Polynomial Lattice Point Sets

Friedrich Pillichshammer

Abstract Polynomial lattice point sets are special types of digital (t, m, s)-nets as introduced by H. Niederreiter in the 1980s. Quasi-Monte Carlo rules using them as underlying nodes are called polynomial lattice rules. In its overall structure polynomial lattice rules are very similar to usual lattice rules due to E. Hlawka and N. M. Korobov. The main difference is that here one uses polynomial arithmetic over a finite field instead of the usual integer arithmetic. In this overview paper we review the research on polynomial lattice rules during the last decade. We touch topics like extensible polynomial lattice rules, higher order polynomial lattice rules, the weighted discrepancy of polynomial lattice point sets, etc. Furthermore we compare polynomial lattice rules with lattice rules and show what results for polynomial lattice rules also have an analogue for usual lattice rules and vice versa.

1 Introduction

Assume we are interested in the approximation of multivariate integrals of the form $I_s(f) = \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ using a quasi-Monte Carlo (QMC) rule of the form $Q_{N,s}(f) = (1/N)\sum_{n=0}^{N-1} f(\boldsymbol{x}_n)$ where $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}$ are fixed sample nodes from the unit-cube $[0,1)^s$. On first sight this approach looks quite simple but the crux of this method is the choice of underlying nodes to obtain good approximations for large classes of functions.

Generally spoken it turned out that point sets with good uniform distribution properties yield a small absolute integration error. This is, for example, reflected in the Koksma-Hlawka inequality which states that

Friedrich Pillichshammer
Institut für Finanzmathematik, Universität Linz, Altenbergerstraße 69, A-4040 Linz, Austria. e-mail: friedrich.pillichshammer(AT)jku.at

$$|I_s(f) - Q_{N,s}(f)| \le V(f) D_N^*(\mathcal{P})$$

where $V(f)$ is the variation of f in the sense of Hardy and Krause and where D_N^* denotes the star discrepancy of the point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$, which can be defined as follows: given a point set $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ of N elements in $[0,1)^s$ the *discrepancy function* of $\mathcal{P}$ is defined by

$$\Delta_{\mathcal{P}}(\boldsymbol{z}) := \frac{\#\{0 \le n < N \,:\, \boldsymbol{x}_n \in [\boldsymbol{0}, \boldsymbol{z})\}}{N} - \lambda_s([\boldsymbol{0}, \boldsymbol{z})) \quad \text{for} \quad \boldsymbol{z} \in (0,1]^s,$$

where λ_s is the s-dimensional Lebesgue measure. The *star discrepancy* of $\mathcal{P}$ is then the L^∞-norm of $\Delta_{\mathcal{P}}$, i.e.,

$$D_N^*(\mathcal{P}) = \sup_{\boldsymbol{z} \in (0,1]^s} |\Delta_{\mathcal{P}}(\boldsymbol{z})|.$$

This is a quantitative measure for the deviation of $\mathcal{P}$ from uniform distribution modulo one. For more information on Koksma-Hlawka inequality and star discrepancy we refer to one of the books [17, 21, 34, 42].

For any point set $\mathcal{P}$ consistion of N points in $[0,1)^s$ it is known that

$$D_N^*(\mathcal{P}) \ge c_s (\log N)^{\kappa_s}/N$$

with a positive c_s independent of $\mathcal{P}$ and where $\kappa_2 = 1$ (see [3, 52]) and $\kappa_s \ge (s-1)/2$ for $s \ge 3$ which follows from a result of Roth [49]. (For $s \ge 3$ the lower bound on κ_s has recently been improved to $\kappa_s \ge (s-1)/2 + \delta_s$ for some unknown $0 < \delta_s < 1/2$; see [4].)

On the other hand, a point set $\mathcal{P}$ whose star discrepancy satisfies an upper bound of the form $D_N^*(\mathcal{P}) \le C_s (\log N)^{\alpha_s}/N$ with a positive c_s independent of $\mathcal{P}$ and where $\alpha_s \ge 0$, is informally called a *low discrepancy point set*. There are several methods to construct low discrepancy point sets:

- Hammersley point sets which are based on the infinite van der Corput sequence (see, e.g., [17, 42]);
- lattice point sets which where introduced independently by Korobov [29] and Hlawka [27] and which are well explained in the books of Niederreiter [42] and of Sloan and Joe [53];
- (t,m,s)-nets in base b which have been introduced by Niederreiter [40, 42] and which are the main topic of the recent book [17]. Very special examples of such nets go back to constructions of Sobol' [58] and Faure [22].

In this article we are concerned with a sub-class of (t,m,s)-nets which has a close relation to lattice point sets. Before we give its definition we recall the definition of (t,m,s)-nets in base b according to Niederreiter [40].

Definition 1. A point set $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ is called *(t,m,s)-net in base b* if every so-called b-adic elementary interval of the form

$$\prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}} \right) \subseteq [0,1)^s$$

of volume b^{t-m} contains exactly b^t points of $\mathcal{P}$.

Some remarks on the definition of (t,m,s)-nets in base b are in order (for more information see [17, 42]).

Remark 1. 1. Definition 1 says that for every b-adic elementary interval J volume b^{t-m} we have $\#\{\boldsymbol{x} \in \mathcal{P} : \boldsymbol{x} \in J\} - b^m \lambda_s(J) = 0$.

2. The uniform distribution quality depends on the so-called *quality parameter* $t \in \{0, \ldots, m\}$. A small t implies good uniform distribution. This is also reflected in Niederreiter's bound on the star discrepancy of a (t,m,s)-net $\mathcal{P}$ in base b which states that

$$D_N^*(\mathcal{P}) = O_{s,b}(b^t (\log N)^{s-1}/N) \tag{1}$$

where $N = b^m$; see [17, 40, 42].

3. The optimal value $t = 0$ is only possible if the parameters b and s satisfy $s \leq b+1$. On the other hand, any point set consisting of b^m elements in $[0,1)^s$ is a (m,m,s)-net in base b since this choice of parameters makes Definition 1 trivial (and also the discrepancy bound (1)).

In this article we are concerned with a sub-class of (t,m,s)-nets. Introduced by Niederreiter [41, 42], today this sub-class is known as polynomial lattice point sets. This name has its origin in a close relation with ordinary lattice point sets. In fact, the research on polynomial lattice point sets and on ordinary lattice point sets often follows two parallel tracks and bears a lot of similarities. It is the aim of this overview to review the, in the author's opinion, most important results on polynomial lattice point sets during the last decade and to demonstrate which of these results have counterparts for lattice point sets or not.

In the following two sections the basic definitions of (polynomial) lattice point sets and their duals are provided. In Section 4–9 we present the results on polynomial lattice point sets and point out their analogs for lattice point sets. The paper closes with a short summary in Section 10.

Notation: Throughout the paper we assume that b is a prime number. By $\mathbb{Z}_b$ we denote the finite field with b elements and with $\mathbb{Z}_b[x]$ the set of polynomials over $\mathbb{Z}_b$. Define $G_{b,m} := \{h \in \mathbb{Z}_b[x] : \deg(h) < m\}$. We have $|G_{b,m}| = b^m$.

The field of formal Laurent series over $\mathbb{Z}_b$ is denoted by $\mathbb{Z}_b((x^{-1}))$. Elements of $\mathbb{Z}_b((x^{-1}))$ are of the form

$$L = \sum_{\ell=w}^{\infty} t_\ell x^{-\ell} \quad \text{where } w \in \mathbb{Z} \text{ and all } t_\ell \in \mathbb{Z}_b.$$

For $n \in \mathbb{N}$ let $\nu_n : \mathbb{Z}_b((x^{-1})) \to [0,1)$ be defined by $\nu_n(L) = \sum_{\ell=\max(1,w)}^{n} t_\ell b^{-\ell}$.

In many results which we are going to present in the following sections there appear constants c which are assumed to be different from case to case. Optionally these constant may depend on the dimension s, on b or on other quantities which are then indicated as sub-scripts. In most cases these constants could be given explicitly.

2 Polynomial lattice point sets

On account of their close relation to polynomial lattice point sets first we recall the maybe more familiar concept of lattice point sets:

Definition 2. For an integer $N \geq 2$ and for $\boldsymbol{g} \in \mathbb{Z}^s$ the point set $\mathcal{P}(\boldsymbol{g}, N)$ consisting of the N elements

$$\boldsymbol{x}_n = \left\{ \frac{n}{N} \boldsymbol{g} \right\} \quad \text{for all } 0 \leq n < N$$

is called a *lattice point set (l.p.s.)*. A QMC rule using $\mathcal{P}(\boldsymbol{g}, N)$ as underlying node set is called a *lattice rule*.

