# GEOMETRIC AND TOPOLOGICAL PROPERTIES OF NET SETS

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Abstract. In this paper we introduce and study *net sets* and *limiting net sets*. Net sets can be constructed with the help of substitutions with *net matrices* which we also introduce here. All these notions are based on the structure of $(t, m, s)$-nets, in our particular case that of $(0, 2, 2)$-nets in base 2. First we study connectivity properties of net sets and limiting net sets, and then we find the Lebesgue measure and Hausdorff dimension of limiting net sets, which in fact are fractal sets that can also be viewed as random fractals.

## 1. Introduction

The starting point for the facts presented in this paper are mathematical objects called $(t, m, s)$-nets. These are sets of points in the unit cube in $\mathbb{R}^s$ which are useful in numerical integration methods, in particular in quasi-Monte Carlo integration. The essential property of these sets is their good distribution in the unit cube.

**Definition 1.** *Let $b \geq 2$, $s \geq 1$ and $0 \leq t \leq m$ be integers. Then a point set $P$ consisting of $b^m$ points in $[0,1)^s$ forms a $(t, m, s)$-net in base $b$, if every cuboid of the form $J = \prod_{j=1}^{s}[a_j b^{-d_j}, (a_j + 1)b^{-d_j})$ of $[0,1)^s$ with integers $d_j \geq 0$ and $0 \leq a_j < b^{d_j}$ for $1 \leq j \leq s$ and of volume $b^{t-m}$ contains exactly $b^t$ points of $P$.*

For more details regarding $(t, m, s)$-nets and uniformly distributed point sets in the unit cube we refer to Niederreiter [9, 10].
In this paper we construct fractal sets in the unit cube with the help of $(0, 2, 2)$-nets in base 2. We call these sets *limiting net sets* where

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"net" always (except Section 5) refers to $(0,2,2)$-nets in base 2. A limiting net set is defined as the limit of a decreasing sequence of *net sets*, where the sequence is constructed by iteration. The structure of net sets is given by the configuration of points of $(0,2,2)$-nets. For example, Figure 1 shows a $(0,2,2)$-net in base 2, while the left net set in Figure 2 corresponds to the $(0,2,2)$-net in Figure 1. In Section 5 we refer to net sets corresponding to $(t,m,s)$-nets in some more general cases.

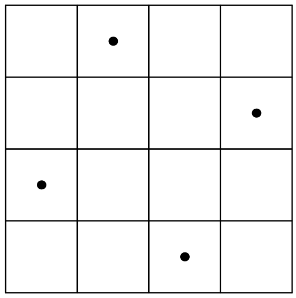

FIGURE 1. The four points of a $(0,2,2)$-net in base 2.

Now we give a short outline of this paper.

In Section 2 we start with the definitions of notions that we use along the paper: we introduce net sets and the corresponding limiting net sets. Another object and useful instrument are net matrices, $4\times 4$ matrices with entries zero or one satisfying certain conditions corresponding to the structure of $(0,2,2)$-nets in base 2. We show how one can define and construct net sets with the help of net matrices and substitutions with net matrices, which we introduce here. We also introduce some notions of connectivity for net sets and net matrices.

In Section 3 we study connectivity properties of net sets. First, we take the case of net sets defined by substitutions with connected net matrices and prove two propositions about how substitutions with connected net matrices "transmit" the strong connectivity along sequences of net sets. The main part of this section is Subsection 3.2, about properties of limiting net sets. We introduce the notion of *net connectivity* for limiting net sets, which is stronger than connectivity with respect to the topology induced by the Euclidean metric on $\mathbb{R}^2$. We prove theorems regarding the strong connectivity or connectivity (with respect to the Euclidean topology) of limiting net sets. Theorems 5 and 6 give sufficient and necessary conditions for the total disconnectivity of a uniform limiting net set. After establishing the existence of different "connectivity degrees" for limiting net sets, the question "How large is the limiting net set?" appears naturally. This is one of the main motivations of the next section.

In Section 4 we approach the fractal structure of limiting net sets. We derive, by using means from fractal geometry, the Hausdorff dimension, as well as the 1-dimensional Lebesgue measure of limiting net sets. We start with the very special case of totally uniform self-similar net sets and get then to the general case when a limiting net set is viewed as a random fractal. Of course, we could have started with the general case, but we had in view mentioning also facts about self-similar fractals and their connection with special net sets. In the context of random fractals the question about fractal percolation arises in a natural way. We define *net percolation*, which is an analogon of fractal percolation in the unit cube, and show that it occurs for special cases of limiting net sets. For details regarding fractal percolation in the cube we refer, e.g., to Falconer [5], Chayes [2] or Dekking and Meester [3]. The section ends with an open problem of net percolation depending on the probabilities of occurrence of connected and, respectively, disconnected matrices in the construction by net substitutions of the net sets that define a limiting net set.

In the last section we mention and very briefly discuss some other cases of $(t, m, s)$-nets, mainly for $s = 2$.

The results proven here provide methods for constructing random fractals with a certain type of uniformity of structure (which is given by the structure of $(t, m, s)$-nets) by using net matrices, but also for constructing percolating fractal sets and sets with certain connectivity properties.

## 2. Definitions

2.1. $(0, 2, 2)$**-nets in base** $2$ **and net sets in the unit square.** Throughout this paper we deal with the particluar case of $(0, 2, 2)$-nets in base 2. Unless specified differrently, by "net" we mean in all the considerations to follow a $(0, 2, 2)$-net in base 2. In order that the reader gets familiar with such point sets let us remind their definition.

**Definition 2.** *A point set $P$ consisting of $2^2$ points in $[0,1)^2$ forms a $(0,2,2)$-net in base $2$ in $[0,1)^2$ if every subinterval $J = [a_1\, 2^{-d_1}, (a_1 + 1)\, 2^{-d_1}) \times [a_2\, 2^{-d_2}, (a_2 + 1)\, 2^{-d_2})$ of $[0,1)^s$ with integers $d_1, d_2 \geq 0$ and $0 \leq a_j < 2^{d_j}$ for $j = 1, 2$ and of volume $2^{-2}$ contains exactly one point of $P$.*

In the following we call *net squares* the sets $[\frac{a}{4}, \frac{a+1}{4}] \times [\frac{b}{4}, \frac{b+1}{4}]$, with $a, b \in \{0, 1, 2, 3\}$. Each such square, with two of its edges subtracted (the right margins of the intervals in the Cartesian products) occurs in the definition of the $(0, 2, 2)$-nets in base 2.

The definition of $(0, 2, 2)$-nets in base 2 enables us to associate exactly one net square to each net point: if $x$ is a net point with $x \in [\frac{a}{4}, \frac{a+1}{4}) \times [\frac{b}{4}, \frac{b+1}{4})$, with $a, b \in \{0, 1, 2, 3\}$, then we associate to $x$ the net square $[\frac{a}{4}, \frac{a+1}{4}] \times [\frac{b}{4}, \frac{b+1}{4}]$. For any net we call *black net square* each net square containing a net point and *white net squares* the rest of the net squares. Thus, for a given $(0,2,2)$-net in base 2 we have 4 black net squares and 12 white squares associated to the net.

**Definition 3.** *Let $P_0$ be a $(0,2,2)$-net in base 2.*
*We call* net set of level 0 *of type $P_0$ the subset $E(P_0)$ of the unit square that is the union of the white squares associated to $P_0$. $\mathcal{E}_0$ is the family of all net sets of level 0.*
*For any integer $k \geq 1$ we inductively define a net set of level $k$:*
*We call* net set of level $k$ *any subset of a net set $E \in \mathcal{E}_{k-1}$ which is obtained by replacing all squares $[\frac{a}{2^{2k}}, \frac{a+1}{2^{2k}}] \times [\frac{b}{2^{2k}}, \frac{b+1}{2^{2k}}]$, with $a, b \in \{0, 1, \ldots 2^{2k}-1\}$, contained in $E$ by net sets of level 0 scaled by the factor $2^{-2k}$. $\mathcal{E}_k$ is the family of all net sets of level $k$.*

The following definition describes a special class of net sets.

**Definition 4.** *A net set $E$ of level $k \geq 1$ is called a* uniform net set *if at each level $j$, with $0 \leq j \leq k$, of the construction of $E$, all net squares $[\frac{a}{2^{2j}}, \frac{a+1}{2^{2j}}] \times [\frac{b}{2^{2j}}, \frac{b+1}{2^{2j}}]$, with $a, b \in \{0, 1, \ldots 2^{2j}-1\}$, are replaced by the same scaled net set.*

Each net set of level 0 is trivially uniform. Uniformity in net sets plays a role only starting with level 1. All net sets represented in Figure 3, 4, 5 and 6 are uniform.
In the following we will call for $k \geq 1$ the squares $[\frac{a}{2^{2k}}, \frac{a+1}{2^{2k}}] \times [\frac{b}{2^{2k}}, \frac{b+1}{2^{2k}}]$ occuring in Definition 3 *white $(k-1)$-squares* or *white net squares of level $k-1$*. Correspondingly, we also have *black $(k-1)$-squares* or *black squares of level $k-1$*. When we do not specify the colour of a net square we mean a white net square. Thus, by Definition 3, in the construction of a net set of level $k$ we replace every white $(k-1)$-square by a net set of level 0 scaled by the factor $2^{-2k}$.

