The quality parameter of cyclic nets and hyperplane nets

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Abstract

Inspired by constructions from coding theory, Niederreiter introduced the concept of cyclic digital nets. Later this concept has been generalized to so-called hyperplane nets and it turned out that in some sense this constructions can be viewed as a generalization of polynomial lattices. We introduce a figure of merit with which one can determine the quality parameter t of hyperplane nets. Furthermore, we show the existence of hyperplane nets of good quality.

1 Introduction

Quasi-Monte Carlo algorithms for numerical integration approximate the integral of a function over the (often high-dimensional) unit-cube by the average of function evaluations over a well-chosen deterministic point set. Here an appropriate choice of the underlying point set becomes increasingly more important as the dimension of the problem grows. It has been shown that point sets chosen from the unit-cube having a very uniform distribution yield small integration errors, at least for functions with bounded variation in the sense of Hardy and Krause (see [7]).

Currently the best constructions of well distributed finite point sets are based on the concept of (t, m, s)-nets in base b as introduced by Niederreiter [5] (see also [7, Chapter 4] for a survey of this theory).

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Definition 1 ((t, m, s)-nets) Let $b \geq 2$, $s \geq 1$ and $0 \leq t \leq m$ be integers. A point set $\mathcal{P}$ consisting of b^m points in $[0,1)^s$ forms a (t, m, s)-net in base b, if every subinterval of the form $J = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1)b^{-d_i})$ of $[0,1)^s$, with integers $d_i \geq 0$ and integers $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$ and of volume b^{t-m}, contains exactly b^t points of $\mathcal{P}$.

Note that for any point set of b^m points there always exists a $t \in \{0, \ldots, m\}$ such that it is a (t, m, s)-net in base b, e.g., we can always choose $t = m$. Smaller values of t imply stronger equidistribution properties for nets. With regard to this fact, t is often called the *quality parameter* of the net.

Concrete constructions of (t, m, s)-nets are based on the digital construction scheme which we recall in the following. To avoid too many technical notions - and since we only deal with this case - in the following we restrict ourselves to digital nets defined over the finite field $\mathbb{F}_q$ of prime-power order q. For a more general definition (over arbitrary finite, commutative rings) see for example Niederreiter [7], Larcher [2], or Larcher, Niederreiter and Schmid [4].

From now on let q be a prime-power and let $\mathbb{F}_q$ be the finite field of q elements. For a positive integer r let $\mathbb{Z}_r = \{0, \ldots, r-1\}$. Let $\varphi : \mathbb{Z}_q \to \mathbb{F}_q$ be a fixed bijection with $\varphi(0) = 0$. We also extend φ to integers in $\mathbb{Z}_{q^m}$ by setting

$$\varphi(k) := (\varphi(\kappa_0), \ldots, \varphi(\kappa_{m-1}))^\top \tag{1}$$

for $k = \kappa_0 + \kappa_1 q + \cdots + \kappa_{m-1} q^{m-1}$ with $\kappa_0, \ldots, \kappa_{m-1} \in \mathbb{Z}_q$. Here $\boldsymbol{x}^\top$ means the transpose of the vector $\boldsymbol{x}$. For $k \in \{0, \ldots, q-1\}$ it will always be clear from the context whether by $\varphi(k)$ we mean an element in $\mathbb{F}_q$ or an m-dimensional vector in $\mathbb{F}_q^m$.

Definition 2 (digital (t, m, s)-nets) Let $s \geq 1$ and $m \geq 1$ be integers. Let $C_1, \ldots, C_s$ be $m \times m$ matrices over $\mathbb{F}_q$. Now we construct q^m points in $[0,1)^s$: For $1 \leq i \leq s$ and for $k \in \mathbb{Z}_{q^m}$ multiply the matrix C_i by the vector $\varphi(k)$, i.e.,

$$C_i \varphi(k) =: (y_{i,1}(k), \ldots, y_{i,m}(k))^\top \in \mathbb{F}_q^m,$$

and set

$$x_{k,i} := \frac{\varphi^{-1}(y_{i,1}(k))}{q} + \cdots + \frac{\varphi^{-1}(y_{i,m}(k))}{q^m}.$$

