from Theorem 2 appeared the term

$$
\begin{aligned}
\Sigma \;:=\; & \frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{\boldsymbol{j}\in\Gamma^*}\mathrm{e}^{-\frac{2\pi\mathtt{i}}{q}\sum_{l=1}^t j_l a_l}\sum_{s=0}^{r(n)}\gamma_s \\
& \times \mathrm{e}^{\frac{2\pi\mathtt{i}}{q}\sum_{l=\bar{l}+1}^t j_l n_{s+l-1}} q^{\bar{l}+s-1}\left(\frac{\mathrm{e}^{\frac{2\pi\mathtt{i}}{q}j_{\bar{l}}n_{\bar{l}+s-1}}-1}{\mathrm{e}^{\frac{2\pi\mathtt{i}}{q}j_{\bar{l}}}-1}+\mathrm{e}^{\frac{2\pi\mathtt{i}}{q}j_{\bar{l}}n_{\bar{l}+s-1}}\left\{\frac{n}{q^{\bar{l}+s-1}}\right\}\right),
\end{aligned}
$$

which we analyze now. To this end we split the sum Σ into two parts Σ_3 and Σ_4. We have

$$
\begin{aligned}
\Sigma_3 \quad := \quad & \frac{1}{q^t} \sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a}) \sum_{\boldsymbol{j}\in\Gamma^*} \mathrm{e}^{-\frac{2\pi \mathtt{i}}{q}\sum_{l=1}^{t} j_l a_l} \sum_{s=0}^{r(n)} \gamma_s \\
& \times \mathrm{e}^{\frac{2\pi \mathtt{i}}{q}\sum_{l=\bar{l}+1}^{t} j_l n_{s+l-1}} q^{\bar{l}+s-1} \mathrm{e}^{\frac{2\pi \mathtt{i}}{q} j_{\bar{l}} n_{\bar{l}+s-1}} \left\{ \frac{n}{q^{\bar{l}+s-1}} \right\} \\
= \quad & \frac{1}{q^t} \sum_{s=0}^{r(n)} \gamma_s \sum_{\bar{l}=1}^{t} \sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a}) \sum_{j_{\bar{l}}=1}^{q-1} \sum_{j_{\bar{l}+1},\ldots,j_t=0}^{q-1} \mathrm{e}^{-\frac{2\pi \mathtt{i}}{q}\sum_{l=\bar{l}}^{t} j_l a_l} \\
& \times \mathrm{e}^{\frac{2\pi \mathtt{i}}{q}\sum_{l=\bar{l}+1}^{t} j_l n_{s+l-1}} q^{\bar{l}+s-1} \mathrm{e}^{\frac{2\pi \mathtt{i}}{q} j_{\bar{l}} n_{\bar{l}+s-1}} \left\{ \frac{n}{q^{\bar{l}+s-1}} \right\} \\
= \quad & \frac{1}{q^t} \sum_{s=0}^{r(n)} \gamma_s \sum_{\bar{l}=1}^{t} q^{\bar{l}+s-1} \left\{ \frac{n}{q^{\bar{l}+s-1}} \right\} \sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a}) \\
& \times \sum_{j_{\bar{l}}=1}^{q-1} \mathrm{e}^{\frac{2\pi \mathtt{i}}{q} j_{\bar{l}} (n_{\bar{l}+s-1} - a_{\bar{l}})} \sum_{j_{\bar{l}+1},\ldots,j_t=0}^{q-1} \mathrm{e}^{\frac{2\pi \mathtt{i}}{q}\sum_{l=\bar{l}+1}^{t} j_l (n_{s+l-1} - a_l)}.
\end{aligned}
$$

We have

$$
\sum_{j_{\bar{l}}=1}^{q-1} \mathrm{e}^{\frac{2\pi \mathtt{i}}{q} j_{\bar{l}} (n_{\bar{l}+s-1} - a_{\bar{l}})} = \begin{cases} q-1 & \text{if } n_{s+\bar{l}-1} = a_{\bar{l}}, \\ -1 & \text{otherwise,} \end{cases}
$$

and

$$
\sum_{j_{\bar{l}+1},\ldots,j_t=0}^{q-1} \mathrm{e}^{\frac{2\pi \mathtt{i}}{q}\sum_{l=\bar{l}+1}^{t} j_l (n_{s+l-1} - a_l)} = \begin{cases} q^{t-\bar{l}} & \text{if } n_{s+l-1} = a_l \qquad \text{for all } l \in \{\bar{l}+1, \ldots, t\}, \\ 0 & \text{otherwise.} \end{cases}
$$

So we get

$$
\begin{aligned}
\Sigma_3 \quad = \quad & \frac{n}{q^t} \sum_{s=0}^{r(n)} \gamma_s \frac{q^{t+s}}{n} \sum_{l=1}^{t} \left\{ \frac{n}{q^{l+s-1}} \right\} \\
& \times \left(f_q(a_1, \ldots, a_{l-1}, n_{s+l-1}, \ldots, n_{s+t-1}) - \frac{1}{q} f_q(a_1, \ldots, a_l, n_{s+l}, \ldots, n_{s+t-1}) \right).
\end{aligned}
$$

Further we have

$$\begin{aligned}
\Sigma_4 &:= \frac{1}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a}) \sum_{\boldsymbol{j}\in\Gamma^*} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{q}\sum_{l=1}^{t} j_l a_l} \sum_{s=0}^{r(n)} \gamma_s \\
&\quad \times \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}\sum_{l=\bar{l}+1}^{t} j_l n_{s+l-1}} q^{\bar{l}+s-1} \left(\frac{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} j_{\bar{l}} n_{\bar{l}+s-1}} - 1}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} j_{\bar{l}}} - 1} \right) \\
&= \frac{1}{q^t}\sum_{s=0}^{r(n)} \gamma_s \sum_{\bar{l}=1}^{t} q^{\bar{l}+s-1} \sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a}) \sum_{j_{\bar{l}}=1}^{q-1} \sum_{j_{\bar{l}+1},\dots,j_t=0}^{q-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{q}\sum_{l=\bar{l}}^{t} j_l a_l} \\
&\quad \times \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}\sum_{l=\bar{l}+1}^{t} j_l n_{s+l-1}} \frac{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} j_{\bar{l}} n_{\bar{l}+s-1}} - 1}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{q} j_{\bar{l}}} - 1} \\
&= \frac{1}{q^t}\sum_{s=0}^{r(n)} \gamma_s \sum_{\bar{l}=1}^{t} q^{\bar{l}+s-1} \sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a}) \\
&\quad \times \sum_{j_{\bar{l}}=1}^{q-1} \sum_{k=0}^{n_{s+\bar{l}-1}-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q} j_{\bar{l}}(k-a_{\bar{l}})} \sum_{j_{\bar{l}+1},\dots,j_t=0}^{q-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{q}\sum_{l=\bar{l}+1}^{t} j_l (n_{s+l-1}-a_l)}.
\end{aligned}$$

Together with the above results we find that

$$\begin{aligned}
\Sigma_4 &= \frac{n}{q^t}\sum_{s=0}^{r(n)} \gamma_s \frac{q^{t+s}}{n} \sum_{l=1}^{t} \\
&\left(-\frac{n_{s+l-1}}{q} f_q(a_1,\dots,a_l,n_{s+l},\dots,n_{s+t-1}) + \sum_{k=0}^{n_{s+l-1}-1} f_q(a_1\dots,a_{l-1},k,n_{s+l},\dots,n_{s+t-1}) \right).
\end{aligned}$$

Adding up Σ_3 and Σ_4, we have

$$\Sigma = \Sigma_4 + \Sigma_3 = \frac{n}{q^t}\sum_{s=0}^{r(n)} \gamma_s \frac{q^{s+t}}{n} \Psi\left(\frac{n}{q^{s+t}}\right),$$

where

$$\Psi(x) := \sum_{l=1}^{t} \Psi_l(x) \tag{3}$$

with

$$\begin{aligned}
\Psi_l(x) &:= -\frac{1}{q} f_q(a_1,\dots,a_l,r(x,l+1),\dots,r(x,t))(r(x,l)+\{xq^{t-l+1}\}) \\
&\quad + f_q(a_1,\dots,a_{l-1},r(x,l),\dots,r(x,t))\{xq^{t-l+1}\} \\
&\quad + \sum_{k=0}^{r(x,l)-1} f_q(a_1,\dots,a_{l-1},k,r(x,l+1),\dots,r(x,t))
\end{aligned}$$

and

$$r(x,l) := \lfloor xq^{t-l+1} \rfloor - q\lfloor xq^{t-l} \rfloor.$$

Altogether we can deduce the following corollary, which is a more concise version of the formula in Theorem 2.

Corollary 1 *For any integer $n \geq 2$ we have*

$$S_q(n,\boldsymbol{\gamma}) \;=\; \frac{n}{q^t}\sum_{\boldsymbol{a}\in\Gamma} g(\boldsymbol{a})\sum_{s=0}^{r(n)}\gamma_s - g(\mathbf{0})\sum_{s=0}^{r(n)}\gamma_s q^s + \frac{n}{q^t}\sum_{s=0}^{r(n)}\gamma_s\frac{q^{s+t}}{n}\Psi\left(\frac{n}{q^{s+t}}\right),$$

where $r(n) := \lfloor \log_q(n)\rfloor$ and $\bar{l} := \min\{l \in \{1,...,t\} \,:\, j_l \neq 0\}$.

Example 3 For $q=2$, $t=1$ and $g:\Gamma\to\mathbb{R}$ the identity function we have $\Psi(x) = -\|x\|$ and we obtain the result from Example 1. For $q=2$, $t=2$ and $g:\Gamma\to\mathbb{R}$ the function given by $g(0,0)=g(1,1)=0$ and $g(0,1)=g(1,0)=1$ we have $\Psi(x) = (-1)^{x_1+1}\|2x\|$, where $x_1 = \lfloor 2x\rfloor - 2\lfloor x\rfloor$. This yields the result from Example 2.

Before we move on, we collect some useful properties of the function $\Psi(x)$.