Polynomial lattice point sets are in their overall structure very similar to l.p.s. The main difference is that l.p.s. are based on number theoretic concepts whereas polynomial lattice point sets are based on algebraic methods (polynomial arithmetic over a finite field). For simplicity we only discuss polynomial lattice point sets in prime base b. For the more general case of prime-power bases we refer to [17, 42].

Definition 3. For $s, m \in \mathbb{N}$, $p \in \mathbb{Z}_b[x]$, with $\deg(p) = m$, and $\boldsymbol{q} \in \mathbb{Z}_b[x]^s$ the point set $\mathcal{P}(\boldsymbol{q}, p)$ consisting of the b^m elements

$$\boldsymbol{x}_h = \nu_m \left(\frac{h(x)}{p(x)} \boldsymbol{q}(x) \right) \quad \text{for all } h \in G_{b,m}$$

is called a *polynomial lattice point set (p.l.p.s.)*. A QMC rule using $\mathcal{P}(\boldsymbol{q}, p)$ as underlying node set is called a *polynomial lattice rule*.

The structural similarity between Definition 2 and Definition 3 is evident. Hence let us compare the two concepts by means of some pictures.

The l.p.s. $\mathcal{P}(\boldsymbol{g}, N)$ shown in the left part of Fig. 1 shows a very regular lattice structure. Such a geometric structure cannot be observed for the p.l.p.s. $\mathcal{P}(\boldsymbol{q}, p)$ shown in the right part of Fig. 1. However also this point set has some inherent structure, namely the (t, m, s)-net structure. In fact, for this example every 2-adic elementary interval of area 2^{-4} contains exactly one element of the point set $\mathcal{P}(\boldsymbol{q}, p)$ and hence we have a $(0, 4, 2)$-net in base 2; cf. Fig. 2.

A further example of a l.p.s. and a p.l.p.s. is shown in Fig. 3.

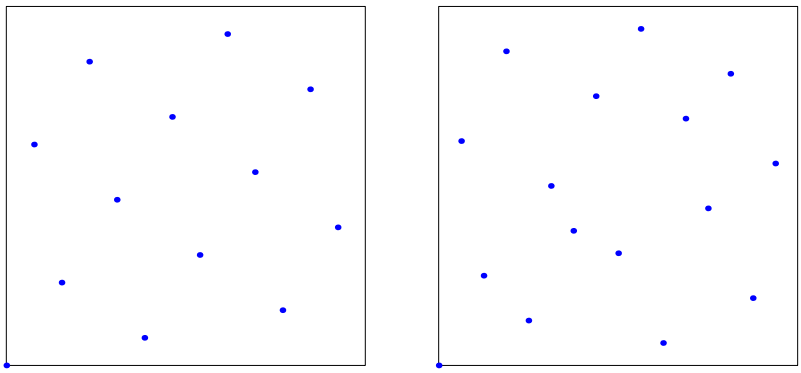

Fig. 1 left: $\mathcal{P}(\boldsymbol{g}, N)$ with $N = 13$ and $\boldsymbol{g} = (1, 8)$; right: $\mathcal{P}(\boldsymbol{q}, p)$ with $p(x) = x^4+x^2+1$ and $\boldsymbol{q} = (1, x^3)$ over $\mathbb{Z}_2$.

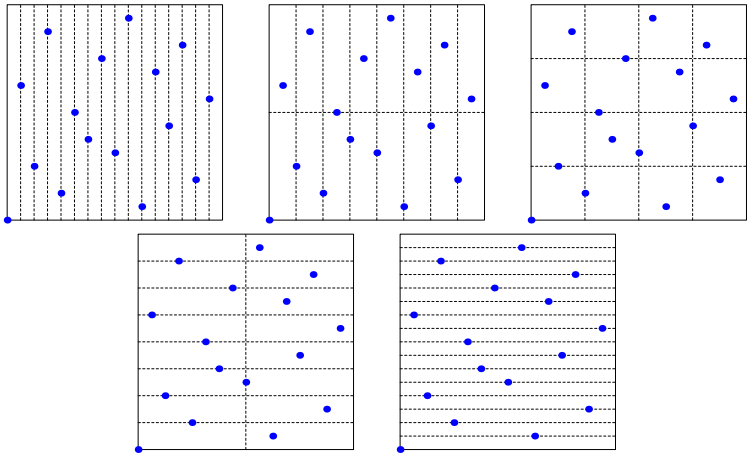

Fig. 2 $\mathcal{P}(\boldsymbol{q}, p)$ from Fig. 1 as $(0, 4, 2)$-net in base 2; every 2-adic elementary interval of area 2^{-4} contains exactly one point.

3 The dual net

For l.p.s. one has the notion of dual lattice which plays a crucial role in the quality analysis of such point sets.

Definition 4. The *dual lattice* of the l.p.s. $\mathcal{P}(\boldsymbol{g}, N)$ from Definition 2 is defined as

$$\mathcal{L}_{\boldsymbol{g},N} = \{\boldsymbol{h} \in \mathbb{Z}^s \,:\, \boldsymbol{h} \cdot \boldsymbol{g} \equiv 0 \pmod{N}\}.$$

An important property of l.p.s. is that

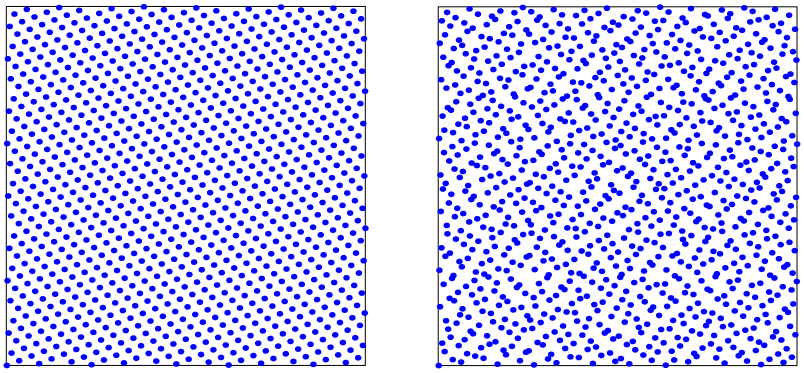

Fig. 3 left: $\mathcal{P}(\boldsymbol{g}, N)$ with $N = 987$ and $\boldsymbol{g} = (1, 610)$; right: $\mathcal{P}(\boldsymbol{q}, p)$ with $p(x) = x^{10} + x^8 + x^4 + x^2 + 1$ and $\boldsymbol{q} = (1, x^9 + x^5 + x)$ over $\mathbb{Z}_2[x]$.

$$\sum_{\boldsymbol{x} \in \mathcal{P}(\boldsymbol{g}, N)} \mathrm{e}_{\boldsymbol{k}}(\boldsymbol{x}) = \begin{cases} N & \text{if } \boldsymbol{k} \in \mathcal{L}_{\boldsymbol{g}, N}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{e}_{\boldsymbol{k}}(x) = \exp(2\pi \mathtt{i} \boldsymbol{k} \cdot \boldsymbol{x})$. This relation is the reason why for the analysis of the integration error of lattice rules it is most convenient to consider one-periodic functions; see [42, 53].

The corresponding definition for p.l.p.s. leads to the notion of a dual net.

Definition 5. The *dual net* of the p.l.p.s. $\mathcal{P}(\boldsymbol{q}, p)$ from Definition 3 is defined as

$$\mathcal{D}_{\boldsymbol{q}, p} = \{\boldsymbol{k} \in G_{b,m}^s \, : \, \boldsymbol{k} \cdot \boldsymbol{q} \equiv 0 \pmod{p}\}.$$

An important property of p.l.p.s. is that

$$\sum_{\boldsymbol{x} \in \mathcal{P}(\boldsymbol{q}, p)} {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \begin{cases} b^m & \text{if } \boldsymbol{k} \in \mathcal{D}_{\boldsymbol{q}, p}, \\ 0 & \text{otherwise,} \end{cases}$$

where ${}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})$ is the $\boldsymbol{k}$th b-adic Walsh function defined by ${}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{i=1}^s {}_b\mathrm{wal}_{k_i}(x_i)$ for $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1)^s$. The one-dimensional kth b-adic Walsh function is defined by ${}_b\mathrm{wal}_k(x) := \exp(2\pi \mathtt{i}(\xi_1 \kappa_0 + \cdots + \xi_{a+1} \kappa_a)/b)$ for $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_a b^a$ with $\kappa_i \in \{0, \ldots, b-1\}$ and $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$ with infinitely many digits $\xi_i \neq b-1$. Many properties of Walsh functions are summarized in [17, Appendix A].