If for some integer t with $0 \leq t \leq m$ the point set consisting of the points

$$\boldsymbol{x}_k = (x_{k,1}, \ldots, x_{k,s}) \quad \text{for} \quad k \in \mathbb{Z}_{q^m},$$

is a (t, m, s)-net in base q, then it is called a digital (t, m, s)-net over $\mathbb{F}_q$, or, in brief, a digital net (over $\mathbb{F}_q$). The C_i are called its *generator matrices*.

Remark 1 Let $C_1, \ldots, C_s$ be the generator matrices of a digital net over $\mathbb{F}_q$ and let $\mathcal{C} = \{\boldsymbol{c}_j^{(i)} \in \mathbb{F}_q^m : 1 \le j \le m, 1 \le i \le s\}$ be the system of the row vectors of these matrices ($\boldsymbol{c}_j^{(i)}$ is the j-th row vector of matrix C_i, $1 \le j \le m$, $1 \le i \le s$). Let $\rho(\mathcal{C})$ be the largest integer d such that any system $\{\boldsymbol{c}_j^{(i)} \in \mathbb{F}_q^m : 1 \le j \le d_i, 1 \le i \le s\}$ with $0 \le d_i \le m$ for $1 \le i \le s$ and $\sum_{i=1}^s d_i = d$ is linearly independent in $\mathbb{F}_q^m$ (the empty system is viewed as linearly independent). Then the digital net with generator matrices $C_1, \ldots, C_s$ is a digital (t, m, s)-net over $\mathbb{F}_q$ with $t = m - \rho(\mathcal{C})$. This was shown by Niederreiter [7, Theorem 4.28]. For an effective method to establish the quality parameter from given matrices $C_1, \ldots, C_s$ we refer to [16].

Many constructions of digital nets are inspired by a close connection between coding theory and the theory of digital nets (see, for example, Niederreiter [8] or [9]). Examples for that are the so-called $(u, u+v)$-construction (see [1, 11]), the matrix-product construction (see [10]) and the Kronecker-product construction (see [1, 12]). Here we deal with a construction for digital nets which is an analog to a special type of codes, namely to cyclic codes which are well known in coding theory. This construction has been introduced by Niederreiter in [8] who used the fact that cyclic codes can be defined by prescribing roots of polynomials. Later this construction has been generalized by Pirsic, Dick and Pillichshammer [15] to so-called hyperplane nets whose definition will be given now.

Definition 3 (hyperplane nets) Let integers $m \ge 1, s \ge 2$ and a prime-power q be given. Let $\mathbb{F}_{q^m}$ be a finite field with q^m elements and fix an element $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in \mathbb{F}_{q^m}^s$. Let $\mathcal{F}$ be the space of linear forms

$$\mathcal{F} := \{f(x_1, \ldots, x_s) = x_1\gamma_1 + \cdots + x_s\gamma_s : \gamma_1, \ldots, \gamma_s \in \mathbb{F}_{q^m}\} \subseteq \mathbb{F}_{q^m}[x_1, \ldots, x_s]$$

and consider the subset

$$\mathcal{F}_{\boldsymbol{\alpha}} := \{f \in \mathcal{F} : f(\alpha_1, \ldots, \alpha_s) = 0\}.$$

For each $1 \le i \le s$ choose an ordered basis $\mathcal{B}_i$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ and define the mapping $\phi : \mathcal{F} \to \mathbb{F}_q^{ms}$ by

$$f(x) = \sum_{i=1}^{s} \gamma_i x^{i-1} \in \mathcal{F} \mapsto (\gamma_{1,1}, \ldots, \gamma_{1,m}, \ldots, \gamma_{s,1}, \ldots, \gamma_{s,m}) \in \mathbb{F}_q^{ms},$$

where $(\gamma_{i,1}, \ldots, \gamma_{i,m})$ is the coordinate vector of γ_i with respect to the chosen basis $\mathcal{B}_i$.