Lemma 3 *Let the function Ψ be defined as in (3). Then we have*

1. *Ψ is periodic with period 1.*

2. *Ψ is continuous on $[0,\infty)$.*

3. *$\Psi(z) = 0$ for any integer $z \in \mathbb{N}_0$.*

4. *For all $m \in \{1,\ldots,t\}$ and $z \in \{1,\ldots,q-1\}$ we have*

$$\Psi\left(\frac{z}{q^m}\right) = \sum_{k=0}^{z-1} f_q(a_1,\ldots,a_{t-m},k,0,\ldots,0) - \frac{z}{q^m}f_q(a_1,\ldots,a_t).$$

5. *For any $m \in \mathbb{N}$ and $z \in \{1,\ldots,q-1\}$ we get*

$$\Psi\left(\frac{z}{q^{t+m}}\right) = \frac{z}{q^m}g(\mathbf{0}) - \frac{z}{q^{t+m}}f_q(a_1,\ldots,a_t).$$

6. *For $j \in \mathbb{N}$ and $m \in \{1,\ldots,t\}$ we have*

$$\begin{aligned}\Psi\left(\frac{q^j+1}{q^{j+m}}\right) \;&=\; -\frac{q^j+1}{q^{j+m}}f_q(a_1,\ldots,a_t) + f_q(a_1,\ldots,a_{t-m},0,\ldots,0)\\ &\quad + f_q(a_1,\ldots,a_{t-(m+j)},\underbrace{0,\ldots,0}_{j-times},1,0,\ldots,0)\min\left\{1,\frac{1}{q^{j+m-t}}\right\}.\end{aligned}$$

7. *For $m,j \in \mathbb{N}$ and $m > t$ we have*

$$\Psi\left(\frac{q^j+1}{q^{j+m}}\right) \;=\; -\frac{q^j+1}{q^{j+m}}f_q(a_1,\ldots,a_t) + g(\mathbf{0})\frac{q^j+1}{q^{m+j-t}}.$$

Proof. 1. This is obviously true since the function $\{xq^{t-l+1}\}$ has period $1/q^{t-l+1}$ and $r(x,l)$ has period $1/q^{t-l}$.

2. First we note that $\Psi_l(0^+) = 0$ because $\lim_{x\to 0^+}\{xq^{t-l+1}\} = 0$ and $\lim_{x\to 0^+} r(x,l) = 0$. Since $\lim_{x\to 1^-}\{xq^{t-l+1}\} = 1$ and $\lim_{x\to 1^-} r(x,l) = q-1$ we also have

$$\begin{aligned}\Psi_l(1^-) &= \sum_{k=0}^{q-2} f_q(a_1,\dots,a_{l-1},k,q-1,\dots,q-1) + f_q(a_1,\dots,a_{l-1},q-1,\dots,q-1) \\ &\quad - f_q(a_1,\dots,a_l,q-1,\dots,q-1) = 0.\end{aligned}$$

Due to the periodicity of Ψ it is therefore enough to prove the continuity of Ψ on the interval $(0,1)$

The function $\Psi_l(x)$ is continuous on the interval $\left(\frac{n}{q^t}, \frac{n+1}{q^t}\right)$ with an arbitrary $n \in \{0,\dots,q^t-1\}$ because $\{xq^{t-l+1}\}$ is continuous and $r(x,l)$ is constant on this interval for all $l \in \{1,\dots,t\}$.

We show that $\Psi_l(x)$ is also continuous in $\frac{n}{q^t}$ with an arbitrary $n \in \{0,\dots,q^t-1\}$. Let $\frac{\overline{n}}{q^m}$ be the reduced fraction with $\frac{\overline{n}}{q^m} = \frac{n}{q^t}$. We have the q-adic expansion $\overline{n} = \overline{n}_{m-1}q^{m-1} + \cdots + \overline{n}_0$ and we know $\overline{n}_0 \neq 0$. Now we have

$$\begin{aligned}
\lim_{x\to \frac{\overline{n}}{q^m}^+} r(x,l) &= \begin{cases} \overline{n}_{m-t-1+l} & \text{if } l > t-m+1, \\ \overline{n}_0 & \text{if } l = t-m+1, \\ 0 & \text{if } l < t-m+1, \end{cases} \\
\lim_{x\to \frac{\overline{n}}{q^m}^-} r(x,l) &= \begin{cases} \overline{n}_{m-t-1+l} & \text{if } l > t-m+1, \\ \overline{n}_0 - 1 & \text{if } l = t-m+1, \\ q-1 & \text{if } l < t-m+1, \end{cases} \\
\lim_{x\to \frac{\overline{n}}{q^m}^+} \{xq^{t-l+1}\} &= \begin{cases} \frac{1}{q^{l-(t-m+1)}} \sum_{s=0}^{m+l-t-2} \overline{n}_s q^s & \text{if } l > t-m+1, \\ 0 & \text{if } l \le t-m+1, \end{cases} \\
\lim_{x\to \frac{\overline{n}}{q^m}^-} \{xq^{t-l+1}\} &= \begin{cases} \frac{1}{q^{l-(t-m+1)}} \sum_{s=0}^{m+l-t-2} \overline{n}_s q^s & \text{if } l > t-m+1, \\ 1 & \text{if } l \le t-m+1. \end{cases}
\end{aligned}$$

With these results we can see, after some tedious but straight forward considerations, that $\Psi_l(x)$ is continuous in $\frac{n}{q^t}$ for all $n \in \{0,\dots,q^t-1\}$ and any $l \in \{1,\dots,t\}$.

3. See the proof of item 2 and item 1.

4. Let $m \in \{1,\dots,t\}$ and let $z \in \{1,\dots,q-1\}$ be arbitrary. We compute $\Psi_l(z/q^m)$ with the results from item 2. We have

$$r(z/q^m, l) = \begin{cases} z & \text{if } l = t-m+1, \\ 0 & \text{if } l \neq t-m+1, \end{cases}$$

and

$$\left\{\frac{z}{q^m} q^{t-l+1}\right\} = \begin{cases} \frac{z}{q^{l-(t-m+1)}} & \text{if } l > t-m+1, \\ 0 & \text{if } l \le t-m+1. \end{cases}$$

Therefore we get for $l < t-m+1$, $\Psi_l(z/q^m) = 0$. For $l = t-m+1$ we get

$$\Psi_l\left(\frac{z}{q^m}\right) = -\frac{z}{q} f_q(a_1,\dots,a_{t-m+1},0,\dots,0) + \sum_{k=0}^{z-1} f_q(a_1,\dots,a_{t-m},k,0,\dots,0)$$

and for $l > t-m+1$ we get

$$\Psi_l\left(\frac{z}{q^m}\right) = -\frac{1}{q}f_q(a_1,\ldots,a_l,0,\ldots,0)\frac{z}{q^{l-(t-m+1)}}+f_q(a_1,\ldots,a_{l-1},0,\ldots,0)\frac{z}{q^{l-(t-m+1)}}.$$

We sum up $\Psi_l(z/q^m)$ for all $l\in\{1,\ldots,t\}$ and the result follows.

5. The proof is similar to the one of item 4.

6. We use the results from the proof of item 2, where $\overline{n}=q^j+1$. We have

$$r\left(\frac{q^j+1}{q^{j+m}},l\right) = \begin{cases} 1 & \text{if } l=t-(m+j)+1 \text{ or } l=t-m+1,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\left\{\frac{q^j+1}{q^{m+j}}q^{t-l+1}\right\} = \begin{cases} 0 & \text{if } l\le t-(m+j)+1,\\ \frac{1}{q^{m+j-(t-l+1)}} & \text{if } t-(m+j)+1<l\le t-m+1,\\ \frac{q^j+1}{q^{m+j-(t-l+1)}} & \text{if } l>t-m+1.\end{cases}$$

So we can compute $\Psi_l\left(\frac{q^j+1}{q^{m+j}}\right)$ for the different values of l.

(a) If $l<t-(m+j)+1$, then $\Psi_l\left(\frac{q^j+1}{q^{m+j}}\right)=0$.

(b) If $l=t-(m+j)+1$, then

$$\begin{aligned}\Psi_l\left(\frac{q^j+1}{q^{m+j}}\right) &= -\frac{1}{q}f_q(a_1,\ldots,a_{t-(m+j)+1},\underbrace{0,\ldots,0}_{j-1-\text{times}},1,0\ldots,0)\\ &\quad +f_q(a_1,\ldots,a_{t-(m+j)},\underbrace{0,\ldots,0}_{j-\text{times}},1,0,\ldots,0).\end{aligned}$$

(c) If $t-(m+j)+1<l<t-m$, then

$$\begin{aligned}\Psi_l\left(\frac{q^j+1}{q^{m+j}}\right) &= -\frac{1}{q^{m+j-(t-l)}}f_q(a_1,\ldots,a_l,\underbrace{0,\ldots,0}_{t-m-l-\text{times}},1,0,\ldots,0)\\ &\quad +\frac{1}{q^{m+j-(t-l+1)}}f_q(a_1,\ldots,a_{l-1},\underbrace{0,\ldots,0}_{t-m-l+1-\text{times}},1,0,\ldots,0).\end{aligned}$$

(d) If $l=t-m+1$, then

$$\begin{aligned}\Psi_l\left(\frac{q^j+1}{q^{m+j}}\right) &= -\frac{1}{q^{j+1}}f_q(a_1,\ldots,a_{t-m+1},0,\ldots,0)-\frac{1}{q}f_q(a_1,\ldots,a_{t-m+1},0,\ldots,0)\\ &\quad +\frac{1}{q^j}f_q(a_1,\ldots,a_{t-m},1,0,\ldots,0)+f_q(a_1,\ldots,a_{t-m},0,\ldots,0).\end{aligned}$$

(e) If $l>t-m+1$, then

$$\begin{aligned}\Psi_l\left(\frac{q^j+1}{q^{m+j}}\right) &= -\frac{q^j+1}{q^{m+j-(t-l)}}f_q(a_1,\ldots,a_l,0,\ldots,0)\\ &\quad +\frac{q^j+1}{q^{m+j-(t-l+1)}}f_q(a_1,\ldots,a_{l-1},0,\ldots,0).\end{aligned}$$

For the computation of $\Psi(x)=\sum_{l=1}^{t}\Psi_l(x)$ one has to differ the cases $m=1$, $j=1$, $j\in\{2,\ldots,t-1\}$ or $j\geq t$ and $m>1$, $j+m<t+1$ or $j+m\geq t+1$. We omit the details.

7. See the proof of item 6. In this case $t-m+1\leq 0$ and we get for all $l\in\{1,\ldots,t\}$

$$\begin{aligned}\Psi_l\left(\frac{q^j+1}{q^{m+j}}\right) &= -\frac{q^j+1}{q^{m+j-(t-l)}}f_q(a_1,\ldots,a_l,0,\ldots,0)\\ &\quad+\frac{q^j+1}{q^{m+j-(t-l+1)}}f_q(a_1,\ldots,a_{l-1},0,\ldots,0)\end{aligned}$$

and the result follows.