Figure 2. Strongly connected net sets of level 0.

**Remark.** Equivalently, a net set of level 0 is obtained by "cutting out" the black net squares associated to the net and then taking the topological closure (with respect to the usual Euclidean topology) of the obtained set. Proceeding inductively as in the above definition one can define the net sets of level $k$, with $k \geq 1$ with the help of cut out black $l$-squares, with $l = 0, 1, \ldots, k$ and taking, at each step $l$, the

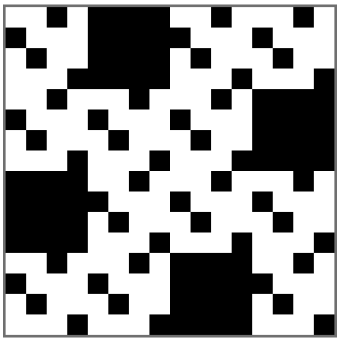

FIGURE 3. The uniform net set of level 1 based on the net sets in Figure 2.

topological closure (in the Euclidian topology).
Each net set is a compact set with respect to the topology induced by the Euclidean metric on $\mathbb{R}^2$.

**Definition 5.** *Let $E_0 \in \mathcal{E}_0$ and $\{E_k\}_{k\geq 0}$, $E_k \in \mathcal{E}_k$ for $k \geq 0$ and $E_0 \supset E_1 \supset \dots E_{k-1} \supset E_k \supset \dots$ be a decreasing sequence of net sets. We call* limiting net set *of the sequence $\{E_k\}_{k\geq 0}$ the set*

$$E_\infty := \bigcap_{k\geq 0} E_k.$$

*If the sets $E_k$ of the above sequence are uniform net sets, then $E_\infty$ is called a* uniform limiting net set.

**Remark.** Since any decreasing sequence of compact sets in $\mathbb{R}^2$ is convergent, the limiting net set of a decreasing sequence of net sets as above is always well defined, and nonempty.

**Definition 6.** *Let $E \in \mathcal{E}_k$, $k \geq 0$. Two net squares (having the same colour or different colours) of the same level $l \leq k$ occuring in $E$ are said to be neighbours (neighbouring net squares) if they share an edge.*

**Definition 7.** *A $k$-path in a set $E \in \mathcal{E}_k$, $k \geq 0$ is a finite sequence $p$ of neighbouring (white) $k$-squares. The union of the net squares of a path $p$ is called the corridor of the path and is denoted by $\Gamma(p)$.*

**Definition 8.** *Let $E \in \mathcal{E}_k$, $k \geq 0$. We say that $E$ is strongly connected if for any $k$-squares $S$ and $T$ included in $E$ there exists a $k$-path $p = p(S,T)$ in $E$ that connects $S$ and $T$.*

2.2. **Net matrices.** In the following we work with net matrices defined by $(0,2,2)$-nets in base 2.

**Definition 9.** *Let $A = (a_{i,j})_{0\leq i,j\leq 3}$ be a $4\times 4$ matrix with entries in $\{0,1\}$. We say that $A$ is a net matrix (corresponding to a $(0,2,2)$-net in base 2) if exactly 4 entries equal 1 and their distribution within the matrix satisfies the following conditions:*

(1) *in each row and each column of $A$ there is exactly one entry with value* $1$,
(2) *in each submatrix* $(a_{i,j})_{\substack{i=k,k+1\\ j=l,l+1}}$, *where* $k,l \in \{0,2\}$ *there is exactly one entry with value* $1$.

We denote by $\mathcal{A}$ the set of all net matrices.

It is easy to see that there are exactly 16 net matrices defined by $(0,2,2)$-nets in base 2.

**Example.** The connected net matrices corresponding to the net sets shown in Figure 2 are

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively.

**Remark.** It is easy to see that each net matrix corresponds to a net set of level zero: the “0” entries correspond to white net squares, and the “1” entries correspond to black net squares. Thus, Definition 6 leads in a natural way to

**Definition 10.** *Two entries of a net matrix are said to be neighbours (or neighbouring entries) if either their column indices coincide and their row indices are consecutive integers or their row indices concide and their column indices are consecutive integers.*

**Definition 11.** *A path in a net matrix is a finite sequence of neighbouring “0” entries of the matrix.*

**Definition 12.** *A net matrix is said to be connected if its set of “0” entries is connected, i.e., for any two “0” entries there exists a path in the matrix that connects them. A net matrix which is not connected is called disconnected. $\mathcal{C}$ is the set of all connected net matrices and $\mathcal{D} = \mathcal{A} \setminus \mathcal{C}$ is the set of the disconnected net matrices.*

**Remark.** The last three definitions can be extended in a natural way to any $\{0,1\}$-valued matrix.

Here we list the disconnected net matrices corresponding to $(0,2,2)$-nets in base 2:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Net matrices are a nice tool for the construction of net sets, as we shall see in the following subsection.

2.3. **Net sets and substitutions with net matrices.** Substitutions are maps which assign words (concatenations of symbols) to symbols. They are also a good instrument for describing certain self-similar sets or generalisations thereof (see, e.g., Dekking and Meester [3], Arnoux and Ito [1] )˙
Here we consider substitutions from the set of symbols $\{0,1\}$ to the set $\mathcal{A} \cup \{\mathbf{1}\}$, where $\mathbf{1}$ denotes the $4 \times 4$ matrix with all entries "1".

**Definition 13.** *We call* net substitution *any mapping* $\sigma : (\mathbb{N} \cup \{0\})^2 \times \{0,1\} \to \mathcal{A} \cup \{\mathbf{1}\}$ *satisfying the conditions* $\sigma(i,j,1) = \mathbf{1}$*, and* $\sigma(i,j,0) \in \mathcal{A}$*, for all* $i,j \in \mathbb{N} \cup \{0\}$.

If a net substitution $\sigma$ assigns to all zeroes of a $\{0,1\}$-matrix $A$ the same net matrix, say $B$, then we say that $\sigma$ is a *uniform net substitution* on $A$ with the matrix $B$.
Now we use net substitutions in order to construct net sets.
We introduce the notation $I^k_{i,j} = I^k_i \times I^k_j$, where $I^k_i = [(\frac{i}{2^{2(k+1)}}, \frac{i+1}{2^{2(k+1)}})]$, for $0 \le i,j \le 2^{2(k+1)} - 1$ and $k \ge 0$.

Let $A_0 = (a^0_{i,j})_{0 \le i,j \le 3} \in \mathcal{A}$ be a net matrix and $E_0$ the corresponding net set. We have $a_{i,j} = 0$ if and only if $I^0_{i,j} \subset E_0$ (i.e., $I^0_{i,j}$ is a white net square of $E_0$). Let $\sigma_1$ be a net substitution. By applying $\sigma_1$ to $A_0$ we get a $16 \times 16$ matrix $A_1$. Let $\sigma_2$ be a net substitution. By applying $\sigma_2$ to $A_1$ we obtain a $64 \times 64$ matrix $A_2$. Inductively, after $k$ steps we have a $2^{2(k+1)} \times 2^{2(k+1)}$ matrix

$$A_k = (a^k_{i,j}) = \begin{pmatrix} a^k_{0,2^{2(k+1)}-1} & a^k_{1,2^{2(k+1)}-1} & \cdots & a^k_{2^{2(k+1)}-1,2^{2(k+1)}-1} \\ \cdots & \cdots & \cdots & \cdots \\ a^k_{0,1} & a^k_{1,1} & \cdots & a^k_{2^{2(k+1)}-1,1} \\ a^k_{0,0} & a^k_{1,0} & \cdots & a^k_{2^{2(k+1)}-1,0} \end{pmatrix}. \tag{2.1}$$

We chose the indices of the matrix elements in this way in order to emphasise the correspondence between the matrix element $a^k_{i,j}$ and the left lower corner of the square $I^k_{i,j}$, for $k \ge 0$ and $0 \le i,j \le 2^{2(k+1)}$.