We denote by $\mathcal{C}_\alpha$ the orthogonal subspace in $\mathbb{F}_q^{ms}$ of the image $\mathcal{N}_\alpha := \phi(\mathcal{P}_\alpha)$. Let

$$C_\alpha = (C_1^\top \cdots C_s^\top) \in \mathbb{F}_q^{m \times sm}$$

be a matrix whose row space is $\mathcal{C}_\alpha$. Then $C_1, \ldots, C_s$ are the generating matrices of a *hyperplane net over* $\mathbb{F}_q$ *with respect to* $\mathcal{B}_1, \ldots, \mathcal{B}_s$ and C_α is its overall generating matrix. This hyperplane net will be denoted by $\mathcal{P}_{\boldsymbol{\alpha}}$ and we say $\mathcal{P}_{\boldsymbol{\alpha}}$ is the hyperplane net associated to $\boldsymbol{\alpha}$. We shall from now on assume a fixed choice of bases $\mathcal{B}_1, \ldots, \mathcal{B}_s$ and will therefore not explicitly mention them anymore.

Remark 2 In Definition 3 above, if $\boldsymbol{\alpha} \in \mathbb{F}_{q^m}^s$ is of the special form $\boldsymbol{\alpha} = (1, \alpha, \alpha^2, \ldots, \alpha^{s-1})$ with some $\alpha \in \mathbb{F}_{q^m}$, then we obtain a *cyclic digital net* as introduced initially by Niederreiter [8]. This cyclic net will be denoted by $\mathcal{P}_\alpha$ and we say $\mathcal{P}_\alpha$ is the cyclic net associated to α.

In this paper we introduce a figure of merit for hyperplane nets which allows to express the quality parameter t in terms of $\boldsymbol{\alpha}$. Based on this figure of merit we will show the existence of $\boldsymbol{\alpha} \in \mathbb{F}_{q^m}^s$ which yield hyperplane nets of good quality with respect to the quality parameter t.

2 Preliminary results

Let $\mathbb{F}_{q^m} = \mathbb{F}_q[\omega]$ such that $\{1, \omega, \ldots, \omega^{m-1}\}$ form a basis of $\mathbb{F}_{q^m}$ as vector space over $\mathbb{F}_q$. Let $\omega^m = \beta_0 + \cdots + \beta_{m-1}\omega^{m-1}$ where $\beta_0, \ldots, \beta_{m-1} \in \mathbb{F}_q$ and let P be the companion matrix of ω, i.e.,

$$P := \begin{pmatrix} 0 & 0 & 0 & \cdots & \beta_0 \\ 1 & 0 & 0 & \cdots & \beta_1 \\ 0 & 1 & 0 & \cdots & \beta_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & \beta_{m-1} \end{pmatrix} \in \mathbb{F}_q^{m \times m}.$$

Now, if we have the representation of α in $\mathbb{F}_{q^m}$ as $\alpha = \sum_{l=0}^{m-1} a_l \omega^l$, where $a_0, \ldots, a_{m-1} \in \mathbb{F}_q$, then we define

$$\psi(\alpha) := (a_0, \ldots, a_{m-1}) \in \mathbb{F}_q^m \quad \text{and} \quad \Psi(\alpha) := \sum_{l=0}^{m-1} a_l P^l \in \mathbb{F}_q^{m \times m}.$$

With these definitions for any $\alpha, \beta \in \mathbb{F}_{q^m}$ we have

$$\psi(\alpha\beta) = \Psi(\alpha)\psi(\beta).$$

This follows by first showing the identity for α and β equal to powers of ω and then using linearity.

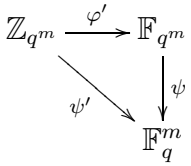
Note that for any $\alpha, \beta \in \mathbb{F}_{q^m}^* := \mathbb{F}_{q^m} \setminus \{0\}$ we have $\Psi(\alpha)\psi(\beta) = \psi(\alpha\beta) \neq \mathbf{0} \in \mathbb{F}_q^m$ as $\alpha\beta \neq 0 \in \mathbb{F}_{q^m}$. Hence it follows that for any $\alpha \in \mathbb{F}_{q^m}^*$ we have that the matrix $\Psi(\alpha)$ is regular.