□

Now we can give the (weak) Delange type result for the first moment of the generalized weighted digit-block-counting function under the assumption of convergent weights $\boldsymbol{\gamma}$.

Theorem 3 *If the sequence $\boldsymbol{\gamma}=(\gamma_s)_{s\geq 0}$ of weights converges, say $\lim_{s\to\infty}\gamma_s=\widetilde{\gamma}$, then for the first moment of the generalized weighted digit-block-counting function we have*

$$\begin{aligned}S_q(n,\boldsymbol{\gamma}) &= \frac{n}{q^t}f_q(a_1,\ldots,a_t)\left(\{\log_q(n)\}\gamma_{r(n)}+\sum_{s=0}^{r(n)-1}\gamma_s\right)\\ &\quad-g(\mathbf{0})\left(\{\log_q(n)\}\gamma_{r(n)}q^{r(n)}+\sum_{s=0}^{r(n)-1}\gamma_s q^s\right)+nF(\log_q(n))+nE(n)+o(n),\end{aligned}$$

where

$$F(x):=\left(\frac{f_q(a_1,\ldots,a_t)}{q^t}-g(\mathbf{0})q^{-\{x\}}\right)(1-\{x\})\widetilde{\gamma}+\frac{1}{q^t}\sum_{s=0}^{\infty}\widetilde{\gamma}\frac{\Psi(q^{\{x\}+s-t})}{q^{\{x\}+s-t}}$$

is a continuous and periodic function with period 1 and $E(n)$ is defined as

$$E(n):=-\frac{1}{q^t}\sum_{s=1}^{t-1}\widetilde{\gamma}\frac{\Psi(nq^{s-t})}{nq^{s-t}}.$$

We have $E(n)=0$ for all $n\equiv 0 \pmod{q^{t-1}}$ and $E(n)=o(1)$.

Furthermore, we have $o(n)=0$ in the above formula if $\gamma_s=\widetilde{\gamma}$ for all $s\in\mathbb{N}_0$.

Proof. The proof is mainly based on Corollary 1 and Lemma 3. We define

$$H(x):=\left(\frac{f_q(a_1,\ldots,a_t)}{q^t}-g(\mathbf{0})q^{-\{x\}}\right)(1-\{x\})\gamma_{\lfloor x\rfloor}+\frac{1}{q^t}\sum_{s=0}^{\lfloor x\rfloor}\gamma_s\frac{\Psi(q^{x-s-t})}{q^{x-s-t}}.$$

With this definition we can re-write the formula from Corollary 1 as

$$\begin{aligned}S_q(n,\boldsymbol{\gamma}) &= \frac{n}{q^t}f_q(a_1,\ldots,a_t)\left(\{\log_q(n)\}\gamma_{r(n)}+\sum_{s=0}^{r(n)-1}\gamma_s\right)\\ &\quad-g(\mathbf{0})\left(\{\log_q(n)\}\gamma_{r(n)}q^{r(n)}+\sum_{s=0}^{r(n)-1}\gamma_s q^s\right)+nH\left(\log_q(n)\right).\end{aligned}$$

With $\widetilde{\gamma} = \lim_{s\to\infty} \gamma_s$ we have

$$\begin{aligned} H(x) &= \left(\frac{f_q(a_1,\dots,a_t)}{q^t} - g(\mathbf{0})q^{-\{x\}}\right)(1-\{x\})(\gamma_{\lfloor x\rfloor} - \widetilde{\gamma}) + \frac{1}{q^t}\sum_{s=0}^{\lfloor x\rfloor}(\gamma_s - \widetilde{\gamma})\frac{\Psi(q^{x-s-t})}{q^{x-s-t}} \\ &\quad + \left(\frac{f_q(a_1,\dots,a_t)}{q^t} - g(\mathbf{0})q^{-\{x\}}\right)(1-\{x\})\widetilde{\gamma} + \frac{1}{q^t}\sum_{s=0}^{\lfloor x\rfloor}\widetilde{\gamma}\frac{\Psi(q^{x-s-t})}{q^{x-s-t}}. \end{aligned}$$

It is easy to see, that

$$\left(\frac{f_q(a_1,\dots,a_t)}{q^t} - g(\mathbf{0})q^{-\{x\}}\right)(1-\{x\})(\gamma_{\lfloor x\rfloor} - \widetilde{\gamma}) + \frac{1}{q^t}\sum_{s=0}^{\lfloor x\rfloor}(\gamma_s - \widetilde{\gamma})\frac{\Psi(q^{x-s-t})}{q^{x-s-t}} = o(1).$$

We define

$$\overline{F}(x) := \left(\frac{f_q(a_1,\dots,a_t)}{q^t} - g(\mathbf{0})q^{-\{x\}}\right)(1-\{x\})\widetilde{\gamma} + \frac{1}{q^t}\sum_{s=0}^{\lfloor x\rfloor}\widetilde{\gamma}\frac{\Psi(q^{x-s-t})}{q^{x-s-t}}.$$

Hence $H(x) = \overline{F}(x) + o(1)$ and $o(1) = 0$ if the sequence of weights is constant.

We show that $\overline{F}(x)$ is continuous on $(0,\infty)$. From Lemma 3 we know that $\Psi(x)$ is continuous on $[0,\infty)$ and therefore $\overline{F}(x)$ is continuous on $[0,\infty)\setminus\mathbb{Z}$. But we also know from Lemma 3 that $\Psi\left(1/q^t\right) = -\frac{f_q(a_1,\dots,a_t)}{q^t} + g(\mathbf{0})$, and therefore one can see easily that $\overline{F}(x)$ is also continuous in any positive integer.

We define

$$F(x) := \overline{F}(x) + \frac{1}{q^t}\sum_{s=1}^{\infty}\widetilde{\gamma}\frac{\Psi(q^{x+s-t})}{q^{x+s-t}}$$

and

$$E(n) := -\frac{1}{q^t}\sum_{s=1}^{t-1}\tilde{\gamma}\frac{\Psi(nq^{s-t})}{nq^{s-t}}.$$

Then it is easy to see, that

$$H(\log_q(n)) = F(\log_q(n)) + E(n) + o(1),$$

where $o(1) = 0$ if the sequence of weights is constant.

For any nonnegative integer $k \in \mathbb{N}_0$ we have

$$\begin{aligned} \overline{F}(x+k) - \overline{F}(x) &= \frac{1}{q^t}\sum_{s=0}^{\lfloor x\rfloor+k}\widetilde{\gamma}\frac{\Psi(q^{x+k-s-t})}{q^{x+k-s-t}} - \frac{1}{q^t}\sum_{s=0}^{\lfloor x\rfloor}\widetilde{\gamma}\frac{\Psi(q^{x-s-t})}{q^{x-s-t}} \\ &= \frac{1}{q^t}\sum_{s=-k}^{-1}\widetilde{\gamma}\frac{\Psi(q^{x-s-t})}{q^{x-s-t}} \\ &= \frac{1}{q^t}\sum_{s=1}^{k}\widetilde{\gamma}\frac{\Psi(q^{x+s-t})}{q^{x+s-t}}, \end{aligned}$$

and hence it follows that

$$\overline{F}(x) = \overline{F}(\{x\}) + \frac{1}{q^t}\sum_{s=1}^{\lfloor x\rfloor}\widetilde{\gamma}\frac{\Psi(q^{\{x\}+s-t})}{q^{\{x\}+s-t}}.$$

Further we have

$$\sum_{s=1}^{\infty} \widetilde{\gamma} \frac{\Psi(q^{x+s-t})}{q^{x+s-t}} = \sum_{s=\lfloor x \rfloor+1}^{\infty} \widetilde{\gamma} \frac{\Psi(q^{\{x\}+s-t})}{q^{\{x\}+s-t}}$$

and

$$\overline{F}(\{x\}) = \left(\frac{f_q(a_1, \ldots, a_t)}{q^t} - g(\mathbf{0}) q^{-\{x\}} \right) (1 - \{x\}) \widetilde{\gamma} + \frac{1}{q^t} \widetilde{\gamma} \frac{\Psi(q^{\{x\}-t})}{q^{\{x\}-t}}.$$

So we can express $F(x)$ the following way

$$F(x) = \left(\frac{f_q(a_1, \ldots, a_t)}{q^t} - g(\mathbf{0}) q^{-\{x\}} \right) (1 - \{x\}) \widetilde{\gamma} + \frac{1}{q^t} \sum_{s=0}^{\infty} \widetilde{\gamma} \frac{\Psi(q^{\{x\}+s-t})}{q^{\{x\}+s-t}}.$$

Because $\Psi(z) = 0$ for any integer z we have $E(n) = 0$ if $n \equiv 0 \pmod{q^{t-1}}$ and because $\Psi(x)$ is bounded it is clear that $E(n) = o(1)$.

The function $F(x)$ is periodic with period 1 since $\{x\}$ has period 1. Furthermore $F(x)$ is continuous on $(0, 1)$ because $\{x\}$ is continuous on $(0, 1)$, $\Psi(x)$ is continuous and

$$\sum_{s=0}^{\infty} \widetilde{\gamma} \frac{\Psi(q^{\{x\}+s-t})}{q^{\{x\}+s-t}}$$

is absolute convergent. But $F(x)$ is also continuous in any arbitrary integer z because

$$F(1^-) = \frac{1}{q^t} \sum_{s=0}^{\infty} \widetilde{\gamma} \frac{\Psi(q^{1+s-t})}{q^{1+s-t}},$$

$$F(0^+) = \left(\frac{f_q(a_1, \ldots, a_t)}{q^t} - g(\mathbf{0}) \right) \widetilde{\gamma} + \frac{1}{q^t} \sum_{s=0}^{\infty} \widetilde{\gamma} \frac{\Psi(q^{s-t})}{q^{s-t}}$$

and

$$\Psi(q^{-t}) = -\left(\frac{f_q(a_1, \ldots, a_t)}{q^t} - g(\mathbf{0}) \right),$$

by Lemma 3. This concludes the proof. □

The following corollary gives the generalization of the results of Delange [2], of Kirschenhofer [6] and of Flajolet & Ramshaw [5]. Furthermore, this reproves a result of Cateland [1] for the unweighted version of the generalized weighted digit-block-counting function.