The above relation is the reason why for the analysis of the integration error of polynomial lattice rules it is most convenient to consider Walsh series. We will come back to this issue in Section 6.

4 Quality measures and existence results

Based on the dual net one can introduce two quality measures for p.l.p.s. (see [42, Chapter 4] or [17, Chapter 10]): for $p \in \mathbb{Z}_b[x]$ and $\boldsymbol{q} \in \mathbb{Z}_b[x]^s$ define

$$\rho(\boldsymbol{q}, p) = s - 1 + \min_{\boldsymbol{h} \in \mathcal{D}_{\boldsymbol{q},p} \setminus \{\boldsymbol{0}\}} \sum_{i=1}^{s} \deg(h_i)$$

and

$$R_b(\boldsymbol{q}, p) = \sum_{\boldsymbol{h} \in \mathcal{D}_{\boldsymbol{q},p} \setminus \{\boldsymbol{0}\}} \prod_{i=1}^{s} r_b(h_i),$$

where $r_b(0) = 1$ and $r_b(h) = b^{-r-1} \sin^{-2}(\pi \kappa_r / b)$ for $h \in G_{b,m}$ of the form $h = \kappa_0 + \kappa_1 b + \cdots + \kappa_r x^r$, $\kappa_r \neq 0$.

We remark here that analogous quality measures also exist for l.p.s.; see [42, Chapter 5]. Based on these quality measures Niederreiter [42] proved the following results:

Theorem 1. *The p.l.p.s. $\mathcal{P}(\boldsymbol{q}, p)$ is a (t, m, s)-net in base b with $m = \deg(p)$, $t = m - \rho(\boldsymbol{q}, p)$ and*

$$D^*_{b^m}(\mathcal{P}(\boldsymbol{q}, p)) \leq \frac{s}{b^m} + R_b(\boldsymbol{q}, p).$$

For example for $p = x^4 + x^2 + 1$ and $\boldsymbol{q} = (1, x^3)$ over $\mathbb{Z}_2$ the "minimal" element of $\mathcal{D}_{\boldsymbol{q},p}$ is $(h_1, h_2) = (x^2 + 1, x)$ and hence $\rho(\boldsymbol{q}, p) = 4$ in this case. Theorem 1 then shows that $\mathcal{P}(\boldsymbol{q}, p)$ is a $(0, 4, 2)$-net in base 2; cf. Fig. 2. Theorem 1 also gives a bound on the star discrepancy of p.l.p.s. which is easier to handle then $D^*_{b^m}$ itself. For an analogous discrepancy bound for l.p.s. we refer to [42, Chapter 5] or [17, Proposition 3.49]. Based on Theorem 1 one can use averaging arguments to obtain the following existence results:

Theorem 2. *Let $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$.*

1. *If p is irreducible, then there exists $\boldsymbol{q} \in G^s_{b,m}$ such that*

$$t \leq (s-1) \log_b m - (s-2) - \log_b \frac{(s-1)!}{(b-1)^{s-1}}.$$

   Hence $D^*_{b^m}(\mathcal{P}(\boldsymbol{q}, p)) = O_{s,b}\left(m^{2s-2} b^{-m}\right)$.
2. *For $0 \leq \varepsilon < 1$ there are more than $\varepsilon |G^s_{b,m}|$ vectors $\boldsymbol{q} \in G^s_{b,m}$ with*

$$D^*_{b^m}(\mathcal{P}(\boldsymbol{q}, p)) \leq \frac{s}{b^m} + R_b(\boldsymbol{q}, p) = O_{s,b,\varepsilon}\left(\frac{m^s}{b^m}\right).$$

Part 1 of Theorem 2 for $b = 2$ has been shown by Larcher et al. [38]; see also [51] or [17, Chapter 10] for general b. Part 2 has been shown by

Niederreiter [42, Chapter 4] and also by Dick et al. [10] and [13]. For an analogous discrepancy bound for l.p.s. we refer to [42, Chapter 5] or [17, Theorem 3.51].

The bound on R_b in Theorem 2 is best possible in the order of magnitude in m. This was shown recently by Kritzer and the author in [33]. (A corresponding result for l.p.s. has been shown by Larcher [36].)

Theorem 3. *There exists $c_{s,b} > 0$ such that for any $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ and any $\boldsymbol{q} \in G_{b,m}^s$, $q_i \neq 0$, $1 \leq i \leq s$, we have*

$$R_b(\boldsymbol{q}, p) \geq c_{s,b} b^{\deg(\delta_s)} \frac{(m - \deg(\delta_s))^s}{b^m} \text{ where } \delta_s := \gcd(q_1, \ldots, q_s, p).$$

On the other hand, the bound on $D_{b^m}^*$ in Theorem 2 is *not* best possible in the order of magnitude in m. For example, in dimension $s = 2$ the so-called Fibonacci p.l.p.s. has a star discrepancy of order $O_b(mb^{-m})$; see [42, Chapter 4] or [17, Chapter 10]. For arbitrary dimension s it was shown by Larcher [37] that for any $m \geq 2$ there exists $\boldsymbol{q} \in G_{b,m}^s$ with

$$D_{b^m}^*(\mathcal{P}(\boldsymbol{q}, x^m)) = O_{s,b}\left(m^{s-1}(\log m) b^{-m}\right).$$

This result has no counterpart for l.p.s.

5 CBC construction of polynomial lattice point sets

According to Theorem 2 for any given irreducible polynomial $p \in \mathbb{Z}_b[x]$ there exist a sufficiently large number of good vectors $\boldsymbol{q}$ of polynomials which yield p.l.p.s. with reasonably low star discrepancy. Now one aims to find such vectors by computer search. Unfortunately a full search is not possible (except maybe for small values of m, s) since one has to check b^{ms} vectors of polynomials.

At this point one gets a cue from the analogy between p.l.p.s. and l.p.s. where the component-by-component (CBC) construction approach works very well. This approach was introduced by Korobov [30] for l.p.s. and later it was re-invented by Sloan and Reztsov [54]. The same idea applies for p.l.p.s. Here we use the more general weighted star discrepancy as introduced by Sloan and Woźniakowski [55] as underlying quality criterion: let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots)$ be a sequence of weights in $\mathbb{R}^+$. Let $\mathcal{I}_s = \{1, \ldots, s\}$ and for $\mathfrak{u} \subseteq \mathcal{I}_s$ let $\gamma_{\mathfrak{u}} = \prod_{i \in \mathfrak{u}} \gamma_i$. The *weighted star discrepancy* of an N-element point set $\mathcal{P}$ in $[0,1)^s$ is given by

$$D_{N,\boldsymbol{\gamma}}^*(\mathcal{P}) = \sup_{\boldsymbol{z} \in (0,1]^s} \max_{\emptyset \neq \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u}} |\Delta_{\mathcal{P}}((z_{\mathfrak{u}}, 1))|.$$

The weights $\boldsymbol{\gamma}$ are additional parameters which model the importance of the different coordinate projections. For the weights $\boldsymbol{\gamma} = \mathbf{1} =: (1, 1, \ldots)$ one has $D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}) = D^*_N(\mathcal{P})$ for any point set $\mathcal{P}$. In the weighted setting the CBC construction has the advantage that the quadrature points $\mathcal{P}$ can be optimized with respect to $\boldsymbol{\gamma}$.

The weighted Koksma-Hlawka inequality then states that

$$|I_s(f) - Q_{N,s}(f)| \le D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}) \|f\|_{s,\boldsymbol{\gamma}}$$

with a certain norm $\|\cdot\|_{s,\boldsymbol{\gamma}}$; see [55, 28] or [17, Chapter 2] for details.