Let $\mathcal{A}_k$ denote the set of all $\{0,1\}$-valued $2^{2(k+1)} \times 2^{2(k+1)}$ matrices that can be obtained by starting with a net matrix and applying $k$ times (i.e., in $k$ steps) net substitutions.
For $k \ge 0$ and $A = (a^k_{i,j}) \in \mathcal{A}_k$ we consider the sets

$$E_k := \bigcup_{i,j=0}^{2^{2(k+1)}-1} \{I^k_{i,j} \mid a^k_{i,j} = 0\}.$$

**Remark.** It is straightforward to see that each such set $E_k$ is a net set of $\mathcal{E}_k$. Moreover, there is a one-to-one correspondence between the elements of $\mathcal{A}_k$ and the elements of $\mathcal{E}_k$. Thus sometimes we can use properties of elements of $\mathcal{A}_k$ in order to derive properties of elements of $\mathcal{E}_k$. We will call the matrices of $\mathcal{A}_k$ *net matrices of level* $k$, $k \ge 0$.

**Definition 14.** *Let* $A = (a_{i,j})_{0\leq i,j\leq 2^{2(k+1)}}$ *be a net matrix of level* $k$*. A* net substitution on the matrix $A$ *is the restriction of a net substitution* $\sigma : (\mathbb{N}\cup\{0\})^2 \times \{0,1\} \to \mathcal{A}\cup\{\mathbf{1}\}$ *to the set* $\{(i,j,a_{i,j}), 0 \leq i,j \leq 2^{2(k+1)}\}$.

Each net matrix $A_0$ of level 0 can be interpreted as the image of the $1\times 1$ matrix $(0)$ by a net substitution. Moreover, each matrix satisfying the conditions of Definition 9 is a net matrix of level zero, i.e., $\mathcal{A} = \mathcal{A}_0$. If a net substitution $\sigma$ assigns to all zeroes of a $\{0,1\}$-matrix $A$ the same net matrix, say $B$, then we say that $\sigma$ is a *uniform net substitution* on $A$ with the matrix $B$.

Definition 4 can easily be reformulated in terms of uniform net substitutions. Thus, a uniform net set $E_k = E_k(P_0,\ldots,P_k) \in \mathcal{E}_k$ corresponds to a net matrix $A_k = A_k(\sigma_0,\ldots,\sigma_k) \in \mathcal{A}_k$, where $\sigma_i$ is the uniform substitution applied at the step (level) $i$ of the construction, $0\leq i\leq k$.

**Remark.** From the definitions it follows that each strongly connected net set of level $k\geq 0$ corresponds to a connected net matrix of level $k$ and vice versa.

## 3. Connectivity properties of net sets.

3.1. **Net sets defined by substitutions with connected net matrices.** In this subsection we will show that substitutions with connected net matrices lead to net sets with nice properties.

For $k\geq 1$ let $\mathcal{A}_k^{\mathcal{C}}$ be the set of matrices of $\mathcal{A}_k$ that have the property that they can be obtained by starting with a connected net matrix and subsequently applying only substitutions with connected net matrices. For $k\geq 0$ we denote by $\mathcal{E}_k^{\mathcal{C}}$ the set of the corresponding net sets of $\mathcal{E}_k$, i.e., the set of strongly connected net sets of level $k$.

Assume that we take an arbitrary net matrix $A\in\mathcal{C}$ (corresponding to a connected net set) and apply to it an arbitrary net substitution $\sigma$ with connected net matrices. That is, each “0” entry of $A$ is replaced by some net matrix of $\mathcal{C}$, and each “1” entry by the matrix $\mathbf{1}$. We get a matrix $A_1\in\mathcal{A}_1^{\mathcal{C}}$. We analyse $A_1$ with respect to its connectivity.
Let us consider two aritrary connected net matrices $S,T\in\mathcal{C}$. We analyse the $(4\times 8)$- or $(8\times 4)$-block built by “sticking together” $S$ and $T$. Let us take, without loss of generality, the case of a $(4\times 8)$-block, with left $(4\times 4)$- submatrix $S$ and right $(4\times 4)$- submatrix $T$. By Definition 9, the fourth column of $S$ and the first column of $T$ respectively, contain exactly one entry with value 1. Therefore we can find at least two and at most three pairs of neighbouring zeroes in the $(4\times 8)$-block that have one element in the right column of $S$ and the other element in the left column of $T$. This, together with the connectivity of $S$ and $T$ implies the connectivity of the block. By

applying this argument for all blocks composed by two images through the substitution $\sigma$ of neighbouring zeroes of $A$, we obtain, since $A$ is connected, the connectivity of the net matrix $A_1$. As $A$ and $\sigma$ have been chosen arbitrarily, it follows by the definition of $\mathcal{A}_k^{\mathcal{C}}$, $k \geq 1$, that all matrices of $\mathcal{A}_1^{\mathcal{C}}$ are connected.

**Example.** A connected block constructed with two connected net matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

In particular, by the one-to-one correspondence of the elements of $\mathcal{A}_k^{\mathcal{C}}$ and $\mathcal{E}_k^{\mathcal{C}}$, for $k \geq 1$, we obtain, in the case $k = 1$, that each element of $\mathcal{E}_1^{\mathcal{C}}$ is a strongly connected net set.

Now we proceed inductively. Assume that all elements of $\mathcal{A}_k^{\mathcal{C}}$ are connected matrices (and, equivalently, all elements of $\mathcal{E}_k^{\mathcal{C}}$ are strongly connected). We take an arbitrary $A_k \in \mathcal{A}_k^{\mathcal{C}}$ and apply to it a net substitution $\tau$ which assigns to each "0" entry of $A_k$ some connected net matrix, such that distinct zeroes of $A_k$ may be mapped into distinct connected net matrices. We get a matrix $A_{k+1} \in \mathcal{A}_{k+1}$. As $A_k$ is connected, we deduce, by the same argument as for the connectivity of $A_1$ above, that $A_{k+1}$ is connected. By the definition of $\mathcal{A}_{k+1}^{\mathcal{C}}$ it immediately follows that all matrices of $\mathcal{A}_{k+1}^{\mathcal{C}}$ are connected, and, equivalently, all net sets of $\mathcal{E}_{k+1}^{\mathcal{C}}$ are strongly connected.

Thus we have inductively proven the following

**Proposition 1.** *Let $E_0 \in \mathcal{E}_0^{\mathcal{C}}$ be a strongly connected net set with corresponding net matrix $A_0$. By applying substitutions with connected net matrices we construct the sequence of net matrices $A_1$, $A_2, \ldots,$ with $A_k \in \mathcal{A}_k$, $k \geq 1$. For every $k \geq 1$ let $E_k$ be the net set of level $k$ defined by $A_k$. Then $E_k$ is strongly connected, for all $k \geq 0$.*

**Proposition 2.** *Let $\{E_k\}_{k\geq 0}$ be a decreasing sequence of net sets and $n \geq 0$ an arbitrary integer. If $E_{n+1}$ ist strongly connected, then $E_n$ also is strongly connected.*

*Proof.* Let $S_n, T_n \subset E_n$ be two arbitrary net squares of level $n$. Then one can choose two arbitrary $(n+1)$-squares, $S, T \subset E_{n+1}$, such that $S \subset S_n$ and $T \subset T_n$. By the strong connectivity of $E_{n+1}$, there exists an $(n+1)$-path $p_{n+1} = p_{n+1}(S, T)$ that connects the net squares $S$ and $T$. Then, by the construction of net sets, there exists a finite sequence of net squares of level $n$, $(B_i)_{i\in\mathcal{J}}$, where $\mathcal{J}$ is a set of indices depending on $S$ and $T$, such that $\Gamma(p_{n+1}) \subset \bigcup_{i\in\mathcal{J}} B_i$. The sequence $(B_i)_{i\in\mathcal{J}}$ is a $n$-path in $E_n$ connecting $S_n$ and $T_n$. $\square$

3.2. **Properties of limiting net sets.** In the following we analyse connectivity properties of limiting net sets.

**Definition 15.** *The limiting set $E_\infty$ of a decreasing sequence of net sets $\{E_k\}_{k\geq 0}$ is net-connected if for each two points $x, y \in E_\infty$ and for all $k \geq 0$ there exist two $k$-squares $S_k, T_k \subset E_k$ with $x \in S_k$ and $y \in T_k$ and a $k$-path in $E_k$ that connects $S_k$ and $T_k$.*

Proposition 1 leads us to the following result.