Corollary 2 *In the case of constant weights, say $\gamma_s = \widetilde{\gamma}$ for all $s \in \mathbb{N}_0$, and $g(\mathbf{0}) = 0$ we get for the first moment of the generalized weighted digit-block-counting function*

$$S_q(n, (\widetilde{\gamma})_{s \geq 0}) = \widetilde{\gamma} \left(\frac{1}{q^t} \sum_{\boldsymbol{a} \in \Gamma} g(\boldsymbol{a}) \right) n \log_q(n) + nF(\log_q(n)) + nE(n),$$

where $F(x)$ is a continuous and periodic function on $[0, \infty)$ with period 1 and where $E(n) = o(1)$. In the case $t = 1$ we have $E(n) = 0$ for all $n \geq 2$. Furthermore $F(x)$ is nowhere differentiable if and only if there exists at least one $n \geq 1$ so that $s_q(nq^{t-1}, (\widetilde{\gamma})_{s \geq 0}) \neq 0$.

Proof. The corollary is a consequence of Theorem 3. The statement on the nowhere differentiability can be shown in the same way as [13, Théorème 3]. □

2.3 Single Converse Results

As already mentioned, for $q=2$, $t=1$ and $g:\Gamma\to\mathbb{R}$ the identity function, the generalized weighted digit-block-counting function equals the weighted sum-of-digits function for binary representation (see Example 1). In this case it was proved by Larcher & Pillichshammer [10] that a (weak) Delange type result holds if and only if the sequence of weights converges. In other words in this special case also the converse of the assertion in Theorem 3 holds. Motivated from their result now the question arises, whether this is true in the more general case. More detailed, assume the formula for the first moment of the generalized weighted digit-block-counting function is *weak Delange type*, i.e., of the form

$$\begin{aligned} S_q(n,\boldsymbol{\gamma}) &= \frac{n}{q^t} f_q(a_1,\ldots,a_t)\left(\{\log_q(n)\}\gamma_{r(n)} + \sum_{s=0}^{r(n)-1}\gamma_s\right) \\ &\quad -g(\mathbf{0})\left(\{\log_q(n)\}\gamma_{r(n)}q^{r(n)} + \sum_{s=0}^{r(n)-1}\gamma_s q^s\right) + nF(\log_q(n)) + nE(n) + o(n), \end{aligned} \tag{4}$$

where $F(x)$ is a continuous and periodic function and $E(n)=0$ for all $n\equiv 0 \pmod{q^{t-1}}$ and $E(n)=o(1)$. (If $o(n)=0$ we say the formula is *Delange type* and not only weak Delange type.)

Is it then true that the sequence of weights has to converge? As it turned out, this question is by no means trivial. Not even in the case $t=1$ we can answer this question for arbitrary functions $g:\Gamma\to\mathbb{R}$ at the moment. However we can answer this question for many special choices of functions $g:\Gamma\to\mathbb{R}$.

So let us assume now that $S_q(n,\boldsymbol{\gamma})$ is of the form (4). From Corollary 1 (see also the first lines of the proof of Theorem 3) we get $F(\log_q(n))+E(n)+o(1)=H(\log_q(n))$, where

$$\begin{aligned} &H(\log_q(n)) \\ &= \left(\frac{f_q(a_1,\ldots,a_t)}{q^t} - g(\mathbf{0})q^{-\{\log_q(n)\}}\right)(1-\{\log_q(n)\})\gamma_{\lfloor\log_q(n)\rfloor} + \frac{1}{q^t}\sum_{s=0}^{\lfloor\log_q(n)\rfloor}\gamma_s\frac{\Psi(nq^{-s-t})}{nq^{-s-t}}. \end{aligned}$$

For any fixed $n\geq 2$ we get an equation, where the weights γ_s are the unknowns. Our method is the following. We use different values for $n\geq 2$ to get a system of equations, which has at least one uniquely determined solution $\gamma_s = const + o(1)$ for arbitrary $s\in\mathbb{N}_0$. From this we obtain that the sequence of weights converges.

To evaluate the equation for different integers, we need the following lemma, whose easy proof will be omitted.

Lemma 4 *For any positive integer $l\in\mathbb{N}$ we have $q^{-\{\log_q(q^l+1)\}} = q^l/(q^l+1)$.*

We will use the following different values of n:

- Let $n=q^r$ with $r\geq t-1$ an arbitrary positive integer. Then we get the equation

$$F(0)+o(1) = \sum_{i=1}^{t-1}\gamma_{r-i}\frac{1}{q^i}\Psi\left(\frac{1}{q^{t-i}}\right).$$

(In particular this shows, that $F(z)=0$ for any positive integer z if $t=1$.)

- For $n = q^r + q^{r-l}$ with $r > t + l$ we get the equation

$$\begin{aligned} & F(\log_q(q^l+1)) + o(1) \\ = \ & \gamma_r \left(\frac{f_q(a_1,\ldots,a_t)}{q^t} - g(\mathbf{0}) \frac{q^l}{q^l+1} \right) (1 - \{\log_q(q^l+1)\}) + \frac{q^l}{q^l+1} \Psi\left(\frac{q^l+1}{q^{t+l}}\right) \gamma_r \\ & + \sum_{i=1}^{t-1} \gamma_{r-i} \frac{q^{l-i}}{q^l+1} \Psi\left(\frac{q^l+1}{q^{t+l-i}}\right) + \sum_{i=0}^{l-1} \gamma_{r-t-i} \frac{q^{l+i-t}}{q^l+1} \Psi\left(\frac{1}{q^{l-i}}\right). \end{aligned}$$

In the following we will use these equations to obtain information on the weights $\boldsymbol{\gamma}$, whenever the generalized weighted digit-block-counting function is (weak) Delange type (4).

2.3.1 The weighted sum-of-digits function

In this case we have $t = 1$ and $g : \Gamma \to \mathbb{R}$ is given by $g(a) = a$ for all $a \in \Gamma$. We use the values $n \in \{q^r + q^{r-2}, q^r + q^{r-1}, q^{r-1} + q^{r-2}\}$ with $r > 3$ an arbitrary integer.

First we evaluate $\Psi(x)$ at the relevant points with the help of Lemma 3. We have

$$\Psi\left(\frac{1}{q}\right) = -\frac{q-1}{2}, \quad \Psi\left(\frac{1}{q^2}\right) = -\frac{q-1}{2q}, \quad \Psi\left(\frac{q+1}{q^2}\right) = -\frac{q}{2} + \frac{3}{2q},$$

and

$$\Psi\left(\frac{q^2+1}{q^3}\right) = -\frac{q}{2} + \frac{1}{2} - \frac{1}{2q} + \frac{3}{2q^2}.$$

Now $2(q^2+1)$ times the equation for $n = q^r + q^{r-2}$ and $2(q+1)$ times the equations for $n = q^r + q^{r-1}$ and $n = q^{r-1} + q^{r-2}$ give the following system of equations with real constants C_1, C_2, C_3:

$$\begin{aligned} & \begin{pmatrix} C_1 + o(1) \\ C_2 + o(1) \\ C_3 + o(1) \end{pmatrix} \\ = & \begin{pmatrix} (q^2+1)(1-q)\log_q(1+\frac{1}{q^2}) + 2 & 1-q & (1-q)q^2 \\ (1-q^2)\log_q(1+\frac{1}{q}) + 2 & 1-q & 0 \\ 0 & (1-q^2)\log_q(1+\frac{1}{q}) + 2 & 1-q \end{pmatrix} \begin{pmatrix} \gamma_r \\ \gamma_{r-1} \\ \gamma_{r-2} \end{pmatrix}. \end{aligned}$$

Since $q \geq 2$ we find that the above 3×3 matrix is regular. Hence we get a solution for γ_r, namely $\gamma_r = C + o(1)$ with a real constant C. Since $r > 3$ was an arbitrary integer we find that the sequence of weights converges. Together with the result from the last section we obtain:

Theorem 4 *The first moment of the weighted sum-of-digits function in arbitrary base q is (weak) Delange type if and only if the sequence of weights is constant (converges).*

2.3.2 The weighted Gray Code sum

For $t = q = 2$, and $g : \Gamma \to \mathbb{R}$ given by $g(0,0) = g(1,1) = 0$ and $g(1,0) = g(0,1) = 1$ our generalized weighted digit-block-counting function equals the Gray Code sum. We use the same values for n as above, $n \in \{2^r + 2^{r-2}, 2^r + 2^{r-1}, 2^{r-1} + 2^{r-2}\}$ with $r > 3$. Since $\Psi(1/2) = 0$ we get a linear system of three equations with the matrix

$$\begin{pmatrix} \frac{-5\log_2(5)+12}{10} & \frac{1}{10} & -\frac{1}{5} \\ \frac{-3\log_2(3)+5}{6} & \frac{1}{6} & 0 \\ 0 & \frac{-3\log_2(3)+5}{6} & \frac{1}{6} \end{pmatrix}.$$

This matrix has rank 3. Hence we get a solution for γ_r, namely $\gamma_r = C + o(1)$ with a real constant C. Since $r > 3$ was an arbitrary integer we find that the sequence of weights converges. Together with the result from the last section we obtain:

Theorem 5 *The first moment of the weighted Gray code sum is (weak) Delange type if and only if the sequence of weights is constant (converges).*

2.3.3 The weighted single-block-occurrence of length 1

We have $t = 1$ and $q \geq 2$ is an arbitrary integer. If $g : \Gamma \to \mathbb{R}$ is given by $g(a) = 1$ for a given integer $a \in \Gamma$ and zero otherwise, then we count single integer occurrences.