Let $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ and let $\boldsymbol{q} \in G^s_{b,m}$. Then it can be shown (see [17, Corollary 10.16]) that

$$D^*_{b^m,\boldsymbol{\gamma}}(\mathcal{P}(\boldsymbol{q}, p)) \le \sum_{\emptyset \ne \mathfrak{u} \subseteq \mathcal{I}_s} \gamma_{\mathfrak{u}} \left(1 - \left(1 - \frac{1}{b^m}\right)^{|\mathfrak{u}|}\right) + R_{b,\boldsymbol{\gamma}}(\boldsymbol{q}, p),$$

where

$$R_{b,\boldsymbol{\gamma}}(\boldsymbol{q}, p) = \sum_{\boldsymbol{h} \in \mathcal{D}_{\boldsymbol{q},p} \setminus \{\mathbf{0}\}} \prod_{i=1}^{s} r_b(h_i, \gamma_i)$$

and where for $h \in G_{b,m}$ we put $r_b(0, \gamma) = 1 + \gamma$ and $r_b(h, \gamma) = \gamma r_b(h)$ if $h \ne 0$, where $r_b(h)$ is as in Section 4. An analogous bound for the weighted star discrepancy of l.p.s. can be found in [28].

Now we deal with the quantity $R_{b,\boldsymbol{\gamma}}(\boldsymbol{q}, p)$ which can be computed in $O(b^m s)$ operations (see [17, Proposition 10.20]).

Algorithm 1 CBC-algorithm

Require: b a prime, $s, m \in \mathbb{N}$, $p \in \mathbb{Z}_b[x]$, with $\deg(p) = m$, and weights $\boldsymbol{\gamma} = (\gamma_i)_{i \ge 1}$.

1: Choose $q_1 = 1$.
2: **for** $d = 2$ to s **do**
3: find $q_d \in G^*_{b,m}$ which minimises the quantity $R_{b,\boldsymbol{\gamma}}((q_1, \ldots, q_{d-1}, z), p)$ as a function of z.
4: **end for**
5: **return** $\boldsymbol{q} = (q_1, \ldots, q_s)$.

Theorem 4. *Let p be irreducible. If $\boldsymbol{q} \in G^s_{b,m}$ is constructed with Algorithm 1, then*

$$R_{b,\boldsymbol{\gamma}}(\boldsymbol{q}, p) \le \frac{1}{b^m - 1} \prod_{i=1}^{s} \left(1 + \gamma_i \left(1 + m \frac{b^2 - 1}{3b}\right)\right),$$

A proof can be found in [13]. A similar result for not necessarily irreducible p has been shown in [10] and a corresponding result for l.p.s. is [28, Theorem 3].

Using an argument from [14, Section 7] one can deduce the following result from Theorem 4; see also [17, Corollary 10.30].

Corollary 1. *Let p be irreducible. If $\sum_{i=0}^{\infty} \gamma_i < \infty$, then for any $\delta > 0$ there exists $c_{\boldsymbol{\gamma},\delta} > 0$, such that for $\boldsymbol{q} \in G_{b,m}^s$ constructed with Algorithm 1 we have*

$$D_{b^m,\boldsymbol{\gamma}}^*(\mathcal{P}(\boldsymbol{q},p)) \le c_{\boldsymbol{\gamma},\delta} b^{-m(1-\delta)}.$$

Let $N \in \mathbb{N}$ with 2-adic expansion $N = 2^{m_1} + \cdots + 2^{m_k}$, where $0 \le m_1 < m_2 < \ldots < m_k$. For $1 \le j \le k$ choose $p^{(j)} \in \mathbb{Z}_2[x]$ irreducible with $\deg(p^{(j)}) = m_j$ and construct $\mathcal{P}(\boldsymbol{q}^{(j)}, p^{(j)})$ with Algorithm 1. Then set $\mathcal{P}_N = \mathcal{P}(\boldsymbol{q}^{(1)}, p^{(1)}) \cup \ldots \cup \mathcal{P}(\boldsymbol{q}^{(k)}, p^{(k)})$. In [26] the following is shown:

Corollary 2. *If $\sum_{i=0}^{\infty} \gamma_i < \infty$, then for any $\delta > 0$ there exists $C_{\boldsymbol{\gamma},\delta} > 0$, such that*

$$D_{N,\boldsymbol{\gamma}}^*(\mathcal{P}_N) \le C_{\boldsymbol{\gamma},\delta} N^{-1+\delta} \text{ for any } N \in \mathbb{N}.$$

The weighted star discrepancy is strongly polynomial tractable with ε-exponent equal to one.

The cost for the CBC-algorithm is of $O(b^{2m}s^2)$ operations. This is comparable with the CBC construction cost of l.p.s.; cf. [28, Section 3]. However, in this form the CBC-algorithm can only be used for not too large cardinality b^m. A breakthrough for this problem was obtained by Nuyens and Cools [46, 47] when they introduced — first for l.p.s. and then for p.l.p.s. — the Fast CBC construction with a significant reduction of cost to $O(smb^m)$ operations with $O(b^m)$ memory space. Only through this reduction of the construction cost does the CBC-algorithm become applicable for the generation of p.l.p.s. (and of l.p.s.) with reasonably large cardinality. See also [17, Section 10.3].

6 Integration of Walsh series

As already mentioned in Section 3 it is most convenient for the error analysis to consider Walsh series. Let $\alpha > 1$ and let $\mathscr{H}_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}$ be the weighted Hilbert function space with reproducing kernel given by

$$K_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \rho_\alpha(\boldsymbol{k},\boldsymbol{\gamma})\, {}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\, \overline{{}_b\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{y})},$$

where for $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ we put $\rho_\alpha(\boldsymbol{k},\boldsymbol{\gamma}) = \prod_{j=1}^s \rho_\alpha(k_j,\gamma_j)$ with $\rho_\alpha(0,\gamma) = 1$ and $\rho_\alpha(k,\gamma) = \gamma b^{-\alpha v}$ if $b^v \le k < b^{v+1}$ for $v \in \mathbb{N}_0$. The norm in this function space is given by

$$\|f\|_{\mathscr{H}_{\mathrm{wal},s,\alpha,\boldsymbol{\gamma}}} = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \rho_\alpha(\boldsymbol{k},\boldsymbol{\gamma})^{-1} |\widehat{f}_{\mathrm{wal}}(\boldsymbol{k})|^2$$

where $\widehat{f}_{\rm wal}(\boldsymbol{k}) = \int_{[0,1]^s} f(\boldsymbol{x})\,\overline{{}_b{\rm wal}_{\boldsymbol{k}}(\boldsymbol{x})}\,{\rm d}\boldsymbol{x}$. For more information on $\mathscr{H}_{{\rm wal},s,\alpha,\boldsymbol{\gamma}}$ we refer to [15]. The counterpart to the function space $\mathscr{H}_{{\rm wal},s,\alpha,\boldsymbol{\gamma}}$ for the analysis of l.p.s. is the so-called Korobov space ([20, 56] or [45, Appendix A.1]) whose reproducing kernel looks similar to $K_{{\rm wal},s,\alpha,\boldsymbol{\gamma}}$ but with the main difference that the Walsh function system is replaced by the trigonometric function system and Walsh coefficients are replaced by Fourier coefficients.

The worst-case integration error of a QMC rule is defined as the worst performance of the QMC algorithm over the unit ball of the function space under consideration, i.e., in our case $e(\mathscr{H}_{{\rm wal},s,\alpha,\boldsymbol{\gamma}},\mathcal{P}) := \sup_{\|f\|_{\mathscr{H}_{{\rm wal},s,\alpha,\boldsymbol{\gamma}}}\le 1} |I_s(f) - Q_{b^m,s}(f)|$. For p.l.p.s. it can be shown that

$$e^2(\boldsymbol{q},p) := e^2(\mathscr{H}_{{\rm wal},s,\alpha,\boldsymbol{\gamma}},\mathcal{P}(\boldsymbol{q},p)) = \sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\setminus\{\boldsymbol{0}\}\\ {\rm tru}_m(\boldsymbol{k})(x)\in\mathcal{D}_{\boldsymbol{q},p}}} \rho_\alpha(\boldsymbol{k},\boldsymbol{\gamma})$$

where ${\rm tru}_m(\boldsymbol{k}) :\equiv \boldsymbol{k} \pmod{b^m}$ (component-wise) and where

$$k = \kappa_0 + \kappa_1 b + \cdots + \kappa_{m-1} b^{m-1} \in \mathbb{N}_0$$

is identified with

$$k(x) = \kappa_0 + \kappa_1 x + \cdots + \kappa_{m-1} x^{m-1} \in \mathbb{Z}_b[x].$$

For the worst-case integration error of a polynomial lattice rule for integration in $\mathscr{H}_{{\rm wal},s,\alpha,\boldsymbol{\gamma}}$ we have the following result which was first proved in [11] for irreducible p and later generalized in [32] to not necessarily irreducible p. The corresponding result for l.p.s. was shown by Kuo [35].