**Theorem 1.** *Let $\{E_k\}_{k\geq 0}$, be a decreasing sequence of strongly connected net sets, $E_k \in \mathcal{E}_k^{\mathcal{C}}$, and $E_\infty$ the corresponding limiting net set. Then $E_\infty$ is connected with respect to the topology induced by the Euclidean metric on $\mathbb{R}^2$.*

*Proof.* Let $x, y \in E_\infty$ be two arbitrary points of the limiting set. Then, by the definitions of $E_\infty$ and $E_k$, $k \geq 0$, there exist two decreasing sequences of net squares, $\{S_k\}_{k\geq 0}$ and $\{T_k\}_{k\geq 0}$, where $k$ indicates the level of the square $S_k$ and $T_k$, such that

$$x \in \bigcap_{k\geq 0} S_k \text{ and } y \in \bigcap_{k\geq 0} T_k.$$

By the strong connectivity of $E_k$, there exists for every $k \geq 0$ a $k$-path $p_k$ in $E_k$ that connects the net squares $S_k$ and $T_k$. Moreover, $\Gamma(p_k)$ is a compact and connected set, for all $k \geq 0$. We have the decreasing sequence of connected sets (with respect to the topology induced by the Euclidean metric on $\mathbb{R}^2$)

$$\Gamma(p_0) \supset \Gamma(p_1) \supset \cdots \supset \Gamma(p_k) \subset \Gamma(p_{k+1}) \supset \ldots,$$

and $x, y \in \Gamma(p_k)$, for all $k \geq 0$. Thus $x, y \in \bigcap_{k\geq 0} \Gamma(p_k)$, which is a connected set. This completes the proof. □

**Proposition 3.** *Let $\{E_k\}_{k\geq 0}$, be a decreasing sequence of strongly connected net sets, $E_k \in \mathcal{E}_k^{\mathcal{C}}$, and $E_\infty$ the corresponding limiting net set. Then $E_\infty$ is locally connected with respect to the topology induced by the Euclidean metric on $\mathbb{R}^2$.*

*Proof.* Let $k \geq 0$ be arbitrarily fixed and $S_k$ be a net square occuring in $E_k$. Let $(E'_l)_{l\geq 0}$ be the decreasing sequence of net sets defined by the condition $E'_l := \phi_k(E_{k+l})$, where $\phi_k$ is a properly chosen similarity of factor $2^{2(k+1)}$ and $E'_\infty$ the limiting netset of $(E'_l)_{l\geq 0}$. Then $S_k \cap E_\infty = \psi_k(E'_\infty)$, where $\psi_k$ is a similarity of factor $2^{-2(k+1)}$ , i.e., the set $S_k \cap E_\infty$ can be viewed as the image by a similarity of the limiting net set of a decreasing sequence of net sets. Then, by Theorem 1 and the continuity of similarities, $S_k \cap E_\infty$ is connected. Let now $\varepsilon > 0$ be some arbitrarily chosen real number, and $k \geq 0$ such that $2^{-2(k+1)}\sqrt{2} < \varepsilon$. Then each $k$-square $S_k$ of $E_k$ has diamenter (with respect to the Euclidian metric) less than $\varepsilon$ and, as $E_\infty = \cup_{S\in\mathcal{S}_k}(S \cap E_\infty)$, where $\mathcal{S}_k = \{S : S \text{ is a } k\text{ -square of } E_k\}$, it follows that the limiting net set $E_\infty$ is a continuum that can be written as a finite union of connected sets of diameter less than $\varepsilon$. As $\varepsilon$ was chosen arbitrarily, this implies, by a theorem of Hahn-Mazurkiewicz-Sierpiński [7, Theorem2, p. 256], that $E_\infty$ is locally connected. □

Hata [6] proved connectivity properties of self-similar sets by using arguments some of which are related to those used in the proofs of the last two results.

Proposition 3 and the mentioned Hahn-Mazurkiewicz-Sierpiński theorem yield

**Corollary 1.** *The limiting net set $E_\infty$ of a decreasing sequence of strongly connected net sets is a continuous image of an interval. In particular, $E_\infty$ is arcwise connected.*

**Remark.** Using similar arguments as above, one can easily prove that every strongly connected net set of some level $k \geq 0$ is locally connected and arcwise connected.

In the considerations to follow we will restrict ourselves to studying properties of connectivity and net-connectivity of limiting net sets. The first part of the proof of Theorem 1 immediately leads to

**Theorem 2.** *Let $\{E_k\}_{k\geq 0}$, be a decreasing sequence of strongly connected net sets, $E_k \in \mathcal{E}_k^{\mathcal{C}}$, and $E_\infty$ the corresponding limiting net set. Then $E_\infty$ is net-connected.*

**Theorem 3.** *Let $E_\infty$ be the limiting net set of a decreasing sequence of net sets. If $E_\infty$ is net-connected then it is also connected with respect to the canonical topology of the Euclidean plane.*

*Proof.* The affirmation follows immediately from the definition of $E_\infty$, of net-connectivity and connectivity with respect to the topology induced by the Euclidean metric on $\mathbb{R}^2$. □

For simplicity, in the following, when talking about (limiting) net sets, by *connected* we mean *connected with respect to the topology induced by the Euclidean metric.*

**Remark.** In general the limiting set of a decreasing sequence of net sets is not necessarily net-connected. An example illustrating this is the case when at a certain level $k_0$ of the construction of the sequence $\{E_k\}_{k\geq 0}$ the matrix $A_{k_0}$ corresponding to the net set $E_{k_0}$ contains a submatrix of the form shown below

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This occurs when at the corresponding step $k_0$ of the construction of the sequence $\{E_k\}_{k\geq 0}$ there has been applied a net substitution that replaces certain “0” entries of the net matrix $A_{k_0-1}$ by a disconnected net matrix. In fact, if the above block is a submatrix of $A_{k_0}$ that contains one of the elemets $a_{0,2^{2(k_0+1)-1}}$ or $a_{2^{2(k_0+1)-1},0}$ of $A_{k_0-1}$ or, e.g., if it is just a submatrix of $A_{k_0}$, boarded from above by a row $(1\ 1\ 1\ 1\ t_1\ t_2\ t_3\ t_4)$ and from the left by a column $(1\ 1\ 1\ 1\ y_1\ y_2\ y_3\ y_4)^T$, $t_i, y_i \in \{0,1\}$, $i = 1,2,3,4$, then we have in $A_{k_0}$ a set of “0” entries which cannot be connected by a path to “0” entries outside this set. Thus $A_{k_0}$ is not connected and the corresponding net set $E_{k_0}$ is not strongly connected.

Are the reciprocal affirmations of the last three theorems also true? The reciprocal of Theorem 1 does not hold, i.e., the fact that the limiting net set of a decreasing net sequence $\{E_k\}_{k\geq 0}$ is connected does not necessarily imply that $E_k$ is strongly connected, for all $k \geq 0$. This is shown, e.g., by the following simple counterexample. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We construct a decreasing sequence of net sets by substitutions as follows. Let $E_0$ be the net set corresponding to the net matrix $A$. $E_0$ is connected, but not strongly connected. Let now $A_1$ be the net matrix that we obtain by applying a uniform net substitution that replaces all zeroes in $A_0$ by the matrix $B$, and let $E_1$ be the corresponding net set of level 1. If we continue the construction with net substitutions, replacing at each level $k \geq 1$ all zeroes of $A_k$ by the matrix $B$, then the limiting net set $E_\infty$ is connected, but none of the sets $E_k$, $k \geq 0$ is strongly connected. Still, $E_k$, for $k \geq 1$, is connected. Let us explain this is a more general case. Suppose we take an arbitrary disconnected matrix $A$ at the first step ($A_0 = A$) and then at the next steps uniform net substitutions with a connected net matrix $B$, chosen with the property: if $(i,j)$ is the corner position of an “1” entry in $A$, then the entries of $B$ at the positions $(i,j)$ and $(3-i, 3-j)$ are zeroes. This property does not let $E_k$, for $k \geq 1$, get disconnected: if the intersection of two net squares of certain level is a point, then this point remains in the net set at the next level.
Let us now prove the reciprocal of Theorem 2.

**Theorem 4.** *Let $\{E_k\}_{k\geq 0}$, be a decreasing sequence of net sets, and $E_\infty$ the corresponding limiting net set. If $E_\infty$ is net-connected then $E_k$ is strongly connected, for every $k \geq 0$.*

*Proof.* Let $E_\infty$ be as above. Suppose there exists a $k \geq 0$ such that $E_k$ is not strongly connected. Then, by the definition of strong connectivity, there exist two distinct arbitrary net squares of level $k$, $S_k, T_k \subset E_k$

which are not connected by any $k$-path in $E_k$. Let now $s \in \mathrm{Int}(S_k) \cap E_\infty$ and $t \in \mathrm{Int}(T_k) \cap E_\infty$ be two points of the limiting net set, where by Int we denote the interior with respect to the topology induced by Euclidean metric on $\mathbb{R}^2$. By the net-connectivity of $E_\infty$ there exist two $(k+1)$-squares $S, T \subset E_{k+1}$, with $s \in S$ and $t \in T$, such that there is a $(k+1)$-path in $E_{k+1}$, $p_{k+1}$, connecting $S$ and $T$. Let us consider a finite sequence of net squares of level $k$, $(B_i)_{i \in \mathcal{J}}$, where $\mathcal{J}$ is a set of indices depending on $S$ and $T$, such that $\Gamma(p_{k+1}) \subset \bigcup_{i \in \mathcal{J}} B_i$. It is easy to see that such a sequence exists and provides a $k$-path which connects $S_k$ and $T_k$ in $E_k$. This contradicts the assumption that $E_k$ is not strongly connected. □

It is easy to show that the reciprocal of Theorem 3 does not hold. Suppose the opposite. Let $E_\infty$ be the limiting net set of a decreasing sequence of net sets, such that $E_\infty$ is connected. Assume that the reciprocal of Theorem 3 holds. This would imply that $E_\infty$ is also net-connected. Then by Theorem 4 the set $E_k$ is strongly connected, for all $k \geq 0$. This would imply that the reciprocal of Theorem 1 holds, which, as it was already shown, is false.