We use the following values $n \in \{q^r + q^{r-2}, q^r + q^{r-1}, q^{r-1} + q^{r-2}\}$ with $r > 3$ an arbitrary integer and evaluate $\Psi(x)$ at the relevant points with the help of Lemma 3. We have

$$\begin{aligned}
\Psi\left(\frac{1}{q}\right) &= \begin{cases} 1 - \frac{1}{q} & \text{if } a = 0, \\ -\frac{1}{q} & \text{if } a > 0, \end{cases} \\
\Psi\left(\frac{1}{q^2}\right) &= \begin{cases} 1 - \frac{1}{q^2} & \text{if } a = 0, \\ -\frac{1}{q^2} & \text{if } a > 0, \end{cases} \\
\Psi\left(\frac{q+1}{q^2}\right) &= \begin{cases} 1 - \frac{q+1}{q^2} & \text{if } a = 0, \\ \frac{1}{q} - \frac{q+1}{q^2} & \text{if } a = 1, \\ -\frac{q+1}{q^2} & \text{if } a > 1, \end{cases} \\
\Psi\left(\frac{q^2+1}{q^3}\right) &= \begin{cases} 1 - \frac{q^2+1}{q^3} & \text{if } a = 0, \\ \frac{1}{q^2} - \frac{q^2+1}{q^3} & \text{if } a = 1, \\ -\frac{q^2+1}{q^3} & \text{if } a > 1. \end{cases}
\end{aligned}$$

Analogously to the previous cases we get for each integer $a \in \Gamma$ a system of three linear equations. For example for $a = 0$ we have the matrix

$$\begin{pmatrix} (q^3 - q^2 - 1)\log_q(1 + \frac{1}{q^2}) & q - 1 & q - 1 \\ (q^2 - q - 1)\log_q(1 + \frac{1}{q}) & q - 1 & 0 \\ 0 & (q^2 - q - 1)\log_q(1 + \frac{1}{q}) & q - 1 \end{pmatrix}.$$

Clearly this matrix is regular. But we can also show that the corresponding matrix is regular for any other digit $a \neq 0$. Hence we have:

Theorem 6 *The first moment of the weighted single-block-occurrence function of length* 1 *is (weak) Delange type if and only if the sequence of weights is constant (converges).*

2.3.4 The weighted single-block-occurrence of length 2

If $t = 2$ and $g : \Gamma \to \mathbb{R}$ is given by $g(a_1, a_2) = 1$ for fixed integers a_1, a_2 and zero otherwise, then the weighted generalized digit-block-counting function counts weighted single-block-occurrences of length 2. In this case we have $\Psi(1/q) = -1/q$ or $\Psi(1/q) = 1 - 1/q$ depending on the given integers a_1 and a_2. In any case we can be sure that $\Psi(1/q) \neq 0$. For $n = q^r$ with $r > 1$ we get the equation

$$\gamma_{r-1}\frac{1}{q}\Psi\left(\frac{1}{q}\right) = F(0) + o(1),$$

which shows that the sequence $\boldsymbol{\gamma}$ converges. Hence we have:

Theorem 7 *The first moment of the weighted single-block-occurrence function of length 2 is (weak) Delange type iff the sequence of weights is constant (converges).*

2.3.5 The weighted single-block-occurrence of arbitrary length

Let $t \geq 1$ and $q \geq 2$ be integers. If $g : \Gamma \to \mathbb{R}$ is given by $g(a_1, \ldots, a_t) = 1$ for integers $a_1, \ldots, a_t \in \{0, 1, \ldots, q-1\}$ and zero otherwise, then the generalized weighted digit-block-counting function counts weighted single-block-occurrences of length t.

The cases $t = 1$ or $t = 2$ have been treated above already. Here we prove similar results for arbitrary $t \geq 3$. To this end we need the following three lemmas, whose easy proofs will be omitted.

Lemma 5 *For any integers $q \geq 2$ and $l \geq 1$ we have* $\log_q(q^l + 1) \notin \mathbb{Q}$.

Lemma 6 *For any nonnegative integers $l \neq k$ we have* $\gcd\left(q^{2^l} + 1, q^{2^k} + 1\right) \in \{1, 2\}$.

Lemma 7 *Let $l \geq 1$ be an integer. The numbers* $1, \log_q(q^{2^1}+1), \log_q(q^{2^2}+1), \ldots, \log_q(q^{2^l}+1)$ *are linearly independent over* $\mathbb{Q}$.

Let $t \geq 3$ and $a_1, \ldots, a_t \in \{0, \ldots, q-1\}$ be fixed. We know from Lemma 3 that $\Psi(1/q^i) \neq 0$ for all $i \in \{1, \ldots, t-1\}$. Further

$$\frac{f_q(a_1, \ldots, a_t)}{q^t} - g(\mathbf{0})\frac{q^l}{q^l + 1} \neq 0$$

for any integer $l \geq 1$.

Now q^t times the equation for $n = q^r$ gives

$$z_1\gamma_{r-1} + \cdots + z_{t-1}\gamma_{r-t+1} = q^t F(0) + o(1), \tag{5}$$

where $z_i \in \mathbb{Z} \setminus \{0\}$. This is an equation in $t - 1$ variables.

$q^t(q^l + 1)$ times the equation for $n = q^r + q^{r-l}$ with $r > t + l$ gives the equation

$$(w_0 \log_q(q^l+1) + w_0')\gamma_r + \sum_{i=1}^{t-1} w_i\gamma_{r-i} + \sum_{i=0}^{l-1} w_{t+i}\gamma_{r-t-i} = q^t(q^l+1)F(\log_q(q^l+1)) + o(1), \tag{6}$$

where $w_0', w_i \in \mathbb{Z}\setminus\{0\}$ for all $i \in \{0, \dots, t+l-1\}$. We remark that $\log_q(q^l+1)$ is irrational according to Lemma 5.

We fix $r \in \mathbb{N}$ large enough and take $l = 2^k$ in Eq. (6) with $k \in \{1, \dots, t-2\}$ and get $t-2$ equations with $t+2^{t-2}$ variables $\gamma_r, \dots, \gamma_{r-t-2^{t-2}+1}$. We also take $n = q^s$ in Eq. (5) with $s \in \{r-2^{t-2}, \dots, r+1\}$ and we get $2^{t-2}+2$ equations in the same variables as above. Together we have $t+2^{t-2}$ equations with $t+2^{t-2}$ variables. We use the last $2^{t-2}+2$ lines to reduce the first $t-2$ lines. This works because of the structure of the matrix of coefficients. After this the first $t-2$ lines of the matrix have at most the first $t-2$ coefficient not equal zero. By Lemma 5 and the structure of the original matrix of coefficients we still have the irrational parts $\log_q(q^{2^i}+1)$ in the first column of the new matrix. We take a look at the left upper $(t-2)\times(t-2)$ sub-matrix. By Gauss elimination we can reduce at least one line of this matrix, such that only the first coefficient of this line is not equal zero. We know that the first coefficients cannot be zero, because of Lemma 7. This line gives the solution for any $\gamma_{r-j} = C + o(1)$, where C is a real constant and $j \in \{0, \dots, t-3\}$. Because $r > t+l$ was arbitrary, we find that the sequence $\boldsymbol{\gamma}$ has to converge. We obtain:

Theorem 8 *The first moment of the weighted single-block-occurrence function of arbitrary length* $t \geq 1$ *is (weak) Delange type if and only if the sequence of weights is constant (converges).*

2.3.6 Open problem

We close Section 2 with the statement of a conjecture. We assume that the converse result of Theorem 3 holds for every function $g : \Gamma \to \mathbb{R}$, which is not the zero-function. (Of course, if $g : \Gamma \to \mathbb{R}$, $g(\boldsymbol{a}) = 0$ for all $\boldsymbol{a} \in \Gamma$, then we get for the first moment $S_q(n, \boldsymbol{\gamma}) = 0$ such that the converse of Theorem 3 cannot hold.)

Conjecture 1 The first moment of the generalized weighted digit-block-counting function, where $g : \Gamma \to \mathbb{R}$ is not the zero function, is (weak) Delange type if and only if the sequence of weights is constant (converges).

However, we think that our method used in the previous subsections would be too complicated to prove the converse for general functions $g : \Gamma \to \mathbb{R}$. The reason for this is that it is not clear in general how to use the different values of $n \in \mathbb{N}$. There will be lots of different cases to consider and it seems to be hard to come to a positive end this way.

3 The generalized weighted digit-block-counting function and uniform distribution modulo one

In the first part of this paper we have been studying in detail the average growth-behavior of generalized weighted digit-block-counting functions. In the following we will study distribution properties of the d-dimensional generalized weighted digit-block-counting sequence as defined in (2).

The condition for the uniform distribution of $(\boldsymbol{s}_q(k, \boldsymbol{\gamma}))_{k=0,1,\dots}$ is rather technical and we need some notations to be able to state it.

For arbitrary $l, b \in \{0, \ldots, q^t - 1\}$ let

$$l = l_0 + l_1 q + \cdots + l_{t-1} q^{t-1} \quad \text{and} \quad b = b_0 + b_1 q + \cdots + b_{t-1} q^{t-1}$$

and for arbitrary $u \in \mathbb{N}$ and $i \in \{1, \ldots, d\}$ let

$$\begin{aligned}
\Lambda^{(i)} &= \Lambda^{(i)}(t, u, l, b) \\
&:= \left(\gamma^{(i)}_{t(u-1)+1} g^{(i)}(b_1, \ldots, b_{t-1}, l_0) + \cdots + \gamma^{(i)}_{tu-1} g^{(i)}(b_{t-1}, l_0, \ldots, l_{t-2})\right) \\
&\quad - \left(\gamma^{(i)}_{t(u-1)+1} g^{(i)}(b_1, \ldots, b_{t-1}, 0) + \cdots + \gamma^{(i)}_{tu-1} g^{(i)}(b_{t-1}, 0, \ldots, 0)\right) \\
&\quad + \left(\gamma^{(i)}_{tu} g^{(i)}(l_0, l_1, \ldots, l_{t-1}) + \cdots + \gamma^{(i)}_{tu+t-1} g^{(i)}(l_{t-1}, 0, \ldots, 0)\right)
\end{aligned}$$

and

$$\boldsymbol{\Lambda} := (\Lambda^{(1)}, \ldots, \Lambda^{(d)}).$$

For $\boldsymbol{h} := (h_1, \ldots, h_d) \in \mathbb{Z}^d$ let

$$v_u = v_{u,t,\boldsymbol{h}} := \max_{l,b \in \{0,\ldots,q^t-1\}} \|\boldsymbol{\Lambda} \cdot \boldsymbol{h}\|.$$

Finally for $\boldsymbol{h}, l, u$ like above let

$$S^*_{\boldsymbol{h}}(l, u) := \sum_{k=lq^{tu}}^{(l+1)q^{tu}-1} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{h} \cdot \boldsymbol{s}_q(k, \boldsymbol{\gamma})}.$$

We will show the following result.

Theorem 9 *The sequence $(\boldsymbol{s}_q(k, \boldsymbol{\gamma}))_{k=0,1,\ldots}$ is uniformly distributed modulo one if and only if for every $\boldsymbol{h} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}$ one of the following conditions holds:*

(i) For every $\delta > 0$, every u large enough and every $l \in \{0, 1, \ldots, q^t - 1\}$ we have

$$|S^*_{\boldsymbol{h}}(l, u)| < \delta^u,$$

or

(ii) $\sum_{u=1}^{\infty} v_u^2 = +\infty$.