Furthermore, for $k = \sum_{l=0}^{m-1} \kappa_l q^l \in \mathbb{Z}_{q^m}$ and a bijection $\varphi : \mathbb{Z}_q \to \mathbb{F}_q$ let

$$\varphi'(k) := \sum_{l=0}^{m-1} \varphi(\kappa_l)\omega^l \quad \text{and} \quad \psi'(k) := \psi(\varphi'(k)).$$

Note that we have $\psi' = \varphi$ when we extend the bijection φ to integers in $\mathbb{Z}_{q^m}$ as in (1).

We summarize the above definition in the following commutative diagram:



Remark 3 Let $m, s, \mathbb{F}_q$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in \mathbb{F}_{q^m} = \mathbb{F}_q[\omega]$ be given as above and define s matrices $B_i = (\psi(b_{i,1}), \ldots, \psi(b_{i,m})))^{-1}$, where the $b_{i,1}, \ldots, b_{i,s}$ constitute the chosen basis $\mathcal{B}_i$ for $1 \leq i \leq s$. Then the generator matrices of the hyperplane net $\mathcal{P}_{\boldsymbol{\alpha}}$ are given by $C_i = (\Psi(\alpha_i)B_i)^\top$ for $1 \leq i \leq s$. Furthermore it follows that C_i is regular whenever $\alpha_i \neq 0$. For a proof we refer to [15]

Remark 4 Another construction of digital nets goes by the name of *polynomial lattices* which have been introduced by Niederreiter [6] (see also [7]). It has been shown by Pirsic [14] that polynomial lattices appear as a special cases of hyperplane nets when we choose the ordered basis $\mathcal{B}_1, \ldots, \mathcal{B}_s$ all equal to $\{1, \omega, \ldots, \omega^{m-1}\}$ if $\mathbb{F}_{q^m} = \mathbb{F}_q[\omega]$ as before.

For hyperplane nets we can express the dual space in terms of $\boldsymbol{\alpha} \in \mathbb{F}_{q^m}^s$. The following lemma has been given implicitly already in [15, Lemma 2.5]. In view of Remark 4 this lemma corresponds to [7, Lemma 4.40].

Lemma 1 *Let the $m \times m$ matrices $C_1, \ldots, C_s$ be the generator matrices of a hyperplane net over $\mathbb{F}_q$ as given in Remark 3. Then for any integers $k_1, \ldots, k_s \in \mathbb{Z}_{q^m}$ we have*

$$C_1^\top \psi'(k_1) + \cdots + C_s^\top \psi'(k_s) = \mathbf{0} \in \mathbb{F}_q^m \tag{2}$$

if and only if

$$\alpha_1 \varphi'(\tau_1(k_1)) + \cdots + \alpha_s \varphi'(\tau_s(k_s)) = 0 \in \mathbb{F}_{q^m} \tag{3}$$

with permutations $\tau_i(k) = \psi'^{-1}(B_i \psi'(k))$, and B_i as in Remark 3 for all $1 \le i \le s$.

Proof. By Remark 3 we have

$$\begin{aligned} \mathbf{0} &= \sum_{i=1}^{s} C_i^\top \psi'(k_i) = \sum_{i=1}^{s} \Psi(\alpha_i) B_i \psi'(k_i) \\ &= \sum_{i=1}^{s} \Psi(\alpha_i) \psi'(\tau_i(k_i)) = \sum_{i=1}^{s} \psi(\alpha_i \varphi'(\tau_i(k_i))) = \psi\Big(\sum_{i=1}^{s} \alpha_i \varphi'(\tau_i(k_i)) \Big), \end{aligned}$$

and this holds if and only if $\sum_{i=1}^{s} \alpha_i \varphi'(\tau_i(k_i)) = 0$ and we are done. □

3 The quality parameter of hyperplane nets

For polynomial lattices there exists a so-called figure of merit [7, Definition 4.39] which is based on the associated dual space and with which one can express the quality parameter t of a polynomial lattice considered as digital net, see [7, Theorem 4.42]. These ideas can be transferred to the more general concept of hyperplane nets.