Theorem 5. *For any $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ one can construct CBC $\boldsymbol{q} \in G_{b,m}^s$ such that (with $N = b^m$)*

$$e(\boldsymbol{q},p) \le c_{s,\alpha,\boldsymbol{\gamma},\delta} N^{-\alpha/2+\delta} \qquad \text{for all } 0 < \delta \le \tfrac{\alpha-1}{2}.$$

If $\sum_{i=1}^\infty \gamma_i^{1/(\alpha-2\delta)} < \infty$, then $c_{s,\alpha,\boldsymbol{\gamma},\delta} \le c_{\infty,\alpha,\boldsymbol{\gamma},\delta} < \infty$, i.e., the above bound can be made independent of the dimension s.

7 Extensible polynomial lattice point sets

A disadvantage of the CBC-algorithm is that the generated vectors $\boldsymbol{q}$ depend on p and hence on $N = b^{\deg(p)}$. If one changes p, then one has to construct a new vector $\boldsymbol{q} \in \mathbb{Z}_b[x]^s$. This means that an extension in the number of points is not possible with the CBC approach. For this reason Niederreiter [43] introduced the notion of extensible p.l.p.s. whose definition will be presented below. For the corresponding notion of extensible l.p.s. we refer to [25].

For $p \in \mathbb{Z}_b[x]$ with $m = \deg(p) \geq 1$, let Y_p be the set of all p-adic polynomials $\sum_{n=0}^{\infty} a_n p^n$ with $\deg(a_n) < m$. Then $Y_p/(p^n) = G_{b,nm}$. Let $\boldsymbol{Q} \in Y_p^s$ and for $n \in \mathbb{N}$ let $\boldsymbol{q}_n \equiv \boldsymbol{Q} \pmod{p^n}$. Then

$$\mathcal{P}(\boldsymbol{q}_1, p) \subseteq \mathcal{P}(\boldsymbol{q}_2, p^2) \subseteq \mathcal{P}(\boldsymbol{q}_3, p^3) \subseteq \ldots.$$

Definition 6. An *extensible p.l.p.s.* is defined as

$$\mathcal{P}(\boldsymbol{Q}, p) := \mathcal{P}(\boldsymbol{q}_1, p) \cup \mathcal{P}(\boldsymbol{q}_2, p^2) \cup \mathcal{P}(\boldsymbol{q}_3, p^3) \cup \ldots.$$

For $\mathcal{P}(\boldsymbol{q}_n, p^n)$ only the first n "digits" in p-adic expansion of each component of $\boldsymbol{Q}$ are important. This observation is used in the following construction algorithm which uses ideas from Korobov [31] for l.p.s.

Algorithm 2 Construction of extensible p.l.p.s.

Require: b a prime, $s, m \in \mathbb{N}$, $p \in \mathbb{Z}_b[x]$ monic and irreducible with $\deg(p) = m$, and weights $\boldsymbol{\gamma} = (\gamma_i)_{i \geq 1}$.

1: Find $\boldsymbol{q}_1 := \boldsymbol{q}$ by minimizing $e^2(\boldsymbol{q}, p)$ over all $\boldsymbol{q} \in G_{b,m}^s$.
2: **for** $n = 2, 3, \ldots$ **do**
3: find $\boldsymbol{q}_n := \boldsymbol{q}_{n-1} + p^{n-1}\boldsymbol{q}$ by minimizing $e^2(\boldsymbol{q}_{n-1} + p^{n-1}\boldsymbol{q}, p^n)$ over all $\boldsymbol{q} \in G_{b,m}^s$.
4: **return** $\boldsymbol{q}_n$.
5: **end for**

Theorem 6. *If $\boldsymbol{q}_n \in G_{b,m}^s$ is constructed according to Algorithm 2, then*

$$e^2(\boldsymbol{q}_n, p^n) \leq c_{s,b,\boldsymbol{\gamma},\alpha} b^{-nm}.$$

If $\sum_{i=1}^{\infty} \gamma_i < \infty$, then $c_{s,\alpha,\boldsymbol{\gamma},\delta} \leq c_{\infty,\alpha,\boldsymbol{\gamma},\delta} < \infty$, i.e., the above bound can be made independent of the dimension s.

A proof of this result and also a corresponding result for l.p.s. can be found in [44]; see also [17]. A disadvantage of the above error bound is that the worst-case error converges only with order $O(N^{-1/2})$.

There exists another algorithm — first introduced for l.p.s. in [18] and then for p.l.p.s. in [8] — which is called CBC sieve algorithm (see [17, Section 10.4]) and which yields better error bounds, but with the disadvantage that the generated p.l.p.s. (and l.p.s. respectively) are only finitely extensible. In this context one also speaks about *embedded p.l.p.s.* (and *embedded l.p.s.* respectively). For embedded l.p.s. we also refer to [6]. A pure existence result for extensible p.l.p.s. with small star discrepancy is due to Niederreiter [43]. For existence results for extensible l.p.s. we refer to Hickernell and Niederreiter [25].

8 Integration in Sobolev spaces

For $x = x_1 b^{-1} + x_2 b^{-2} + \cdots$ and $\sigma = \sigma_1 b^{-1} + \sigma_2 b^{-2} + \cdots$ with $x_i, \sigma_i \in \{0, \ldots, b-1\}$ the digitally shifted point $y = x \oplus \sigma$ is given by $y = y_1 b^{-1} + y_2 b^{-2} + \cdots$, where $y_i = x_i + \sigma_i \pmod b$. For vectors $\boldsymbol{x}$ and $\boldsymbol{\sigma}$ we define the digitally shifted point $\boldsymbol{y} = \boldsymbol{x} \oplus \boldsymbol{\sigma}$ component wise. This digital shift can be used to randomize a p.l.p.s.

Definition 7. For $\boldsymbol{\sigma} \in [0,1)^s$ the point set $\mathcal{P}_{\boldsymbol{\sigma}}(\boldsymbol{q}, p) := \mathcal{P}(\boldsymbol{q}, p) \oplus \boldsymbol{\sigma}$ is called a *digitally shifted p.l.p.s.*.

In the context of l.p.s. one often uses a "geometric" shift instead of the digital shift to randomize the point set and speaks then about shifted l.p.s.

Similar results to those from Section 6 hold for the mean square worst-case error of digitally shifted polynomial lattices for integration in the Sobolev space $\mathscr{H}^{(1)}_{\mathrm{sob},s,\boldsymbol{\gamma}}$ with reproducing kernel

$$K^{(1)}_{\mathrm{sob},s,\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{s} \left(1 + \gamma_i B_1(x_i) B_1(y_i) + \frac{\gamma_i}{2} B_2(|x_i - y_i|)\right).$$

The function space $\mathscr{H}^{(1)}_{\mathrm{sob},s,\boldsymbol{\gamma}}$ contains all functions $f : [0,1]^s \to \mathbb{R}$ whose mixed partial derivatives up to order one in each variable are square integrable. See [19, 57] and [45, Appendix A.2.3.] for more information on $\mathscr{H}^{(1)}_{\mathrm{sob},s,\boldsymbol{\gamma}}$.

The *mean square worst-case error* of digitally shifted p.l.p.s. for integration in $\mathscr{H}^{(1)}_{\mathrm{sob},s,\boldsymbol{\gamma}}$ is defined by

$$\widehat{e}^2(\boldsymbol{q}, p) = \int_{[0,1]^s} e^2(\mathscr{H}^{(1)}_{\mathrm{sob},s,\boldsymbol{\gamma}}, \mathcal{P}_{\boldsymbol{\sigma}}(\boldsymbol{q}, p))\, \mathrm{d}\boldsymbol{\sigma}.$$

We have the following result whose proof can be found in [17, Theorem 12.14]; see also [11]. The corresponding result for shifted l.p.s. was shown by Kuo [35].