Remark 2 Note that - from a heuristic point of view - condition (i) is a very rare event, whereas (ii) is an event of high probability.

Remark 3 As is easily checked, for $t = 1$ and $g(a) = a$ for all $a \in \Gamma$ the condition of Theorem 9 coincides with the condition of [11, Theorem 1].

Since the conditions (i) and (ii) are rather technical we consider two important examples.

Example 4 First we consider linear functions $g^{(i)} : \Gamma \to \mathbb{R}$, i.e.,

$$g^{(i)}(k_0, \dots, k_{t-1}) = a_0^{(i)} k_0 + \cdots + a_{t-1}^{(i)} k_{t-1}$$

for all $i \in \{1, \dots, d\}$ and for given reals $a_j^{(i)}$, $j \in \{0, 1, \dots, t-1\}$ and $i \in \{1, \dots, d\}$.

Inserting the $g^{(i)}$ in the definition of $S_{\boldsymbol{h}}^*(l, u)$ we find

$$|S_{\boldsymbol{h}}^*(l, u)| = \left| \prod_{j=-t+1}^{tu-t} \sum_{k=0}^{q-1} \mathrm{e}^{2\pi \mathtt{i} \left(\sum_{i=1}^{d} h_i \theta_i^{(j)} \right) k} \right|,$$

where

$$\theta_i^{(j)} = \begin{pmatrix} \gamma_j^{(i)} \\ \vdots \\ \gamma_{j+t-1}^{(i)} \end{pmatrix} \cdot \begin{pmatrix} a_{t-1}^{(i)} \\ \vdots \\ a_0^{(i)} \end{pmatrix} \quad \text{for all } j = -t+1, \dots, t(u-1)$$

(with $\gamma_l^{(i)} = 0$ if $l < 0$).

A necessary condition for (i) to hold therefore is that either

(ia) $\sum_{i=1}^{d} h_i \theta_i^{(j)} = \frac{A}{q}$ with $A \in \mathbb{Z}$, $A \not\equiv 0 \pmod{q}$ for some j, or

(ib) $\sum_{j=1}^{\infty} \left\| \sum_{i=1}^{d} h_i \theta_i^{(j)} \right\|^2 = +\infty$.

We will show that (ib) implies (ii), so that our sequence is uniformly distributed if and only if (ia) or (ii) holds for every $\boldsymbol{h} \neq \boldsymbol{0}$, and (ia) is equivalent to condition (i') *There is an integer u such that for all $l \in \{0, 1, \dots, q^t - 1\}$ we have $S_{\boldsymbol{h}}^*(l, u) = 0$.*

First we find that $\Lambda^{(i)} = \Lambda^{(i)}(t, u, l, b)$ is independent of the choice of b and is given by

$$\Lambda^{(i)} = \begin{pmatrix} \gamma_{t(u-1)+1}^{(i)} \\ \vdots \\ \gamma_{tu+t-1}^{(i)} \end{pmatrix} \cdot \begin{pmatrix} \sum_{k=0}^{t-1} a_k^{(i)} l_{0-(t-1)+k} \\ \vdots \\ \sum_{k=0}^{t-1} a_k^{(i)} l_{2(t-1)-(t-1)+k} \end{pmatrix},$$

where $l_k = 0$ for $k < 0$ and for $k > t-1$.

We finally show that (ib) implies

$$\sum_{u=1}^{\infty} \max_{l, b \in \{0, \dots, q^t - 1\}} \|\boldsymbol{\Lambda} \cdot \boldsymbol{h}\|^2 = +\infty.$$

Let $\widetilde{l}^{(w)} = \widetilde{l}_0^{(w)} + \widetilde{l}_1^{(w)} q + \cdots + \widetilde{l}_{t-1}^{(w)} q^{t-1}$ with $\widetilde{l}_j^{(w)} = 0$ if $j \neq w$ and $\widetilde{l}_j^{(w)} = 1$ if $j = w$. Then

$$\Lambda^{(i)}(t, u, \widetilde{l}^{(w)}, b) = \sum_{z=0}^{t-1} \gamma_{t(u-1)+w+z+1}^{(i)} a_{(t-1)-z}^{(i)} = \theta_i^{(t(u-1)+w+1)}.$$

Let $w \in \{0, \dots, t-1\}$ be such that

$$\sum_{u=0}^{\infty} \left\| \sum_{i=1}^{d} h_i \theta_i^{(tu+w+1)} \right\|^2 = +\infty$$

(such a w exists since (ib) holds). Then

$$\sum_{u=1}^{\infty}\left\|\mathbf{\Lambda}(t,u,\widetilde{l}^{(w)},b)\cdot\boldsymbol{h}\right\|^2=\sum_{u=0}^{\infty}\left\|\sum_{i=1}^{d}h_i\theta_i^{(tu+w+1)}\right\|^2=+\infty$$

and hence (ii) holds. This closes our example.

Motivated by this example and by the result in [11] one may assume that the rare event (i) in Theorem 9 can be replaced by the even more rare event (i') *There is an integer* u *such that for all* $l\in\{0,1,\ldots,q^t-1\}$ *we have* $S^*_{\boldsymbol{h}}(l,u)=0$. This, however, is in general not the case. To illustrate this, we consider the weighted Gray code sequence.

Example 5 As second example we consider the weighted Gray code sequence, i.e., $d=1$, $q=2$, $t=2$ and $g:\Gamma\to\mathbb{R}$ given by $g(k_0,k_1)=k_0\oplus k_1$ where $\oplus$ denotes addition modulo 2. Here for $l=l_0+l_1 2$ and $b=b_0+b_1 2$ we have (we omit the superscript for the dimension)

$$\Lambda(2,u,l,b)=\gamma_{2u-1}((b_1\oplus l_0)-b_1)+\gamma_{2u}(l_0\oplus l_1)+\gamma_{2u+1}l_1$$

and therefore

$$\begin{aligned}v_u &= \max\{\|h(\gamma_{2u}+\gamma_{2u+1})\|,\|h(\gamma_{2u-1}+\gamma_{2u})\|,\|h(\gamma_{2u-1}+\gamma_{2u+1})\|,\\ &\qquad \|h(-\gamma_{2u-1}+\gamma_{2u})\|,\|h(-\gamma_{2u-1}+\gamma_{2u+1})\|\}.\end{aligned}$$

For $l=l_0+l_1 2$ and $k=k_0+k_1 2+\cdots+k_{2u-1}2^{2u-1}$ we have

$$s_2(k+lq^{tu})=(k_0\oplus k_1)\gamma_0+\cdots+(k_{2u-2}\oplus k_{2u-1})\gamma_{2u-2}+(k_{2u-1}\oplus l_0)\gamma_{2u-1}+(l_0\oplus l_1)\gamma_{2u}+l_1\gamma_{2u+1}.$$

Therefore

$$|S^*_h(l,u)|=\left|\sum_{k_0,\ldots,k_{2u-1}=0}^{1}\mathrm{e}^{2\pi\mathtt{i}h((k_0\oplus k_1)\gamma_0+\cdots+(k_{2u-2}\oplus k_{2u-1})\gamma_{2u-2}+(k_{2u-1}\oplus l_0)\gamma_{2u-1})}\right|.$$

As for fixed l_0 the mapping $\{0,1\}^{2u}\to\{0,1\}^{2u}$, $(k_0,\ldots,k_{2u-1})\mapsto(k_0\oplus k_1,\ldots,k_{2u-2}\oplus k_{2u-1},k_{2u-1}\oplus l_0)$ is bijective, we obtain

$$|S^*_h(l,u)|=\left|\prod_{j=0}^{2u-1}\sum_{k=0}^{1}\mathrm{e}^{2\pi\mathtt{i}hk\gamma_j}\right|.$$

A necessary condition for (i) to hold therefore is that either

(ia) $h\gamma_j=\frac{A}{2}$ with $A\in\mathbb{Z}$, $A\equiv 1 \pmod 2$ for some j, or

(ib) $\sum_{j=1}^{\infty}\|h\gamma_j\|^2=+\infty$.

Now we construct a sequence $\boldsymbol{\gamma}$ such that (ib) does not imply (ia) or (ii). (Note that (ia) is equivalent to (i') mentioned above.)

Let $h\neq 0$ be fixed. We define the sequence $\boldsymbol{\gamma}=(\gamma_j)_{j\geq 1}$ in such a way that $h\gamma_j\neq\frac{1}{2}$ but $\left|h\gamma_j-\frac{1}{2}\right|<j^{-2}$ for all $j\geq 1$. Then we have $\|h\gamma_j\|\geq\frac{1}{2}-j^{-2}$ and hence $\|h\gamma_j\|^2\geq\frac{1}{4}-j^{-2}$. Therefore

$$\sum_{j=1}^{\infty}\|h\gamma_j\|^2=+\infty$$

such that (ib) is satisfied.

Of course, condition (ia) is not satisfied by the construction of the γ_j's. Now we show that also (ii) cannot hold. For $i, j \in \mathbb{N}$ we have

$$\|h(\gamma_i+\gamma_j)\| \le |h(\gamma_i+\gamma_j)-1| \le \left|h\gamma_i - \frac{1}{2}\right| + \left|h\gamma_j - \frac{1}{2}\right| < \frac{1}{i^2} + \frac{1}{j^2} \le \frac{2}{\min\{i,j\}^2}.$$

We also have

$$\|h(-\gamma_i+\gamma_j)\| \le \|h(\gamma_i+\gamma_j)\| + \|2h\gamma_i\| < \frac{2}{\min\{i,j\}^2} + \frac{2}{i^2} \le \frac{4}{\min\{i,j\}^2}.$$

Therefore it follows that $v_u < \frac{4}{(2u-1)^2}$ for all $u \ge 1$ and from this we obtain $\sum_{u=1}^{\infty} v_u^2 < +\infty$. Hence (ii) is not satisfied. This shows that (i) cannot be replaced by (i') in general.

We conclude this example by noting that for the sequence $\boldsymbol{\gamma}$ given by $\gamma_j = \alpha$ for all $j \in \mathbb{N}$, the conditions (ib) and (ii) are equivalent to $\max\{\|h\alpha\|, \|2h\alpha\|\} \ne 0$. Compare this with [9, Remark 3].