Definition 4 For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in \mathbb{F}_{q^m}^s$ the figure of merit $\rho(\boldsymbol{\alpha})$ is defined as

$$\rho(\boldsymbol{\alpha}) = s - 1 + \min \sum_{i=1}^{s} \lfloor \log_q(k_i) \rfloor,$$

where the minimum is extended over all $k_1, \ldots, k_s \in \mathbb{Z}_{q^m}$, not all zero, such that $\alpha_1 \varphi'(\tau_1(k_1)) + \cdots + \alpha_s \varphi'(\tau_s(k_s)) = 0 \in \mathbb{F}_{q^m}$. Here $\log_q$ denotes the logarithm to base q and we use the convention $\lfloor \log_q(0) \rfloor := -1$.

With this figure of merit at hand we may now give a formula for the quality parameter t of a hyperplane net.

Theorem 5 *The hyperplane net $\mathcal{P}_{\boldsymbol{\alpha}}$ associated to $\boldsymbol{\alpha} \in \mathbb{F}_{q^m}^s$ is a digital (t, m, s)-net over $\mathbb{F}_q$ with $t = m - \rho(\boldsymbol{\alpha})$.*

Proof. Let $\boldsymbol{c}_j^{(i)}$ be the j-th row vector of the i-th generator matrix C_i of the hyperplane net. Then by Remark 1, $t = m - \rho(\mathcal{C}) =: m - \rho_1$, where $\rho_1 + 1$ is the smallest integer such that there exist $d_1, \ldots, d_s \in \mathbb{N}_0$ with $\sum_{i=1}^s d_i = \rho_1 + 1$ and $\lambda_{i,j} \in \mathbb{F}_q$, $1 \le i \le s$, $1 \le j \le m$ such that

$$\sum_{i=1}^{s} \sum_{j=1}^{d_i} \lambda_{i,j} \boldsymbol{c}_j^{(i)} = 0. \tag{4}$$

Equivalently, by the isomorphism ψ', the integer $\rho_1 + 1$ is minimal such that there exist $k_i \in \mathbb{Z}_{q^m}$, $i = 1, \ldots, s$ such that (2) holds, where $k_i = \sum_{j=1}^m \nu_{i,j} q^{j-1}$, with

$$\rho_1 + 1 = \sum_{i=1}^{s} d_i := \sum_{i=1}^{s} \max\{j : \nu_{i,j} \neq 0\} = \sum_{i=1}^{s} \lfloor \log_q(k_i) + 1 \rfloor$$

(choose $k_i = (\psi')^{-1}(\,(\lambda_{i,1}, \ldots, \lambda_{i,m})\,)$ with the $\lambda_{i,j}$ from above).

Finally, by Lemma 1

$$\begin{aligned} &\min \left\{ \sum_{i=1}^{s} \lfloor \log_q(k_i) + 1 \rfloor : \ (2) \text{ holds} \right\} \\ &= \ \min \left\{ \sum_{i=1}^{s} \lfloor \log_q(k_i) + 1 \rfloor : \ (3) \text{ holds} \right\} = \rho(\boldsymbol{\alpha}) + 1, \end{aligned}$$

so that altogether

$$t = m - \rho(\mathcal{C}) = m - \left(\min \left\{ \sum_{i=1}^{s} d_i : \ (4) \text{ holds} \right\} - 1 \right) = m - \rho(\boldsymbol{\alpha})$$

as desired. □

From the definition of the figure of merit $\rho(\boldsymbol{\alpha})$ and from Theorem 5 we see that it is enough to consider vectors $\boldsymbol{\alpha}$ of the form $\boldsymbol{\alpha} = (1, \alpha_2, \ldots, \alpha_s)$ only, where 1 denotes the neutral element with respect to multiplication in $\mathbb{F}_q$. The following result is the analogon of [3, Theorem 1] and [17, Theorem 6] for hyperplane nets.