Theorem 7. *For any $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ we can construct CBC $\boldsymbol{q} \in G^s_{b,m}$ such that (with $N = b^m$)*

$$\widehat{e}(\boldsymbol{q}, p) \le c_{s,b,\boldsymbol{\gamma},\varepsilon} N^{-1+\varepsilon} \quad \text{for all } 0 < \varepsilon \le 1/2.$$

If $\sum_{i=1}^{s} \gamma_i^{1/(2(1-\varepsilon))} < \infty$, then $c_{s,b,\boldsymbol{\gamma},\varepsilon} \le c_{\infty,b,\boldsymbol{\gamma},\varepsilon} < \infty$, i.e., the above bound can be made independent of the dimension s.

Remark 2. Baldeaux and Dick [1] showed that in the *randomized setting* one can obtain an improved error bound by using Owen's scrambling (see [48] or [17, Chapter 13]). For scrambled p.l.p.s. one has

$$\mathbb{E}\left[|I_s(f) - Q_{N,s}(f)|^2\right] \le c_{s,b,\boldsymbol{\gamma},\varepsilon} N^{-3+\varepsilon} \quad \text{for } \varepsilon > 0$$

where $N = b^m$ and where the expectation is with respect to all random scrambles of a p.l.p.s. Such a result is not known for l.p.s.

Now we assume more smoothness for integrands. Consider the Sobolev space $\mathscr{H}^{(2)}_{\mathrm{sob},s,\boldsymbol{\gamma}}$ with reproducing kernel

$$K^{(2)}_{\mathrm{sob},s,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y}) = \prod_{i=1}^{s}\left(1+\gamma_i B_1(x_i)B_1(y_i)+\frac{\gamma_i^2}{4}B_2(x_i)B_2(y_i)-\frac{\gamma_i^2}{24}B_4(|x_i-y_i|)\right).$$

The function space $\mathscr{H}^{(2)}_{\mathrm{sob},s,\boldsymbol{\gamma}}$ contains all functions $f:[0,1]^s\to\mathbb{R}$ whose mixed partial derivatives up to order two in each variable are square integrable. See [17, Section 14.6] for more information.

Using an idea from Hickernell [24] we use the tent transformation $\phi(x) = 1-|2x-1|$. For vectors $\boldsymbol{x}$ we apply ϕ component-wise and for a point set $\mathcal{P}$, $\phi(\mathcal{P})$ means that the tent transformation is applied the every element of $\mathcal{P}$. We call $\phi(\mathcal{P})$ the *folded* point set $\mathcal{P}$. Define the mean square worst-case error of folded digitally shifted p.l.p.s. by

$$\widehat{e}^2_\phi(\boldsymbol{q},p) = \int_{[0,1]^s} e^2(\mathscr{H}^{(2)}_{\mathrm{sob},s,\boldsymbol{\gamma}},\phi(\mathcal{P}_{\boldsymbol{\sigma}}(\boldsymbol{q},p)))\,\mathrm{d}\boldsymbol{\sigma}.$$

The following result, proved in [5], shows that one can obtain an improved convergence rate for the mean square worst-case error of folded digitally shifted p.l.p.s. for functions $f\in\mathscr{H}^{(2)}_{\mathrm{sob},s,\boldsymbol{\gamma}}$. A corresponding result for l.p.s. has been shown by Hickernell [24].

Theorem 8. *For any $p\in\mathbb{Z}_2[x]$ with $\deg(p)=m$ we can construct CBC $\boldsymbol{q}\in G^s_{2,m}$ such that (with $N=2^m$)*

$$\widehat{e}_\phi(\boldsymbol{q},p)\le c_{s,\boldsymbol{\gamma},\varepsilon}N^{-2+\varepsilon}\quad \textit{for all } 0<\varepsilon\le 3/2.$$

If $\sum_{i=1}^s \gamma_i^{1/(2(2-\varepsilon))}<\infty$, then $c_{s,\boldsymbol{\gamma},\varepsilon}\le c_{\infty,\boldsymbol{\gamma},\varepsilon}<\infty$, i.e., the above bound can be made independent of the dimension s.

9 Higher order polynomial lattice rules

Now we go a step further and consider functions with arbitrary smoothness for integrands. For a more detailed definition of the functions spaces under consideration we need some notation:

For $k=\kappa_1 b^{a_1-1}+\kappa_2 b^{a_2-1}+\cdots+\kappa_v b^{a_v-1}$, where $1\le a_v<\cdots<a_1$, $v\in\mathbb{N}$ and $\kappa_1,\ldots,\kappa_v\in\{1,\ldots,b-1\}$, and for $\alpha\ge 1$ define

$$\mu_\alpha(k) := a_1+\cdots+a_{\min(v,\alpha)}.$$

Furthermore, for $\gamma > 0$ put $r_\alpha(0,\gamma) = 1$ and $r_\alpha(k,\gamma) = b^{-\mu_\alpha(k)}$ for $k \in \mathbb{N}$. For $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots)$, set $r_\alpha(\boldsymbol{k}, \boldsymbol{\gamma}) := \prod_{i=1}^s r_\alpha(k_i, \gamma_i)$.

Let $\mathscr{W}_{\alpha,s,\boldsymbol{\gamma}} \subseteq L_2([0,1]^s)$ be the space consisting of all Walsh series $f = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}_{\rm wal}(\boldsymbol{k})\, {}_b\mathrm{wal}_{\boldsymbol{k}}$ for which

$$\|f\|_{\mathscr{W}_{\alpha,s,\boldsymbol{\gamma}}} := \sup_{\boldsymbol{k} \in \mathbb{N}_0^s} \frac{|\widehat{f}_{\rm wal}(\boldsymbol{k})|}{r_\alpha(\boldsymbol{k}, \boldsymbol{\gamma})} < \infty$$

For $\alpha \geq 2$ the function space $\mathscr{W}_{\alpha,s,\boldsymbol{\gamma}}$ contains all functions $f : [0,1]^s \to \mathbb{R}$ whose mixed partial derivatives up to order α in each variable are square integrable; see [9]. We call α the *smoothness parameter* of the function space.

Of course one would expect that the higher smoothness of integrands is reflected in the convergence rate of the integration error. Higher smoothness should lead to improved convergence rates. However, it turned out that this is *not* the case when the concept of (digitally shifted) p.l.p.s. as introduce in Definition 3 is used as underlying nodes. For this reason the following suitable generalization has been introduced in [16]; see also [17, Section 15.7].

Definition 8. For $s, m, n \in \mathbb{N}$, $m \leq n$, $p \in \mathbb{Z}_b[x]$, with $\deg(p) = n$, and $\boldsymbol{q} \in \mathbb{Z}_b[x]^s$ the point set $\mathcal{P}_{m,n}(\boldsymbol{q}, p)$ consisting of the b^m points

$$\boldsymbol{x}_h = \nu_n\left(\frac{h(x)}{p(x)}\boldsymbol{q}(x)\right) \quad \text{for all } h \in G_{b,m}$$

is called a *polynomial lattice point set (p.l.p.s.)*. A QMC rule using $\mathcal{P}_{m,n}(\boldsymbol{q}, p)$ is called a *polynomial lattice rule*.

Remark 3. For $m = n$ we have $\mathcal{P}_{m,m}(\boldsymbol{q}, p) = \mathcal{P}(\boldsymbol{q}, p)$.

Definition 9. The *dual net* of the p.l.p.s. $\mathcal{P}_{m,n}(\boldsymbol{q}, p)$ from Definition 8 is defined as

$$\mathcal{D}_{\boldsymbol{q},p} = \{\boldsymbol{k} \in G_{b,n}^s : \boldsymbol{k} \cdot \boldsymbol{q} \equiv u \pmod p \text{ with } \deg(u) < n - m\}.$$

For $\alpha \geq 2$ the worst-case error for integration in $\mathscr{W}_{\alpha,s,\boldsymbol{\gamma}}$ using $\mathcal{P}_{m,n}(\boldsymbol{q}, p)$ is given by (see [2, Proposition 2.1])

$$e_\alpha^2(\boldsymbol{q}, p) := e_\alpha^2(\mathscr{W}_{\alpha,s,\boldsymbol{\gamma}}, \mathcal{P}_{m,n}(\boldsymbol{q}, p)) = \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\} \\ tru_n(\boldsymbol{k})(x) \in \mathcal{D}_{\boldsymbol{q},p}}} r_\alpha(\boldsymbol{k}, \boldsymbol{\gamma}).$$

The following result has been shown in [2].