**Example.** It is not difficult to see that consecutively applying substitutions with the same disconnected net matrix lets the diagonals of the net squares unchanged, but "splits" the set: at each second step where we apply such a substitution in the construction of the sequence $\{E_k\}$, the number of connected components of $E_k$ increases. Thus $E_\infty$ is disconnected. Actually the disconnectivity of $E_\infty$ is already provided after applying a uniform net substitution with a disconnected net matrix $A$ followed by a (not necessarily uniform) net substitution which replaces each zero by some net matrix $B$ that satisfies the following condition: if $(i, j)$ is the "corner" position of a "1" entry in $A$, then $B$ has a "1" entry in at least on of the positions $(i, j)$ and $(3 - i, 3 - j)$ (see also Figure 4). One can find even weaker conditions that provide the disconnectivity of a net set of some level $k$ and thus also of $E_\infty$.

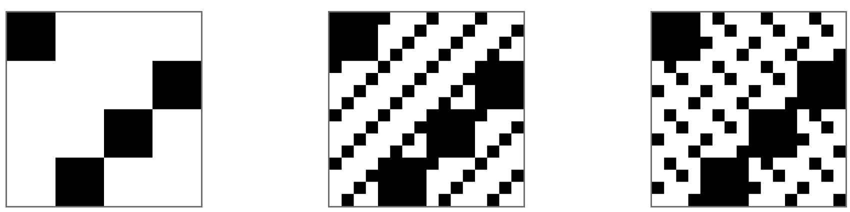

FIGURE 4. Construction of disconnected net sets of level 1.

Let us now consider the special situation when in the construction of the decreasing sequence of net sets we apply, starting with a level $k$, with $k \geq 1$, only uniform substitutions with one and the same disconnected matrix at all levels $k+1$, $k+2$,.... Thus all (white) net squares existing at level $k$ are "cut" into "strips" getting at each level thinner, separated by black squares (see also Figure 5 and 6). $E_\infty$ is again disconnected. Later on we give sufficient and necessary conditions for the limiting net set to be disconnected.

**Conclusion.** There are the following "degrees" of connectivity that a limiting net set can have:
(a) net-connected,
(b) connected,
(c) disconnected,
(d) totally disconnected,
where (a) and (d) are particular cases of (b), and (c) respectively.

The case when the limiting net set is totally disconnected is approached in the considerations to follow.

**Definition 16.** *Let* $A = (a_{i,j})$, *with* $0 \leq i, j \leq 3$ *be a net matrix.* $A$ *is* of type I *if* $(a_{0,3}-1)(a_{3,0}-1) = 0$. $A$ *is* of type II *if* $(a_{0,0}-1)(a_{3,3}-1) = 0$. *We say that two net matrices are from different families when they are of different type.*

Each disconnected net matrix is either of type $I$ or of type $II$. Some of the connected net matrices are neither of type $I$ nor of type $II$.

**Remark.** Let $(E_k)_{k\geq 0}$ be a decreasing sequence of uniform net sets. Consider some integers $l \geq 0$, $0 \leq m < n$ and $A(m), A(n)$ two (not necessarily distinct) disconnected net matrices of same type (say $a \in \{I, II\}$) occurring at the $m$-th and $n$-th step of the construction of the sequence by uniform net substitutions, respectively. Let us assume that $A(m)$ is the $(2l+1)$-th matrix of type $a$ that occurs in the construction of the sequence and $A(n)$ be the $(2l+2)$-th matrix of type $a$ that occurs in the construction of the sequence. Then after applying the uniform net substitution with the matrix $A(n)$ to the net matrix $A_{n-1}$ of level $n-1$ corresponding to the net set $E_{n-1}$ there are created new "strips" containing only "1" entries that "cross" the net matrix $A_n$ of level $n$ that corresponds to $E_n$. For example, the matrix

$$\begin{pmatrix}
\cdots & \cdots & \cdots & & & & & \cdots \\
& 1 & 1 & & & & & \\
& & 1 & 1 & & & & \\
\cdots & & & 1 & 1 & & & \cdots \\
& & & & 1 & 1 & & \\
& & & & & 1 & 1 & \\
& & & & & & & \\
\cdots & \cdots & & & & & \cdots & \cdots
\end{pmatrix}$$

corresponds to the case when $A(m)$ and $A(n)$ are disconnected matrices of type $II$. Here "strips" constisting only of "1" entries separate regions of $E_n$ consisting only of "0" entries. Correspondingly, the union of the black $n$-squares corresponding to the "1" entries in these "strips" separate the net set $E_n$ into connected components that occur at level $n$ as strict subsets of the connected components of $E_{n-1}$. Consequently, these "black strips" in $E_n$ also separate connected components of $E_\infty$. The above remark is useful in the proofs of the theorems to follow.

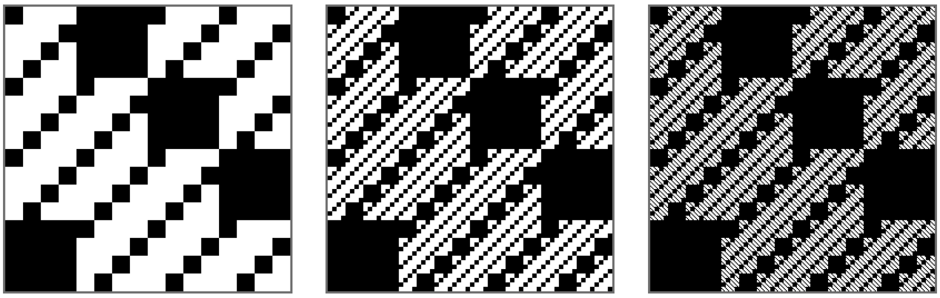

FIGURE 5. Construction of a uniform net set of level 3 by net substitutions with the four disconnected matrices.

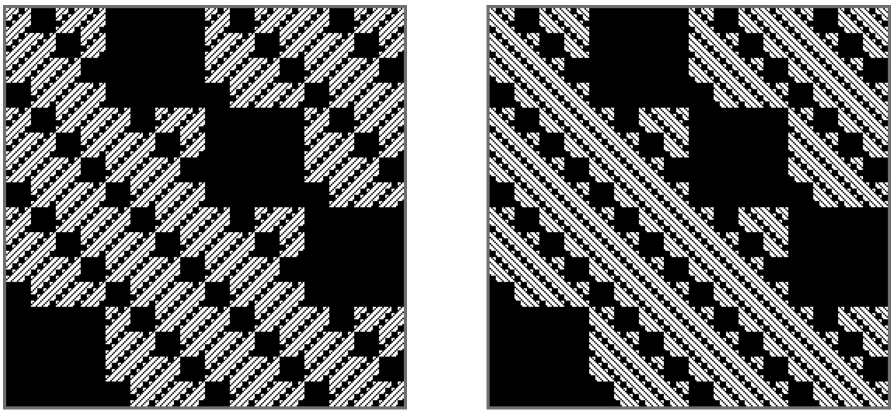

FIGURE 6. The uniform net set of level 3 is constructed by net substitutions with two disconnected matrices from different families, and with one and the same disconnected matrix, respectively.