Theorem 6 *Let $s \geq 2$, $m \geq 1$ and let q be a prime-power. Choose ordered bases $\mathcal{B}_1, \ldots, \mathcal{B}_s$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. For $\rho \in \mathbb{Z}$ define*

$$\Delta_q(s,\rho) = \sum_{d=0}^{s-1} \binom{s}{d} (q-1)^{s-d} \sum_{\gamma=0}^{\rho+d} \binom{s-d+\gamma-1}{\gamma} q^{\gamma} + 1 - q^{\rho+s}.$$

1. *If $\Delta_q(s,\rho) < q^m$, then there exists an $\boldsymbol{\alpha} = (1, \alpha_2, \ldots, \alpha_s) \in \mathbb{F}_{q^m}^s$ with $\rho(\boldsymbol{\alpha}) \geq s + \rho$. Therefore the hyperplane net $\mathcal{P}_{\boldsymbol{\alpha}}$ is a digital (t, m, s)-net over $\mathbb{F}_q$ with $t \leq m - s - \rho$.*

2. *If $\Delta_q(s,\rho) < \frac{q^m}{s-1}$, then there exists an element $\alpha \in \mathbb{F}_{q^m}$ such that $\boldsymbol{\alpha} = (1, \alpha, \ldots, \alpha^{s-1})$ satisfies $\rho(\boldsymbol{\alpha}) \geq s + \rho$. Therefore the cyclic net $\mathcal{P}_{\alpha}$ is a digital (t, m, s)-net over $\mathbb{F}_q$ with $t \leq m - s - \rho$.*

The proof of Theorem 6 is nearly the same as that of [3, Theorem 1]. However, to see the differences we present the proof of the first part of Theorem 6. First of all we need the following result.

Lemma 2 *For a prime-power q and integers $l \geq 1$ and k the number $A_q(l,k)$ of $(h_1, \ldots, h_l) \in (\mathbb{Z}_{q^m}^*)^l$ such that $\sum_{i=1}^{l} \lfloor \log_q(h_i) \rfloor \leq k$ is given by*

$$A_q(l,k) = (q-1)^l \sum_{\gamma=0}^{k} \binom{l+\gamma-1}{\gamma} q^{\gamma}.$$

Proof. We have $A_q(l,k) = \sum_{\gamma=0}^{k} D_q(l,\gamma)$ where $D_q(l,\gamma)$ denotes the number of $(h_1, \ldots, h_l) \in (\mathbb{Z}_{q^m}^*)^l$ such that $\sum_{i=1}^{l} \lfloor \log_q(h_i) \rfloor = \gamma$. Now for $\gamma \geq 0$ there are $\binom{l+\gamma-1}{\gamma}$ l-tuples $(d_1, \ldots, d_l)$ with integers $d_i \geq 0$ for $1 \leq i \leq l$ and $\sum_{i=1}^{l} d_i = \gamma$. For each such l-tuple $(d_1, \ldots, d_l)$ there are $(q-1)^l q^{d_1 + \cdots + d_l} = (q-1)^l q^{\gamma}$ tuples $(h_1, \ldots, h_l) \in (\mathbb{Z}_{q^m}^*)^l$ such that $\lfloor \log_q(h_i) \rfloor = d_i$ for $1 \leq i \leq l$. The result follows. □

We give the proof of Theorem 6.

Proof. Let $M_q(s,\rho)$ be the number of $(k_1,\ldots,k_s) \in \mathbb{Z}_{q^m}^s$ with $(k_2,\ldots,k_s) \neq (0,\ldots,0)$ and $\sum_{i=1}^s \lfloor \log_q(k_i) \rfloor \leq \rho$. Using the notation and the result of Lemma 2, we get

$$M_q(s,\rho) = \sum_{d=0}^{s-1} \binom{s}{d} A_q(s-d,\rho+d) + 1 - q^{\rho+s} = \Delta_q(s,\rho).$$

(Recall the convention that $\lfloor \log_q(0) \rfloor = -1$.)