Theorem 9. *For any irreducible $p \in \mathbb{Z}_b[x]$ with $\deg(p) = n$ we can construct CBC $\boldsymbol{q} \in G_{b,n}^s$ such that*

$$e_\alpha(\boldsymbol{q}, p) \leq c_{s,\alpha,\boldsymbol{\gamma},\tau} b^{-\min(\tau m, n)} \quad \textit{for all } 1 \leq \tau < \alpha.$$

If $\sum_{i=1}^{\infty} \gamma_i^{1/\tau} < \infty$ *then* $c_{s,\alpha,\boldsymbol{\gamma},\tau} \le c_{s,\alpha,\boldsymbol{\gamma},\tau} < \infty$*, i.e., the above bound can be made independent of the dimension* s*.*

Remark 4. Choosing n large we obtain a convergence order of $N^{-\alpha+\varepsilon}$ for $\varepsilon > 0$ where $N = b^m$. By a result of Šarygin [50] this convergence rate is essentially best possible.

The result from Theorem 9 holds for a fixed smoothness parameter $\alpha \ge 2$. However, in practical applications the smoothness parameter is in general not known a priori. Hence it is reasonable to ask for constructions of p.l.p.s. which achieve almost optimal convergence rates for a range of smoothness parameters and which adjust themselves to the smoothness of a given integrand.

The basic idea in [2] can be roughly explained as follows: given $p \in \mathbb{Z}_b[x]$. If there exists a large enough amount of p.l.p.s. $\mathcal{P}(\boldsymbol{q}, p)$ which do well for the smoothness parameter α and if there exists a large enough amount of p.l.p.s. $\mathcal{P}(\boldsymbol{q}, p)$ which do well for the smoothness parameter α', then there must be a p.l.p.s. $\mathcal{P}(\boldsymbol{q}, p)$ which does well for both smoothness parameters α and α'. The underlying mathematical argument is the following "sieve principle": let X be some finite set and $A, B \subseteq X$. If $|A|, |B| > |X|/2$, then $|A \cap B| > 0$.

Algorithm 3 Sieve Algorithm

Require: b a prime, $s, m, \beta \in \mathbb{N}$, $\beta \ge 2$, $p \in \mathbb{Z}_b[x]$ irreducible with $\deg(p) = m$, weights $\boldsymbol{\gamma} = (\gamma_i)_{i \ge 1}$,

1: Set $n = \beta m$.

2: Find $\lfloor (1 - \beta^{-1}) b^{\beta m s} \rfloor + 1$ vectors $\boldsymbol{q}$ in $G_{b,\beta m}^s$ which satisfy

$$e_2(\boldsymbol{q}, p) \le c_{s,b,\boldsymbol{\gamma},s,m,\beta,2,\tau_2} b^{-\tau_2 m} \quad \text{for all } 1 \le \tau_2 < 2,$$

and label this set $\mathcal{T}_2$.

3: **for** $\alpha = 3, \ldots, \beta$ **do**

4: find $\lfloor (1 - (\alpha - 1)\beta^{-1}) b^{\beta m s} \rfloor + 1$ vectors $\boldsymbol{q}$ in $\mathcal{T}_{\alpha-1}$ which satisfy

$$e_\alpha(\boldsymbol{q}, p) \le c_{s,b,\boldsymbol{\gamma},s,m,\beta,\alpha,\tau_\alpha} b^{-\tau_\alpha m} \quad \text{for all } 1 \le \tau_\alpha < \alpha$$

and label this set $\mathcal{T}_\alpha$.

5: **end for**

6: **return** Select $\boldsymbol{q}^*$ to be any vector from $\mathcal{T}_\beta$.

Algorithm 3 only presents the basic idea of a construction for p.l.p.s. which do well for a range of smoothness parameters. In practice this algorithm would not be applicable since it is much to time consuming. However, in [2, Algorithm 2] it has been show how Algorithm 3 can be combined with the CBC approach. This leads then to the following result which is [2, Theorem 4.2]:

Theorem 10. *Let* $s, m, \beta \in \mathbb{N}$*,* $\beta \ge 2$*, then one can construct a vector* $\boldsymbol{q} \in G_{b,\beta m}^s$ *such that*

$$e_\alpha(\boldsymbol{q}, p) \le c_{s,b,\alpha,\beta,\boldsymbol{\gamma},\tau_\alpha} b^{-\tau_\alpha m} \quad \textit{for all } 1 \le \tau_\alpha < \alpha$$

and for all $2 \le \alpha \le \beta$.

If $\sum_{i=1}^{\infty} \gamma_i^{1/\tau_\alpha} < \infty$, *then* $c_{s,b,\alpha,\beta,\boldsymbol{\gamma},\tau_\alpha} \le c_{\infty,b,\alpha,\beta,\boldsymbol{\gamma},\tau_\alpha} < \infty$, *i.e., the above bound can be made independent of the dimension* s.

Until now there exists no counterpart of the results from this section for l.p.s.

10 Summary and further comments

In this paper we reviewed the main progress in the analysis of p.l.p.s. over the last decade and we pointed out several connections to the theory of l.p.s.

For both concepts we have comparable discrepancy bounds and tractability properties, and the worst-case error analysis in several reproducing kernel Hilbert spaces follows parallel tracks. P.l.p.s. and l.p.s. can both be constructed with the (Fast) CBC approach and both can be made extensible in the number of elements. The tent transformation together with a suitable randomization leads in both cases to improved error bounds for smoother integrands.

However, there are also some differences. For example, with a slight generalization of the concept of p.l.p.s. one can achieve almost optimal convergence rates for smooth integrands (even with varying smoothness from a finite range) together with strong tractability which means that the error bound is independent of the dimension. Such a result is not known for l.p.s. until now. (But it is known that with l.p.s. one can obtain almost optimal convergence rates together with strong tractability for smooth *periodic* functions.)

A further difference is that for p.l.p.s. it makes sense to apply Owen's scrambling scheme since this preserves the (t, m, s)-net structure of a point set but not the geometric lattice structure. This leads to an improved error bound in the randomized setting, a result which is not known for l.p.s.

Also the consideration of the quality parameter t of l.p.s. makes in general little sense since these point sets are not constructed to have a good (t, m, s)-net structure. Nevertheless, the analog of the quality measure $\rho(\boldsymbol{q}, p) = m - t$ from Section 4 has some interpretation, namely it is the enhanced trigonometric degree of a lattice rule [7, 39]. A cubature rule of enhanced trigonometric degree δ is one that integrates all trigonometric polynomials of degree less then δ exactly. However, in this vein $\rho(\boldsymbol{q}, p) = m - t$ from Section 4 can also be interpreted as the enhanced Walsh degree of a polynomial lattice rule since any (t, m, s)-net in base b integrates all Walsh polynomials of degree $\le m - t$ exactly (this follows from [23, Lemma 1]).

A further point which was not discussed so far but which is worth to be mentioned is that with l.p.s. one can even obtain exponential convergence for

the worst-case error of infinitely times differentiable periodic functions; see [12]. Such a result in turn is not known for p.l.p.s. until now.

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References

1. Baldeaux, J. and Dick, J.: A construction of polynomial lattice rules with small gain coefficients. arXiv:1003.4785, 2010.
2. Baldeaux, J., Dick, J., Greslehner, J. and Pillichshamemr, F.: Construction algorithms for higher order polynomial lattice rules. To appear in J. Complexity, 2011.
3. Béjian, R.: Minoration de la discrépance d'une suite quelconque sur T. *Acta Arith.*, 41:185–202, 1982.
4. Bilyk, D., Lacey, M. T., and Vagharshakyan, A.: On the small ball inequality in all dimensions. *J. Funct. Anal.*, 254:2470–2502, 2008.
5. Cristea, L. L., Dick, J., Leobacher, G. and Pillichshammer, F.: The tent transformation can improve the convergence rate of quasi-Monte Carlo algorithms using digital nets. Numer. Math. 105: 413–455, 2007.
6. Cools, R., Kuo, F. Y. and Nuyens, D.: Constructing embedded lattice rules for multivariable integration. SIAM J. Sci. Comput. 28: 2162–2188, 2006.
7. Cools, R. and Lyness, J. N.: Three- and four-dimensional K-optimal lattice rules of moderate trigonometric degree. Math. Comp. 70: 1549–1567, 2001.
8. Dick, J.: The construction of extensible polynomial lattice rules with small weighted star discrepancy. Math. Comp. 76: 2077–2085, 2007.
9. Dick, J.: Walsh spaces containing smooth functions and quasi-Monte Carlo rules of arbitrary high order, SIAM J. Numer. Anal. 46 (2008) 1519–1553.
10. Dick, J., Kritzer, P., Leobacher, G. and Pillichshammer, F.: Constructions of general polynomial lattice rules based on the weighted star discrepancy. Finite Fields Appl. **13**: 1045–1070, 2007.
11. Dick, J., Kuo, F. Y., Pillichshammer, F. and Sloan, I. H.: Construction algorithms for polynomial lattice rules for multivariate integration. Math. Comp. 74: 1895–1921, 2005.
12. Dick, J., Larcher, G., Pillichshammer, F. and Woźniakowski, H.: Exponential convergence and tractability of multivariate integration for Korobov spaces. To appear in Math. Comp., 2011.
13. Dick, J., Leobacher, G. and Pillichshammer, F.: Construction algorithms for digital nets with low weighted star discrepancy. SIAM J. Numer. Anal. **43**: 76–95, 2005.
14. Dick, J., Niederreiter, H. and Pillichshammer, F.: Weighted star discrepancy of digital nets in prime bases. In *Monte Carlo and quasi-Monte Carlo methods 2004*, pages 77–96. Springer, Berlin, 2006.
15. Dick, J. and Pillichshammer, F.: Multivariate integration in weighted Hilbert spaces based on Walsh functions and weighted Sobolev spaces. J. Complexity 21: 149–195, 2005.
16. Dick, J. and Pillichshammer, F.: Strong tractability of multivariate integration of arbitrary high order using digitally shifted polynomial lattices rules. J. Complexity 23: 436–453, 2007.