For the proof of Theorem 9 we will use the following lemmas.

Lemma 8 *Let $Q \in \mathbb{N}$,*

$$\sigma(0,0) = \sigma(1,0) = \cdots = \sigma(Q-1,0) = 1,$$

further, for $l, j \in \{0, \ldots, Q-1\}$ and $w \in \mathbb{N}_0$ let $\alpha(l,j,w) \in \mathbb{C}$ with $|\alpha(l,j,w)| = 1$ and $\alpha(0,j,w) = 1$ for all j and w.

Finally, for $l \in \{0, \ldots, Q-1\}$ and $w \in \mathbb{N}_0$ let

$$\sigma(l, w+1) = \sum_{j=0}^{Q-1} \alpha(l,j,w)\sigma(j,w).$$

Then with an absolute constant $c > 0$ (depending only on Q) and with

$$\nu_w := \max_{l,j\in\{0,\ldots,Q-1\}} \left\| \frac{\arg \alpha(l,j,w)}{2\pi} \right\|$$

(where arg *means the argument of a complex number;* $\arg r\mathrm{e}^{\mathrm{i}\gamma} := \gamma$ *for $-\pi < \gamma \le \pi$) we have for all $l \in \{0, \ldots, Q-1\}$, $u \in \mathbb{N}$ and $x, y \in \{0,1,2\}$, that*

$$|\sigma(l, 3u+x)| \le Q^4 \prod_{w=0}^{u-1} \left(Q^3 - c\nu_{3w+y}^2\right).$$

Proof. In the following we use the fact that there exists a positive constant c_1 (depending only on Q) such that

$$\left|\sum_{l=0}^{Q-1} \alpha(l,j,w)\right| = \left|1 + \sum_{l=1}^{Q-1} \alpha(l,j,w)\right| \le Q - c_1\nu_w^2$$

for all j and w.

We have

$$
\begin{aligned}
|\sigma(0,w+1)| &= \left|\sum_{j=0}^{Q-1}\sigma(j,w)\right| \\
&= \left|\sum_{j=0}^{Q-1}\sum_{k=0}^{Q-1}\alpha(j,k,w-1)\sigma(k,w-1)\right| \\
&= \left|\sum_{k=0}^{Q-1}\left(\sum_{j=0}^{Q-1}\alpha(j,k,w-1)\right)\sigma(k,w-1)\right| \\
&\le Q(Q-c_1\nu_{w-1}^2)T_{w-1} \\
&= (Q^2-c\nu_{w-1}^2)T_{w-1},
\end{aligned}
$$

where $T_{w-1} := \max_{k\in\{0,\dots,Q-1\}}|\sigma(k,w-1)|$.

Trivially we have

$$|\sigma(l,w+1)| \le Q^2 T_{w-1} \quad \text{for all } l\in\{0,\dots,Q-1\},$$

and hence

$$|\sigma(l,w+2)| \le (Q-1)Q^2T_{w-1} + (Q^2-c\nu_{w-1}^2)T_{w-1} = (Q^3-c\nu_{w-1}^2)T_{w-1}$$

and

$$|\sigma(l,w+z)| \le Q^{z-2}(Q^3-c\nu_{w-1}^2)T_{w-1}$$

for $z\in\mathbb{N}$, $z\ge 2$. Hence finally

$$
\begin{aligned}
|\sigma(l,3u+x)| &\le Q^{x+2-y}(Q^3-c\nu_{3(u-1)+y}^2)T_{3(u-1)+y} \\
&\le Q^{x+2-y}\prod_{w=0}^{u-1}(Q^3-c\nu_{3w+y}^2)T_y \\
&\le Q^{x+2}\prod_{w=0}^{u-1}(Q^3-c\nu_{3w+y}^2),
\end{aligned}
$$

and the result follows. □

Lemma 9 *Let $z_1,\dots,z_Q$ be complex numbers with $|\arg z_i - \arg z_j| < \pi/8$ for all $i,j\in\{1,\dots,Q\}$. Let $\rho_1,\dots,\rho_Q$ be complex numbers with $|\rho_i|=1$ and with $|\arg\rho_i|<\pi/8$ for all $i\in\{1,\dots,Q\}$.*

Let $B = z_1+\dots+z_Q$ and $B_\rho = \rho_1 z_1+\dots+\rho_Q z_Q$. Then we have

$$|\arg B_\rho - \arg B| \le 2\max_{i\in\{1,\dots,Q\}}|\arg\rho_i|.$$

Proof. This is [9, Lemma 3]. □

Lemma 10 *There is a constant $c>0$ such that for all $\mu>0$ and all $z_1,\dots,z_Q\in\mathbb{C}$ ($Q\ge 2$) we have*

$$|z_1+\dots+z_Q| \ge (Q-c\mu^2)\min_{j\in\{1,\dots,Q\}}|z_j|$$

provided that $\max_{i,j\in\{1,\dots,Q\}}\left\|\frac{\arg z_i}{2\pi}-\frac{\arg z_j}{2\pi}\right\|\le\mu$.

We omit the easy proof of this lemma.

Now we can give the proof of Theorem 9.

Proof. By Weyl's criterion (see for example [3, 8]) it suffices to investigate under which conditions

$$\frac{1}{N}S_{\boldsymbol{h}}(N) := \frac{1}{N}\sum_{k=0}^{N-1} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{h}\cdot\boldsymbol{s}_q(k,\boldsymbol{\gamma})}$$

tends to zero as $N\to\infty$ for all $\boldsymbol{h}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}$.

Let $N\in\mathbb{N}$ have the following representation in base q^t,

$$N=\sum_{j=0}^{r} N_j q^{tj}$$

with $N_j\in\{0,1,\dots,q^t-1\}$ and $N_r\neq 0$. Let further

$$L_j := N_{j+1}q^{t(j+1)}+\cdots+N_r q^{tr}$$

for $j\in\{0,\dots,r-1\}$ and $L_r:=0$.

Then (for simplicity we write $\boldsymbol{s}(k)$ instead of $\boldsymbol{s}_q(k,\boldsymbol{\gamma})$ and $\mathrm{e}(x)$ instead of $\mathrm{e}^{2\pi\mathrm{i}x}$ in the following)

$$\begin{aligned}
S_{\boldsymbol{h}}(N) &= \sum_{k=0}^{N-1}\mathrm{e}(\boldsymbol{h}\cdot\boldsymbol{s}(k))\\
&= \sum_{j=0}^{r}\sum_{\epsilon=0}^{N_j-1}\sum_{k=L_j+\epsilon q^{tj}}^{L_j+(\epsilon+1)q^{tj}-1}\mathrm{e}(\boldsymbol{h}\cdot\boldsymbol{s}(k))\\
&= \sum_{j=0}^{r}\sum_{\epsilon=0}^{N_j-1}\sum_{k=0}^{q^{tj}-1}\mathrm{e}(\boldsymbol{h}\cdot\boldsymbol{s}(k+L_j+\epsilon q^{tj}))\\
&= \sum_{j=0}^{r}\sum_{\epsilon=0}^{N_j-1}\sum_{l=0}^{q^t-1}\sum_{k=lq^{t(j-1)}}^{(l+1)q^{t(j-1)}-1}\mathrm{e}(\boldsymbol{h}\cdot\boldsymbol{s}(k+L_j+\epsilon q^{tj})).
\end{aligned}$$

Consider now fixed ϵ, l and k with q-adic representations

$$\begin{aligned}
\epsilon &= \epsilon_0+\epsilon_1 q+\cdots+\epsilon_{t-1}q^{t-1},\\
l &= l_0+l_1 q+\cdots+l_{t-1}q^{t-1} \quad \text{and}\\
k &= k_0+k_1 q+\cdots+k_{t(j-1)-1}q^{t(j-1)-1}+lq^{t(j-1)}.
\end{aligned}$$

Then the argument $k+L_j+\epsilon q^{tj}$ has a q-adic representation of the form

$$k_0k_1\dots k_{t(j-1)-1}l_0l_1\dots l_{t-1}\epsilon_0\epsilon_1\dots\epsilon_{t-1}\underbrace{N_{j+1}}_{t\text{ digits}}\dots\underbrace{N_r}_{t\text{ digits}}$$

and hence for the i-th coordinate $s^{(i)}$ of $\boldsymbol{s}$ we have

$$s^{(i)}(k+L_j+\epsilon q^{tj}) = s^{(i)}(k)+s^{(i)}(\epsilon q^{tj}+L_j)+\Omega^{(i)}(t,j,l,\epsilon),$$

where

$$\begin{array}{rcl}\Omega^{(i)}(t,j,l,\epsilon) & := & -\gamma^{(i)}_{t(j-1)+1}g(l_1,\dots,l_{t-1},0)-\cdots-\gamma^{(i)}_{tj-1}g(l_{t-1},0,\dots,0)\\ & & -\gamma^{(i)}_{t(j-1)+1}g(0,\dots,0,\epsilon_0)-\cdots-\gamma^{(i)}_{tj-1}g(0,\epsilon_0,\dots,\epsilon_{t-1})\\ & & +\gamma^{(i)}_{t(j-1)+1}g(l_1,\dots,l_{t-1},\epsilon_0)+\cdots+\gamma^{(i)}_{tj-1}g(l_{t-1},\epsilon_0,\dots,\epsilon_{t-1})\end{array}$$

and therefore with $\boldsymbol{\Omega} := (\Omega^{(1)},\dots,\Omega^{(d)})$ we have

$$\begin{array}{rcl}\mathrm{e}(\boldsymbol{h}\cdot\boldsymbol{s}(k+L_j+\epsilon q^{tj})) & = & \mathrm{e}(\boldsymbol{h}\cdot\boldsymbol{s}(k))\mathrm{e}(\boldsymbol{h}\cdot(\boldsymbol{s}(\epsilon q^{tj}+L_j)+\boldsymbol{\Omega}))\\ & =: & \mathrm{e}(\boldsymbol{h}\cdot\boldsymbol{s}(k))\varphi_{\boldsymbol{h}}(t,j,l,\epsilon)\end{array}$$

and

$$\begin{array}{rcl}S_{\boldsymbol{h}}(N) & = & \displaystyle\sum_{j=0}^{r}\sum_{\epsilon=0}^{N_j-1}\sum_{l=0}^{q^t-1}\varphi_{\boldsymbol{h}}(t,j,l,\epsilon)\sum_{k=lq^{t(j-1)}}^{(l+1)q^{t(j-1)}-1}\mathrm{e}(\boldsymbol{h}\cdot\boldsymbol{s}(k))\\ & = & \displaystyle\sum_{j=0}^{r}\sum_{\epsilon=0}^{N_j-1}\sum_{l=0}^{q^t-1}\varphi_{\boldsymbol{h}}(t,j,l,\epsilon)S^*_{\boldsymbol{h}}(l,j-1).\end{array}$$