**Proposition 4.** *Let $E_\infty$ be the limiting net set of a decreasing sequence of uniform net sets $\{E_k\}_{k\geq 0}$ and, for $k \geq 0$, let $A_0, A_1, \ldots, A_k$ denote the sequence of net matrices that defines the set $E_k$. For every $k \geq 0$ let $m_1(k)$ denote the number of disconnected matrices of type I, and $m_2(k)$ the number of disconnected matrices of type II that occur in the finite sequence $A_0, A_1, \ldots, A_k$. If $A_k$ is a disconnected matrix for all $k \geq 0$, and*

$$\lim_{k\to\infty} m_1(k) = \infty \textit{ and } \lim_{k\to\infty} m_2(k) = \infty, \tag{3.1}$$

*then $E_\infty$ is totally disconnected with respect to the topology induced by the Euclidean metric on $\mathbb{R}^2$.*

*Proof.* Let $x, y \in E_\infty$ be two distinct points of the uniform limiting net set $E_\infty$ mentioned in the theorem. Let $d = d(x, y) > 0$ be the Euclidean distance between $x$ and $y$. As $x \in E_\infty$, it follows that $x$ lies in a connected component of $E_k$, for all $k \geq 0$. We consider the open ball $B(x, \frac{d}{2})$ (with respect to the Euclidean metric), centered at $x$ and with radius $\frac{d}{2}$. As $y \notin B(x, \frac{d}{2})$, it is enough to show that there exists a connected component of $E_\infty$ which contains $x$ and is contained in $B(x, \frac{d}{2})$.
Let $j, l \geq 0$, $0 \leq q_1 < q_2$ and $0 \leq n_1 < n_2$ be integers, $A(q_1), A(q_2)$ disconnected matrices of type $I$, and $A(n_1)$, $A(n_2)$ disconnected matrices such that $A(i)$ occurs in the last step of the construction of the net set $E_i$, for $i \in \{q_1, q_2, n_1, n_2\}$. Suppose that $l$ and $j$ have the property that at the levels $q_1$ and $q_2$ respectively we have the $(2j+1)$-th and $(2j+2)$-th occurance of a disconnected matrix of type $I$ (not necessarily the same), and that at the levels $n_1$ and $n_2$ we have the $(2l + 1)$-th and $(2l + 2)$-th, respectively, occurrence of a disconnected matrix of type $II$ in the construction of the sequence of net sets defining $E_\infty$. Let $h := \max\{q_2, n_2\}$, and $g := \min\{q_1, n_1\} - 1$, if $q_1 \cdot n_1 \neq 0$ and $g := 0$ if $q_1 \cdot n_1 = 0$. Let $\delta(E_h)$ and $\delta(E_g)$ be the diameter of the largest connected component of $E_h$ and of $E_g$, respectively. Then $\delta(E_h) < \frac{3}{4} \cdot \delta(E_g)$, where the constant $\frac{3}{4}$ can actually be replaced by a smaller one, close to $\frac{1}{2}$ (see also Figure 5 and 6). As the optimality of this constant does not change the essence of things, we will not get into more detail here.
Let now $t > 0$ and let $E(t)$ be some net set obtained after having already applied $t_1 \geq 2t$ times substitutions with net matrices of type $I$ and $t_2 \geq 2t$ times substitutions with net matrices of type $II$ in the construction of the net set sequence, where at least one of the numbers $t_1, t_2$ equals $2t$. Of course, $E(t)$ is not uniquely determined by these properties, but the existence of such a set is sufficient for our argument. The diameter $\delta(E(t))$ of the largest connected component of $E(t)$ satisfies the inequality $\delta(E(t)) \leq (\frac{3}{4})^t \cdot \delta(E_0) = \sqrt{2} \cdot (\frac{3}{4})^t$. Hence, for $t$ chosen large enough, $\delta(E(t)) \leq \frac{d}{2}$ and thus the connected component of $E(t)$ which contains $x$ is contained in $B(x, \frac{d}{2})$. Consequently the connected component of $E_\infty$ which contains $x$ is contained in $B(x, \frac{d}{2})$. □

Based on the ideas of the above proof, and taking into account that the substitutions with connected net matrices do not "cut" already existing connected components into smaller connected components (see Theorem 1), one can prove the following more general result.

**Theorem 5.** *Let $E_\infty$ be the limiting net set of a decreasing sequence of uniform net sets $\{E_k\}_{k \geq 0}$ and, for $k \geq 0$, let $A_0, A_1, \ldots, A_k$ denote the sequence of net matrices that defines the set $E_k$. For every $k \geq 0$ let*

*$m_1(k)$ denote the number of disconnected matrices of type I, and $m_2(k)$ the number of disconnected matrices of type II that occur in the finite sequence $A_0, A_1, \dots, A_k$. If*

$$\lim_{k\to\infty} m_1(k) = \infty \text{ and } \lim_{k\to\infty} m_2(k) = \infty,$$

*then $E_\infty$ is totally disconnected with respect to the topology induced by the Euclidean metric on $\mathbb{R}^2$.*

Suppose we are in the setting of Theorem 5. Then we have the following

**Theorem 6.** *Condition* (3.1) *is a necessary condition for a uniform limiting net set $E_\infty$ to be totally disconnected.*

*Proof.* Let $E_\infty$ be a totally disconnected uniform limiting net set. Then, by the facts already shown, the infinite sequence of net matrices $A_0, A_1, \dots, A_k, \dots$ contains at least two disconnected matrices of the same type.
From the configuration of the disconnected net matrices one can infere that the connsecutive occurences of uniform net substitutions with disconnected net matrices of the same type in the construction of the sequence $\{E_k\}_{k\geq 0}$ does not change the length of the diameter of the largest connected component of the net set. This is based on the fact that disconnected net matrices have only zero entries on exactly one of their diagonals, where the diagonal with this property is the same for disconnected net matrices of the same type.
The idea of constructing a totally disconnected uniform limiting net set is to reduce, along the construction of the sequence of net sets defining $E_\infty$, the diameters of the connected components, such that $\lim_{k\to\infty} \delta(E_k) = 0$, where $\delta(E_k)$ denotes the diameter of the largest connected component of $E_k$, for $k \geq 0$. See also Figure 6. □

With the same notations as in the above theorems we state two more results, whose proofs we omit, since they are based on facts that have already been mentioned here.

**Theorem 7.** *A uniform limiting net set is disconnected, but not totally disconnected, if and only if there exists a constant $M \geq 0$ and an index $j \in \{1, 2\}$ such that $m_j(k) < M$ for all $k \geq 0$, and one of the following conditions is satisfied:*

(1) *there exists an integer $k_0 \geq 1$ such that*

$$\big(m_1(k_0) - 2\big)\big(m_2(k_0) - 2\big) = 0,$$

(2) *there exists an integer $k_0 \geq 0$ such that $k_0$ is the first step in the construction of the sequence $\{E_k\}_{k\geq 0}$ when a uniform net substitution with a disconnected matrix occurs, say $A = (a_{i,j})_{i,j\in\{0,\dots,3\}}$, and an $l \geq 1$ such that at step $k_0 + l$ a uniform*

*net substitution with a matrix* $B = (b_{i,j})_{i,j\in\{0,\dots,3\}}$ *is applied, where* $B$ *has the same type as* $A$.

Let $A_0, \dots A_k, \dots$ denote the sequence of net matrices occurring in the construction of a sequence of uniform net sets $\dots \subset E_k \subset \dots \subset E_0$, such that $A_k$ occurs at step $k$ of the construction. We denote by $m_1'(k)$ the number of matrices $A_l$ of type $I$, and $m_2'(k)$ the number of matrices $A_l$ of type $II$, with $l \geq k$.

**Theorem 8.** *A uniform limiting net set is connected (in the usual Euclidean sense) but not net-connected if and only if there exists an integer* $k_0$ *such that* $m_1(k_0) + m_2(k_0) \geq 1$ *and if for some* $k \geq 0$ *and some* $j \in \{1, 2\}$ *we have* $m_j(k) = 1$ *then* $m_j'(n) = 1$*, for all* $n \geq k$.

### 3.3. How "large" is $E_\infty$ ?

**Proposition 5.** *Let* $\{E_k\}_{k\geq 0}$ *be a decreasing sequence of net sets, with limiting set* $E_\infty$. *Then the* 2*-dimensional Lebesgue measure of* $E_\infty$ *is zero.*

*Proof.* Obviously, $\lambda^2(E_0) = \frac{3}{4}$. At each step of the construction of a net set we "cut out" of each (white) square of side length $2^{-j}$ exactly four (black) $(j+1)$-squares in order to obtain a net set of level $j+1$, hence we inductively get $\lambda^2(E_k) := (\frac{3}{4})^{-(k+1)}$, for $k \geq 0$. Passing to the limit for $k \to \infty$ completes the proof. □

In the next section we will show, by using techniques of fractal geometry, that the Lebesgue measure of dimension 1 of $E_\infty$ is $\infty$.

## 4. NET SET FRACTALS

Since in the following we study limiting net sets by means of fractal geometry, we first give a few definitions.
Let $|\cdot|$ denote the Euclidean norm in the space $\mathbb{R}^n$, and, for a subset $U \subset \mathbb{R}^n$, let $|U|$ denote the diameter of $U$ with respect to the Euclidean metric. A *$\delta$-cover* of a set $F \subset \mathbb{R}^n$ is a countable (or finite) collection of sets $\{U_i\}$ of diameter at most $\delta$, with $F \subset \bigcup_{i=1}^\infty U_i$. For $F \subset \mathbb{R}^n$ and $s \geq 0$ we can define for any $\delta > 0$

$$\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s \ : \ \{U_i\} \text{ is a } \delta\text{-cover of } F \right\} \tag{4.1}$$

and, subsequently, the *$s-$dimensional Hausdorff measure* of $F$ by

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(F).$$

The above limit exists for every Borel set $F$.
$\mathcal{H}^s$ is a monotonically decreasing function in $s$ and there exists a unique value of $s$ for which $\mathcal{H}^s$ "jumps" from $\infty$ to 0, namely the *Hausdorff dimension* of $F$

$$\dim_H F = \inf\{s \ : \ \mathcal{H}^s(F) = 0\} = \sup\{s \ : \ \mathcal{H}^s(F) = \infty\}.$$

A set $F \subset \mathbb{R}^n$ is called a *fractal set* if its Hausdorff dimension is larger than its topological dimension, which is always an integer.