For a given nonzero s-tuple $(k_1,\ldots,k_s) \in \mathbb{Z}_{q^m}^s$ the equation $\varphi'(\tau_1(k_1)) + \alpha_2\varphi'(\tau_2(k_2)) + \cdots + \alpha_s\varphi'(\tau_s(k_s)) = 0$ has no solution if $k_2 = \cdots = k_s = 0$ (note that $\varphi(\tau_i(0)) = 0$ for all $1 \leq i \leq s$), and it has exactly $q^{m(s-2)}$ solutions $\boldsymbol{\alpha} = (1,\alpha_2,\ldots,\alpha_s) \in \mathbb{F}_{q^m}^s$ otherwise (note that $\varphi' \circ \tau_i$ are bijections for all $1 \leq i \leq s$). Therefore, to all nonzero $(k_1,\ldots,k_s)$ with $\sum_{i=1}^s \lfloor \log_q(k_i) \rfloor \leq \rho$ there are assigned altogether at most $M_q(s,\rho)q^{m(s-2)}$ different solutions $\boldsymbol{\alpha} = (1,\alpha_2,\ldots,\alpha_s) \in \mathbb{F}_{q^m}^s$ of the above equation. Now the total number of $\boldsymbol{\alpha} = (1,\alpha_2,\ldots,\alpha_s) \in \mathbb{F}_{q^m}^s$ is $q^{m(s-1)}$. Thus, if $M_q(s,\rho)q^{m(s-2)} < q^{m(s-1)}$, that is, if $\Delta_q(s,\rho) < q^m$, then there exists at least one $\boldsymbol{\alpha} = (1,\alpha_2,\ldots,\alpha_s) \in \mathbb{F}_{q^m}^s$ such that $\varphi'(\tau_1(k_1)) + \alpha_2\varphi'(\tau_2(k_2)) + \cdots + \alpha_s\varphi'(\tau_s(k_s)) \neq 0$ for all nonzero $(k_1,\ldots,k_s) \in \mathbb{Z}_{q^m}^s$ with $\sum_{i=1}^s \lfloor \log_q(k_i) \rfloor \leq \rho$. For this $\boldsymbol{\alpha}$ we have then $\rho(\boldsymbol{\alpha}) \geq s + \rho$. By Theorem 5 the point set $\mathcal{P}_{\boldsymbol{\alpha}}$ is a digital (t,m,s)-net over $\mathbb{F}_q$ with $t \leq m - s - \rho$. □

We obtain the following corollary whose proof is identical to that of [3, Corollary 1].

Corollary 1 *Let $s \geq 2$ be an integer, let q be a prime-power and let m be a sufficiently large integer.*

1. *There exists a vector $\boldsymbol{\alpha} = (1,\alpha_2,\ldots,\alpha_s) \in \mathbb{F}_{q^m}^s$ with*

$$\rho(\boldsymbol{\alpha}) \geq \lfloor m - (s-1)(\log_q m - 1) + \log_q(s-1)! \rfloor.$$

2. *There exists an element $\alpha \in \mathbb{F}_{q^m}$ such that $\boldsymbol{\alpha} = (1,\alpha,\ldots,\alpha^{s-1})$ satisfies*

$$\rho(\boldsymbol{\alpha}) \geq \lfloor m - (s-1)(\log_q m - 1) + \log_q(s-2)! \rfloor.$$

The following result on the star discrepancy of hyperplane nets and cyclic nets can be obtained from Corollary 1 in combination with [7, Theorem 4.10].