17. Dick, J. and Pillichshammer, F.: *Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration.* Cambridge University Press, Cambridge, 2010.
18. Dick, J., Pillichshammer, F. and Waterhouse, B. J.: The construction of good extensible rank-1 lattices. Math. Comp. 77: 2345–2373, 2008.
19. Dick, J., Sloan, I. H., Wang, X. and Woźniakowski, H.: Liberating the weights. J. Complexity 20: 593–623, 2004.
20. Dick, J., Sloan, I. H., Wang, X. and Woźniakowski, H.: Good lattice rules in weighted Korobov spaces with general weights. Numer. Math. 103: 63–97, 2006.
21. Drmota, M. and Tichy, R. F.: *Sequences, Discrepancies and Applications.* Lecture Notes in Mathematics 1651, Springer, Berlin, 1997.
22. Faure, H.: Discrépance de suites associées à un système de numération (en dimension s). *Acta Arith.*, 41:337–351, 1982. (In French).
23. Hellekalek, P.: Digital (t, m, s)-nets and the spectral test. Acta Arith. 105: 197–204, 2002.
24. Hickernell, F. J.: Obtaining $O(N^{-2+\epsilon})$ convergence for lattice quadrature rules. In: *Monte Carlo and Quasi-Monte Carlo Methods 2000.* pages 274–289, Springer, Berlin, 2002.
25. Hickernell, F. J. and Niederreiter, H.: The existence of good extensible rank-1 lattices. J. Complexity 19: 286–300, 2003.
26. Hinrichs, A., Pillichshammer, F. and Schmid, W. Ch.: Tractability properties of the weighted star discrepancy. J. Complexity **24**: 134–143, 2008.
27. Hlawka, E.: Zur angenäherten Berechnung mehrfacher Integrale. Monatsh. Math. **66**: 140–151, 1962.
28. Joe, S.: Construction of good rank-1 lattice rules based on the weighted star discrepancy. In *Monte Carlo and quasi-Monte Carlo methods 2004*, pages 181–196. Springer, Berlin, 2006.
29. Korobov, N. M.: The approximate computation of multiple integrals. Dokl. Akad. Nauk SSSR **124**: 1207–1210, 1959. (Russian)
30. Korobov, N. M.: *Number-theoretic methods in approximate analysis* Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1963. (Russian)
31. Korobov, N. M.: On the calculation of optimal coefficients. Soviet Math. Dokl. **26**: 590–593, 1982.
32. Kritzer, P. and Pillichshammer, F.: Constructions of general polynomial lattices for multivariate integration. Bull. Austral. Math. Soc. 76: 93–110, 2007.
33. Kritzer, P. and Pillichshammer, F.: A lower bound on a quantity related to the quality of polynomial lattices. To appear in Funct. Approx. Comment. Math., 2011.
34. Kuipers, L. and Niederreiter, H.: *Uniform Distribution of Sequences.* John Wiley, New York, 1974; reprint, Dover Publications, Mineola, NY, 2006.
35. Kuo, F. Y.: Component-by-component constructions achieve the optimal rate of convergence for multivariate integration in weighted Korobov and Sobolev spaces. J. Complexity 19: 301–320, 2003.
36. Larcher, G.: A best lower bound for good lattice points. Monatsh. Math. 104: 45–51, 1987.
37. Larcher, G.: Nets obtained from rational functions over finite fields. Acta Arith. 63: 1–13, 1993.
38. Larcher, G., Lauss, A., Niederreiter, H. and Schmid, W. Ch.: Optimal polynomials for (t, m, s)-nets and numerical integration of multivariate Walsh series. SIAM J. Numer. Anal., **33**: 2239–2253, 1996.
39. Lyness, J.: Notes on lattice rules. J. Complexity 19: 321–331, 2003.
40. Niederreiter, H.: Point sets and sequences with small discrepancy. Monatsh. Math. **104**: 273–337 1987.
41. Niederreiter, H.: Low-discrepancy point sets obtained by digital constructions over finite fields. Czechoslovak Math. J. **42**: 143–166, 1992.

42. Niederreiter, H.: *Random Number Generation and Quasi-Monte Carlo Methods.* No. 63 in CBMS-NSF Series in Applied Mathematics. SIAM, Philadelphia, 1992.
43. Niederreiter, H.: The existence of good extensible polynomial lattice rules. Monatsh. Math. **139**: 295–307, 2003.
44. Niederreiter, H. and Pillichshammer, F.: Construction algorithms for good extensible lattice rules. Constr. Approx. 30: 361–393, 2009.
45. Novak, E. and Woźniakowski, H.: *Tractability of Multivariate Problems. Volume I: Linear Information.* EMS Tracts in Mathematics, 6. European Mathematical Society (EMS), Zürich, 2008.
46. Nuyens, D. and Cools, R.: Fast algorithms for component-by-component construction of rank-1 lattice rules in shift-invariant reproducing kernel Hilbert spaces. Math. Comp. 75: 903–920, 2006.
47. Nuyens, D. and Cools, R.: Fast component-by-component construction, a reprise for different kernels. In *Monte Carlo and Quasi-Monte Crlo Methods 2004*, pages 373–387. Springer, Berlin, 2006.
48. Owen, A. B.: Randomly permuted (t, m, s)-nets and (t, s)-Sequences. In *Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing*, pages 299–317. Springer, New York, 1995.
49. Roth, K. F.: On irregularities of distribution. *Mathematika*, 1:73–79, 1954.
50. Šarygin, I. F.: Lower bounds for the error of quadrature formulas on classes of functions. Ž. Vyčisl. Mat. i Mat. Fiz. 3: 370–376, 1963 (Russian).
51. Schmid, W. Ch.: Improvements and extensions of the "Salzburg Tables" by using irreducible polynomials. In *Monte Carlo and Quasi-Monte Carlo Methods 1998*, pages 436–447. Springer, Berlin, 2000.
52. Schmidt, W. M.: Irregularities of distribution VII. *Acta Arith.*, 21:45–50, 1972.
53. Sloan, I. H. and Joe, S.: *Lattice Methods for Multiple Integration.* Clarendon Press, Oxford, 1994.
54. Sloan, I. H. and Reztsov, A. V.: Component-by component construction of good lattice rules. Math. Comp. 71: 263–273, 2002.
55. Sloan, I. H. and Woźniakowski, H.: When are quasi-Monte Carlo algorithms efficient for high dimensional integrals? J. Complexity **14**: 1–33, 1998.
56. Sloan, I. H. and Woźniakowski, H.: Tractability of multivariate integration for weighted Korobov classes. J. Complexity 17: 697–721, 2001.
57. Sloan, I. H. and Woźniakowski, H.: Tractability of integration in non-periodic and periodic weighted tensor product Hilbert spaces. J. Complexity 18: 479–499, 2002.
58. Sobol', I. M.: Distribution of points in a cube and approximate evaluation of integrals. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 7:784–802, 1967.