**Definition 17.** *Let $F$ be a subset of $\mathbb{R}^n$. Then the box counting dimension of $F$ is given, if the following limit exists, by*

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}, \tag{4.2}$$

*where $N_\delta(F)$ denotes the number of $\delta$-mesh cubes that intersect $F$.*

As we will see, limiting net sets are actually fractals, and thus we call them *net set fractals* when we wish to emphasise their fractal structure. Here the word "net" refers to the fact that the construction of these sets is based on the structure of $(t, m, s)$-nets, and here in particular on $(0, 2, 2)$-nets in base 2. On the other hand, in literature (see, e.g., Falconer [4]) one can find the notion of "net fractals", where the word "net" does not refer to $(t, m, s)$-nets, but to a construction of fractal sets based on trees, closely related to net measures (see Falconer [4, Sections 2 and 7], and [5]). Allthough at *net set fractals* the word *net* has the about mentioned meaning, it is worth remarking that one still can see net set fractals as a particular case of the *net fractals* studied by Falconer [4].

For further details regarding fractals, iterated functions systems and fractals defined by these, the Hausdorff and box counting dimension and the Hausdorff measure we refer, e.g., to Falconer [5].

4.1. **Totally uniform net set fractals. Measures of net set fractals.** A uniform net sequence and the corresponding limiting net set are called *totally uniform* if throughout the construction of the net sets $E_0, E_1, \dots, E_k, \dots$ at all steps the same uniform net substitution is applied, i.e., all zeroes are replaced, at each arbitrary step, and on the other hand at all consecutive steps, by the same net matrix.

**Proposition 6.** *Let $\sigma$ be a uniform net substitution and $E_\infty(\sigma)$ the (totally uniform) limiting net set of the decreasing sequence of totally uniform net sets constructed with $\sigma$. Then $E_\infty(\sigma)$ is a self-similar fractal with Hausdorff and box counting dimensions $\dim_H(E_\infty(\sigma)) = \dim_B(E_\infty(\sigma)) = 1 + \log 3/\log 4$.*

*Proof.* Let $A = (a_{i,j})_{0 \le i,j \le 3}$ be a net matrix. Then applying a uniform substitution $\sigma$ with the matrix $A$ to a net matrix $A_k$ of level $k$, $k \ge 0$, is equivalent to applying to the net set $E_k$ that corresponds to $A_k$ the similarities $(\phi^A_{i,j})_{(i,j) \in J_A}$, where $J_A = \{(i,j) \in \{0, \dots, 3\}^2 \mid a_{i,j} = 0\}$ and the similarities (which are in particular contractions) are defined by the relations:
$\phi^A_{i,j} : [0,1]^2 \to [0,1]^2$, $\phi^A_{i,j}(x,y) := (\frac{x+i}{4}, \frac{y+j}{4})$, for all $(i,j) \in J_A$. Then the totally uniform limiting net set $E_\infty(\sigma)$ is the attractor of the iterated functions system $(\phi^A_{i,j})_{(i,j) \in J_A}$. By a theorem on the Hausdorff and

box counting dimensions of a self-similar set defined by a set of similarities verifying the open set condition (see, e.g. Falconer [5, Theorem 9.3]) we obtain the aimed result. □

In the following we call (totally) uniform limiting net sets also *(totally) uniform net set fractals.*
Proposition 6 leads us to the following result.

**Corollary 2.** *Let $E_\infty(\sigma)$ be the limiting net set of a sequence of totally uniform net sets, and $\lambda^1$ the 1-dimensional Lebesgue measure. Then $\lambda^1(E_\infty(\sigma)) = \infty$.*

*Proof.* By Proposition 6, we have $1 < \dim_H(E_\infty(\sigma)) < 2$. The definition of the Hausdorff dimension implies that $\mathcal{H}^1(E_\infty(\sigma)) = \infty$ and $\mathcal{H}^2(E_\infty(\sigma)) = 0$ (which we have already proven for the general case of limiting net sets). It is known that for any Borel set $F \subset \mathbb{R}^n$ we have $\mathcal{H}^n(F) = c_n \lambda^n(F)$, where $c_n$ is a constant depending only on $n$ (see, e.g., Falconer [5, Chapter 2]). Taking $n = 1$ in the previous relation completes the proof. □

Our next aim is to verify whether the last two results also hold for the limiting net set $E_\infty$ in the general case, when the net sets of the decreasing sequence $\{E_k\}_{k\geq 0}$ that defines $E_\infty$ are constructed by applying arbitrary net substitutions.

**Proposition 7.** *Let $E_\infty$ be the limiting net set of a decreasing sequence of net sets $\{E_k\}_{k\geq 0}$. Then $E_\infty$ is a fractal set with*

$$1 \leq \dim_H(E_\infty) \leq \dim_B(E_\infty) = 1 + \log 3/\log 4. \tag{4.3}$$

*Proof.* We take $\delta_k := \frac{1}{4^k}$ in (4.2) and $s := \lim_{k\to\infty} \frac{\log N_{\delta_k}(E_\infty)}{-\log \delta_k}$. Since $N_{\delta_k}(E_\infty) = 12^k$, we get $s = 1 + \log 3/\log 4$, which implies $\dim_B(E_\infty) = 1 + \log 3/\log 4$ (see Falconer [5, Chapter 3]). The second inequality in (4.3) is a wellknown relation between the two dimensions (see Falconer [5, Chapter 2]). We prove the first inequality. Assume $\dim_H(E_\infty) < 1$. This would imply, by the definition of Hausdorff dimension, that $\mathcal{H}^1(E_\infty) = 0$ and consequently (by an argument already mentioned above) $\lambda^1(E_\infty) = 0$. In particular, this would also hold for the special case when $E_\infty$ is a totally uniform limiting net set, in contradiction with Corollary 2. □

Unfortunately Proposition 7 does not provide the value of $\dim_H(E_\infty)$ or $\lambda^1(E_\infty)$. In order to find the Hausdorff dimension of a limiting net set in the general case, we use other techniques, as it will be shown in the following considerations.

4.2. **Net set fractals as random fractals. Net percolation.** In the previous subsection we have encountered net set fractals and among them self-similar fractals, namely the totally uniform limiting net sets.

Limiting net sets, as we have defined them, are also a species of random fractals.

In the case of random fractals the self-similarity property is often replaced by "statistical self-similarity", in the sense that enlargements of small parts have the same statistical distribution as the whole set. Random fractals look rather irregular and are, thanks to their non-uniform appearance, closer to natural objects such as coastlines, topographical surfaces, cloud boundaries. For more details and examples regarding random fractals we refer, e.g., to Falconer [4, 5].

By their definition, net set fractals show a certain form of "statistical self-similarity". In fact, it can be checked that they are a particular case of *random net fractals*, introduced by Falconer [4], and thus can be regarded as random fractals. This will provide tools for the computation of the Hausdorff dimension of net set fractals.
In the following we approach net set fractals as a species of random net fractals. Since at each step of the construction by means of net substitutions, each zero entry of the corresponding net matrix is replaced by a net matrix "with no preferences", we may assume that each of the 16 net matrices occurs, independently, with the same probability $\frac{1}{16}$, each time when we apply a net substitution. This approach is somehow natural and at hand.
Falconer [4] has proven several results regarding the Hausdorff dimension of random net fractals. The following theorem can be obtained as a direct consequence of a theorem proven by Falconer [4, Corollary 8.6].