Corollary 2 *Let $s \geq 2$ be an integer, let q be a prime-power and let m be a sufficiently large integer.*

1. *There exists a vector $\boldsymbol{\alpha} = (1, \alpha_2, \ldots, \alpha_s) \in \mathbb{F}_{q^m}^s$ such that the star discrepancy of the hyperplane net $\mathcal{P}_{\boldsymbol{\alpha}}$ satisfies*

$$D_{q^m}^*(\mathcal{P}_{\boldsymbol{\alpha}}) = O(m^{2s-2} q^{-m})$$

   with an implied constant only depending on q and s.

2. *There exists an element $\alpha \in \mathbb{F}_{q^m}$ such that the star discrepancy of the cyclic net $\mathcal{P}_\alpha$ satisfies*

$$D_{q^m}^*(\mathcal{P}_\alpha) = O(m^{2s-2} q^{-m})$$

   with an implied constant only depending on q and s.

More detailed results on the discrepancy of hyperplane nets will be given in the forthcoming paper [13].

References

[1] Bierbrauer, J., Edel, Y. and Schmid, W. Ch.: Coding-theoretic constructions for (t, m, s)-nets and ordered orthogonal arrays. J. Combin. Des., **10**: 403–418, 2002.

[2] Larcher, G.: Digital point sets: analysis and application. In: *Random and Quasi-Random Point Sets*, pages 167–222. Springer Lecture Notes in Statistics 138, New York, 1998.

[3] Larcher, G., Lauss, A., Niederreiter, H. and Schmid, W. Ch.: Optimal polynomials for (t, m, s)-nets and numerical integration of multivariate Walsh series. SIAM J. Numer. Anal., **33**: 2239–2253, 1996.

[4] Larcher, G., Niederreiter, H., and Schmid, W.Ch.: Digital nets and sequences constructed over finite rings and their application to quasi-Monte Carlo integration. Monatsh. Math., **121**: 231–253, 1996.

[5] Niederreiter, H.: Point sets and sequences with small discrepancy. Monatsh. Math., **104**: 273–337, 1987.

[6] Niederreiter, H.: Low-discrepancy point sets obtained by digital constructions over finite fields. Czechoslovak Math. J., **42**: 143–166, 1992.

[7] Niederreiter, H.: *Random Number Generation and Quasi-Monte Carlo Methods.* No. 63 in CBMS-NSF Series in Applied Mathematics. SIAM, Philadelphia, 1992.

[8] Niederreiter, H.: Digital nets and coding theory. In *Coding, Cryptography and Combinatorics*, pages 247–257. Birkhäuser, Basel, 2004.

[9] Niederreiter, H.: Nets, (t, s)-sequences and codes. In *Monte Carlo and quasi-Monte Carlo methods 2006*, pages 83–100. Springer, Berlin, 2008.

[10] Niederreiter, H. and Özbudak, F.: Matrix-product constructions of digital nets. Finite Fields Appl., **10**: 464–479, 2004.

[11] Niederreiter, H. and Pirsic, G.: Duality for digital nets and its applications. Acta Arith., **97**: 173–182, 2002.

[12] Niederreiter, H. and Pirsic, G.: A Kronecker product construction for digital nets. In *Monte Carlo and quasi-Monte Carlo methods 2000*, pages 396–405. Springer, Berlin, 2002.

[13] Pillichshammer, F. and Pirsic, G.: Discrepancy of hyperplane nets and cyclic nets. In preparation. 2008.

[14] Pirsic, G.: A small taxonomy of integration node sets. Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II, **214** (2005): 133–140, (2006).

[15] Pirsic, G., Dick, J. and Pillichshammer, F.: Cyclic digital nets, hyperplane nets, and multivariate integration in Sobolev spaces. SIAM J. Numer. Anal., **44**: 385–411, 2006.

[16] Pirsic, G. and Schmid, W. Ch.: Calculation of the quality parameter of digital nets and application to their construction. J. Complexity, **17**: 827–839, 2001.

[17] Schmid, W. Ch.: Improvements and extensions of the "Salzburg tables" by using irreducible polynomials. In *Monte Carlo and quasi-Monte Carlo methods 1998*, pages 436–447. Springer, Berlin, 2000.

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