Now if condition (i) is satisfied, then of course $\frac{1}{N}S_{\boldsymbol{h}}(N)$ tends to zero as $N\to\infty$. Otherwise, and if condition (ii) is satisfied, then for $l=0,\dots,q^t-1$ and $u\in\mathbb{N}_0$ we study now

$$S^*_{\boldsymbol{h}}(l,u)=\sum_{b=0}^{q^t-1}\sum_{k=bq^{t(u-1)}}^{(b+1)q^{t(u-1)}-1}\mathrm{e}(\boldsymbol{h}\cdot\boldsymbol{s}(k+lq^{tu})).$$

Since $s^{(i)}(k+lq^{tu})=s^{(i)}(k)+\Lambda^{(i)}(t,u,l,b)$, we have

$$S^*_{\boldsymbol{h}}(l,u)=\sum_{b=0}^{q^t-1}\psi_{\boldsymbol{h}}(t,u,l,b)S^*_{\boldsymbol{h}}(b,u-1),$$

where $\psi_{\boldsymbol{h}}(t,u,l,b):=\mathrm{e}(\boldsymbol{h}\cdot\boldsymbol{\Lambda})$. Note that $\|\boldsymbol{h}\cdot\boldsymbol{\Lambda}\|=\left\|\frac{\arg\psi_{\boldsymbol{h}}(t,u,l,b)}{2\pi}\right\|$.

Since by assumption $\sum_{u=1}^{\infty}v_u^2=+\infty$, there exists a $y\in\{0,1,2\}$ with

$$\sum_{w=0}^{\infty}v^2_{3w+y}=+\infty.$$

We can apply Lemma 8 and we obtain, with a positive constant c depending only on q,

$$|S^*_{\boldsymbol{h}}(l,3u+x)|\le q^{4t}\prod_{w=0}^{u-1}(q^{3t}-cv^2_{3w+y})$$

for all $x\in\{0,1,2\}$.

Altogether we obtain

$$|S_{\boldsymbol{h}}(N)|\le q^{2t}\sum_{j=0}^{r}\max_{l\in\{0,\dots,q^t-1\}}|S^*_{\boldsymbol{h}}(l,j-1)|\le q^{2t}q^{4t}\sum_{j=0}^{r}\prod_{w=0}^{\left\lfloor\frac{j-4}{3}\right\rfloor}(q^{3t}-cv^2_{3w+y})$$

and

$$\left|\frac{S_{\boldsymbol{h}}(N)}{N}\right| \le q^{6t} \sum_{j=0}^{r} \frac{q^{jt}}{q^{rt}} \prod_{w=0}^{\left\lfloor \frac{j-4}{3} \right\rfloor} \left(1 - \frac{c}{q^{3t}} v_{3w+y}^2\right).$$

We have $\prod_{w=0}^{A} \left(1 - \frac{c}{q^{3t}} v_{3w+y}^2\right) \to 0$ if A tends to infinity. For arbitrary $\varepsilon > 0$ let j_0 be such that

$$\prod_{w=0}^{\left\lfloor \frac{j-4}{3} \right\rfloor} \left(1 - \frac{c}{q^{3t}} v_{3w+y}^2\right) < \varepsilon \quad \text{for all } j \ge j_0.$$

Then

$$\left|\frac{S_{\boldsymbol{h}}(N)}{N}\right| \le q^{6t} \left(\sum_{j=0}^{j_0-1} \frac{q^{jt}}{q^{rt}} + \sum_{j=j_0}^{r} \frac{q^{jt}}{q^{rt}} \varepsilon\right) < 4q^{6t}\varepsilon$$

for r large enough. Now the first direction of our result follows.

Assume now that neither (i) nor (ii) is satisfied. Since (i) is not satisfied, there exists a $\delta > 0$ such that for infinitely many u there is an l such that

$$|S_{\boldsymbol{h}}^*(l,u)| > \frac{\delta}{2} \max_{b \in \{0,\ldots,q^t-1\}} |S_{\boldsymbol{h}}^*(b,u-1)|. \tag{7}$$

Let now $0 < \varepsilon < 1/32$ be so small that $\frac{8\pi q^t}{\delta}\varepsilon < \frac{1}{3}$ and let u_0 be such that $v_u < \varepsilon$ for all $u \ge u_0$ (note that such u_0 exists since (ii) is not satisfied), and let $u_1 \ge u_0$ be such that (7) is satisfied. Then $|S_{\boldsymbol{h}}^*(l,u_1)| > 0$ and for every $l' \in \{0,\ldots,q^t-1\}$ we have

$$\begin{aligned}
|S_{\boldsymbol{h}}^*(l,u_1) - S_{\boldsymbol{h}}^*(l',u_1)| &\le \sum_{b=0}^{q^t-1} |\psi_{\boldsymbol{h}}(t,u_1,l,b) - \psi_{\boldsymbol{h}}(t,u_1,l',b)| \cdot |S_{\boldsymbol{h}}^*(b,u_1-1)| \\
&\le q^t 4\pi v_{u_1} \max_{b \in \{0,\ldots,q^t-1\}} |S_{\boldsymbol{h}}^*(b,u_1-1)| \\
&\le \frac{8\pi q^t}{\delta} v_{u_1} |S_{\boldsymbol{h}}^*(l,u_1)|.
\end{aligned}$$

Hence

$$\left|1 - \frac{S_{\boldsymbol{h}}^*(l',u_1)}{S_{\boldsymbol{h}}^*(l,u_1)}\right| < \frac{8\pi q^t}{\delta} v_{u_1} < \frac{8\pi q^t}{\delta}\varepsilon < \frac{1}{3}$$

and therefore

$$|\arg S_{\boldsymbol{h}}^*(l',u_1) - \arg S_{\boldsymbol{h}}^*(l,u_1)| < \arcsin\frac{1}{3} < \frac{\pi}{8}.$$

So we can apply Lemma 9 (note that $\varepsilon < 1/32$ and hence $4\pi v_{u_1} < \pi/8$) and obtain that

$$|\arg S_{\boldsymbol{h}}^*(l',u) - \arg S_{\boldsymbol{h}}^*(l,u)| < 4\pi v_u \quad \text{for all } u > u_1.$$

Further we have

$$|S_{\boldsymbol{h}}^*(l',u_1)| \ge |S_{\boldsymbol{h}}^*(l,u_1)| \left(1 - \frac{8\pi q^t}{\delta}\varepsilon\right) > \frac{2}{3}|S_{\boldsymbol{h}}^*(l,u_1)| =: B > 0$$

for all $l' \in \{0,\ldots,q^t-1\}$.

By Lemma 10 we obtain for all $u > u_1$ and all $l' \in \{0, \ldots, q^t - 1\}$,

$$|S^*_{\boldsymbol{h}}(l', u)| \geq (q^t - c'' v_u^2) \min_{b \in \{0, \ldots, q^t - 1\}} |S^*_{\boldsymbol{h}}(b, u - 1)|,$$

hence

$$|S^*_{\boldsymbol{h}}(0, u)| \geq \left(\prod_{w=u_1}^{u} (q^t - c'' v_w^2) \right) B$$

and therefore

$$\frac{1}{q^{ut}} |S_{\boldsymbol{h}}(q^{ut})| = \frac{1}{q^{ut}} |S^*_{\boldsymbol{h}}(0, u)| \geq \frac{1}{q^{t(u_1 - 1)}} \prod_{w=u_1}^{u} \left(1 - \frac{c''}{q^t} v_w^2 \right),$$

with the last expression not tending to zero as $u \to \infty$, since $\sum_{u=1}^{\infty} v_u^2$ is finite. □

References

[1] Cateland, E.: *Suites digitales et suites k-régulières.* Thèse, Université Bordeaux 1, 1992.

[2] Delange, H.: Sur les fonctions q-additives ou q-multiplicatives. Acta Arith. **21**: 285–298, 1972.

[3] Drmota, M., and Tichy, R.F.: *Sequences, Discrepancies and Applications.* Lecture Notes in Mathematics 1651, Springer-Verlag, Berlin, 1997.

[4] Flajolet, P., Grabner, P., Kirschenhofer, P., Prodinger, H., and Tichy, R.F.: Mellin transforms and asymptotics: Digital sums. Theor. Comput. Sci. **123**: 291–314, 1994.

[5] Flajolet, P., and Ramshaw, L.: A note on Gray code and odd-even merge. SIAM J. Comput. **9**: 142–158, 1980.

[6] Kirschenhofer, P.: Subblock occurrences in the q-ary representation of n. SIAM J. Algebraic Discrete Methods **4**: 231–236, 1983.

[7] Kirschenhofer, P., Prodinger, H., and Tichy, R.F.: Über die Ziffernsumme natürlicher Zahlen und verwandte Probleme. In: *Zahlentheoretische Analysis* (E. Hlawka, ed.). Lecture Notes in Mathematics 1114, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1985, pp. 55–65.

[8] Kuipers, L., and Niederreiter, H.: *Uniform Distribution of Sequences.* John Wiley, New York, 1974.

[9] Larcher, G., and Tichy, R.F.: Some number-theoretical properties of generalized sum-of-digits functions. Acta Arith. **52**: 183–196, 1989.

[10] Larcher, G., and Pillichshammer, F.: Moments of the weighted sum-of-digits function. Quaest. Math. **28**: 321–336, 2005.

[11] Pillichshammer, F.: Uniform distribution of sequences connected with the weighted sum-of-digits function. Uniform Distribution Theory **2**: 1–10, 2007.

[12] Stolarsky, K.B.: Power and exponential sums of digital sums related to binomial coefficient parity. SIAM J. Appl. Math. **32**: 717–730, 1977.

[13] Tenenbaum, G.: Sur la non-dérivabilité de fonctions périodiques associées à certaines fonctions sommatoires. In R. L. Graham, J. Nesetril (eds), The mathematics of Paul Erdős, Algorithms and combinatorics **13**, Springer Verlag: 117–128, 1997.

[14] Trollope, J. R.: An explicit expression for binary digital sums. Math. Mag. **41**: 21–25, 1968.

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