**Theorem 9.** *Let $E_\infty$ be a net set fractal. If at each step of the construction of the decreasing sequence of net sets defining $E_\infty$ each zero entry is replaced by a net matrix, such that each of the* 16 *net matrices occurs, independently, with the same probability $\frac{1}{16}$, then*

$$\dim_H(E_\infty) = 1 + \log 3/\log 4, \tag{4.4}$$

*with probability one.*

**Remark.** The above result can also be derived by approaching $E_\infty$ as a net fractal in the sense of the definition given by Falconer [4], namely by using the corresponding net measures and the results regarding the relation between the net dimension and the Hausdorff dimension of net fractals. Here we have prefered to work with random fractals, as they also occur when studying fractal percolation, but also out of "didactical" reasons, in order to avoid introducing here net measures and related matters.
From Theorem 9 we easily get

**Corollary 3.** *Let $E_\infty$ be a net set fractal. Then $\lambda^1(E_\infty) = \infty$, with probability one.*

The definition and construction method of limiting net sets leads in a natural way to the question whether and when there occurs percolation for net set fractals.
The process of fractal percolation was first described by Mandelbrot in the 1970's. Its study has been intensified and extended in the 1980's. Fractal percolation has been used as a model for various physical processes such as intermittency in turbulence or distribution of galaxies in the universe (see Mandelbrot [8]).
Let us give the following

**Definition 18.** *Let $(E_k)_{k\geq 0}$ be a decreasing sequence of net sets with limiting net set $E_\infty$. We say that net percolation in the unit square occurs if $E_k$ connects two opposite sides of the unit square, for all $k \geq 0$.*

Equivalenty, we say that *net percolation* in the unit square occurs for a decreasing sequence of net sets $(E_k)_{k\geq 0}$ if there exists a path (i.e., the image of the interval $[0,1]$ by a continuous real function) in the corresponding limiting net set $E_\infty$ that connects two opposite sides of the unit square.

The geometrical structure of net sets, in particular that of strongly connected net sets, and Theorem 2 immediately yield the following

**Proposition 8.** *If $(E_k)_{k\geq 0}$ is a decreasing sequence of strongly connected net sets, then net percolation in the unit square occurs for this sequence.*

The above proposition states a sufficient condition for net percolation. By the counterexample that we gave with respect to the reciprocal of Theorem 1, the condition in the proposition is not also necessary for net percolation. The question, whether and when net percolation occurs if we allow in the construction (by means of net substitutions) of $\{E_k\}_{k\geq 0}$ also disconnected matrices, arises in a natural way. Since the facts that we have already shown about net sets do not give a complete answer to this question, we reformulate it in terms of random fractals.

Net percolation can be regarded as a type of fractal percolation in the unit square (see, e.g., Falconer [5], Chayes [2], Dekking and Meester [3]). In this context, let $p \in [0,1]$ and $q = 1-p$. We construct a decreasing sequence of net sets by means of substitutions with net matrices, such that the following conditions are fulfilled:

(1) Every time when we replace a zero entry of a net matrix by some net matrix $A$, the probability that $A$ is a connected net matrix is $p$, for all zero entries, independently.
(2) Any of the 12 connected net matrices occurs with probability $\frac{p}{12}$ and each disconnected matrix with probability $\frac{q}{4}$.
(3) The net matrix that corresponds to the net set $E_0 = E_0(p)$ is with proability $\frac{p}{12}$ one of the connected matrices and with probability $\frac{q}{4}$ one of the disconnected matrices.

Let $E_\infty(p)$ denote the limiting net set of the sequence $E_k(p)$ of net sets introduced above.
In this setting, any limiting net set can be viewed as a random fractal and thus net percolation becomes a a certain type of fractal percolation in the unit square. Analogously to the “classical” case of fractal percolation in the unit square one can formulate the following questions.

**Problem.** How does, in terms of probability, the structure of $E_\infty(p)$ variate in dependency of the parameter $p$? Does there exist a critical value $p_c$, such that for $p > p_c$ there is positive probability that $E_\infty(p)$ connects opposite sides of the unit square?
Of course, the cases $p = 1$ and $p = 0$ are trivial. We have in view the above problem for $0 < p < 1$ for further research.

## 5. Properties of net sets for other $(0, m, s)$-nets.

Analogously to (limiting) net sets whose construction is based on $(0, 2, 2)$-nets in base 2, one can also define (limiting) net sets corresponding to $(0, m, s)$-nets in base $b$ for larger values of the parameters $m$, $s$ and $b$, if such $(0, m, s)$-nets exist. In this section we mention only a few facts regarding (limiting) net sets associated to $(0, m, s)$-nets in base $b$ in more general cases.
Niederreiter [9, 10] has proven that for $m \geq 2$ a $(0, m, s)$ net in base $b$ can only exist for $s \leq b + 1$. In the following we assume this condition to be fulfilled.
Let us take the case of $(0, m, 2)$-nets in base $b$, with $m \geq 2$ and $b \geq 2$. The corresponding net sets can be described in terms of net substitutions with net matrices with dimension $b^m \times b^m$ over the set $\{0, 1\}$. One can immediately see that in this case the proportion of ones among the entries of a net matrix is less or equal to the proportion of ones in the case of net matrices associated to $(0, 2, 2)$-nets in base 2, and decreases when $m$ or $b$ increase. It is easy to check that in the case when $b > 2$ or $m > 2$ all net matrices are connected and thus all net sets are strongly connected. Correspondingly to the analogous situations in the case of $(0, 2, 2)$-nets we have the following results, whose proofs can be obtained in a straightforward manner from those presented in the previous sections.

**Proposition 9.** *Let $m \geq 2$ and $b \geq 2$. If $m > 2$ or $b > 2$, then every limiting net set defined by a decreasing sequence of net sets corresponding to $(0, m, 2)$-nets in base $b$ is net-connected.*

**Theorem 10.** *Let $m \geq 2$ and $b \geq 2$. Suppose that in the construction (with net matrices corresponding to $(0, m, 2)$-nets in base $b$) of a decreasing sequence $(E_k)_{k\geq 0}$ of net sets each net matrix appears, independently, with the same probability. If $m > 2$ or $b > 2$, then the* *limiting net set* $E_\infty$ *is a fractal set with*

$$\dim_H(E_\infty) = 1 + \frac{\log(b^m - 1)}{\log(b^m)}, \text{ with probability one.}$$

In this case net percolation always occurs, by Proposition 9.
In the case $s \geq 3$ one obviously has to replace the substitutions with net matrices by other tools, suitable for the higher-dimensional case. We do not study this case here, we just make a few remarks:
For the case of $(0, m, s)$-nets in base $b$ with $m, s \geq 2$ it is straightforward to find that $\lambda^d(E_\infty) = 0$, for $2 \leq d \leq s$, where $\lambda^d$ is the $d$-dimensional Lebesgue measure. Moreover, in the probabilistic setting corresponding to that of the above theorem, $\dim_H(E_\infty) = 1 + \log(b^m - 1)/\log(b^m)$, and $\lambda^1(E_\infty) = \infty$.
Finally, a remark on $(0, 1, 2)$-nets in base $b$. The case $b = 2$ is trivial. If $b$ is a prime number, then we have $2b$ disconnected net matrices, and $b! - 2b$ connected ones. The case when $b$ is neither a prime nor a prime power becomes more involved and we do not disscuss it here.
We leave further problems regarding net sets and limiting net sets corresponding to $(0, m, s)$-nets in base $b$ as topics of later research.

One could also consider other classes of matrices in order to construct fractals, analogously as in the construction of net sets. For example, in the case of $(4 \times 4)$-matrices one could take instead of net matrices corresponding to $(0, 2, 2)$-nets in base 2 the larger class of matrices having in each line and each column exactly one "1" entry and the rest of entries equal zero (the permutation matrices). Several results proven for net matrices hold in this case, for some of the criteria and results some of the conditions have to be (slightly) changed, in order to make them hold. Still, net matrices have the nice property of providing a "well distributed" structure of the fractals, which is due to the good distribution of the points of $(0, 2, 2)$-nets in the unit cube. We leave further questions regarding other classes of matrices for future research.

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## References

[1] P. Arnoux, S. Ito, *Pisot substitutions and Rauzy fractals*, Bull. Belg. Math. Soc. Simon Stevin **8** (2001), 181–207.

[2] L. Chayes, *Aspects of the fractal percolation process*, Progress in Probability, **37** (1995), 113–143.

[3] R. Dekking, R. Meester, *On the Structure of Mandelbrot's Percolation Process and Other Random Cantor Sets*, J. Stat. Phys. **58** (1990), 1109–1126.

[4] K.J. Falconer, *Random Fractals*, Math. Proc. Camb. Phil. Soc. **100** (1986), 559–582.

[5] K.J. Falconer, *Fractal Geometry. Mathematical Foundations and Applications*, John Wiley & Sons, Chichester 1990.

[6] M. Hata, *On the Structure of Self-Similar Sets*, Japan J. Appl. Math. **2** (1985), 381-414.

[7] K. Kuratowski, *Topology*, Volume II, Academic Press, New York and London, 1968.

[8] B. Mandelbrot, *The Fractal Geometry of Nature*, W.H. Freeman & Co., San Francisco, 1983.

[9] H. Niederreiter, *Point Sets with Small Discrepancy*, Mh. Math. **104** (1987), 273–337.

[10] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, No. **63** in CBMS-NSF Series in Applied Mathematics, Philadelphia, 1992.