Number systems and tilings over Laurent series

By TOBIAS BECK†
Franziskanerstrasse 15, D-81669 München, GERMANY,
tobias.beck@gmx.de

HORST BRUNOTTE
Haus-Endt-Strasse 88, D-40593 Düsseldorf, GERMANY,
brunoth@web.de

KLAUS SCHEICHER‡
Institut für Mathematik, Universität für Bodenkultur,
Gregor Mendel Strasse 33, A-1180 Wien, AUSTRIA,
klaus.scheicher@boku.ac.at

and JÖRG M. THUSWALDNER §
Institut für Mathematik und Informationstechnologie,
Abteilung für Mathematik und Statistik, Montanuniversität Leoben,
Franz Josef Strasse 18, A-8700 Leoben, AUSTRIA,
joerg.thuswaldner@unileoben.ac.at

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Dedicated to Professor Robert F. Tichy on the occasion of his 50$^{\text{th}}$ *birthday*

Abstract

Let $\mathbb{F}$ be a field and $\mathbb{F}[x,y]$ the ring of polynomials in two variables over $\mathbb{F}$. Let $f \in \mathbb{F}[x,y]$ and consider the residue class ring $R := \mathbb{F}[x,y]/f\mathbb{F}[x,y]$. Our first aim is to study digit representations in R, i.e., we ask for which f each element of R admits a digit representation of the form $d_0 + d_1 x + \cdots + d_\ell x^\ell$ with digits $d_i \in \mathbb{F}[y]$ satisfying $\deg_y(d_i) < \deg_y(f)$. These digit systems are motivated by the well-known notion of canonical number systems. In a next step we enlarge the ring in order to allow representations including negative powers of the "base" x. In particular, we define and characterize digit representations for the ring $S := \mathbb{F}((x^{-1}, y^{-1}))/f\,\mathbb{F}((x^{-1}, y^{-1}))$ and give easy to handle criteria for finiteness and periodicity of such representations. Finally, we attach fundamental domains to our digit systems. The fundamental domain of a digit system is the set of all elements having only negative powers of x in their "x-ary" representation. The translates of the fundamental domain induce a tiling of S. Interestingly, the funda-

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mental domains of our digit systems turn out to be unions of boxes. If we choose $\mathbb{F} = \mathbb{F}_q$ to be a finite field, these unions become finite.

1. *Introduction*

Since the end of the 19th century various generalizations of the usual radix representation of the integers to other algebraic structures have been introduced and extensively investigated. A prominent class of number systems are *canonical number systems* in residue class rings of polynomials over $\mathbb{Z}$. A special instance of a canonical number system has first been investigated by KNUTH [**14**]. Later on, they have been studied thoroughly by GILBERT, GROSSMAN, KÁTAI, KOVÁCS and SZABÓ (cf. for instance [**6**, **7**, **11**, **12**, **13**]). The most general definition of canonical number systems is due to PETHŐ [**15**] and reads as follows. Let $P \in \mathbb{Z}[x]$ and consider the residue class ring $\mathbb{Z}[x]/P\mathbb{Z}[x]$. Then $(x, \{0, 1, \ldots, |P(0)| - 1\})$ is called a *canonical number system* in $\mathbb{Z}[x]/P\mathbb{Z}[x]$, if each $z \in \mathbb{Z}[x]/P\mathbb{Z}[x]$ can be represented by an element

$$\sum_{j=0}^{\ell} d_j x^j \in \mathbb{Z}[x] \quad \text{with “digits”} \;\; 0 \le d_j < |P(0)|.$$

More recently, canonical number systems gained considerable interest and were studied extensively. We refer the reader for instance to AKIYAMA et al. [**1**] where they are embedded in a more general framework.

The present paper is motivated by the definition of canonical number systems. Indeed, we replace $\mathbb{Z}$ in the definition of canonical number systems by $\mathbb{F}[y]$, where $\mathbb{F}$ is an arbitrary field. For finite fields this concept has been introduced and studied by the third and fourth authors [**18**] (similar generalizations have been investigated in recent years; see e.g. [**4**, **8**, **16**]). Indeed, let $\mathbb{F} = \mathbb{F}_q$ be a finite field. In [**18**], digit systems in the residue class ring $R := \mathbb{F}_q[x, y]/f\,\mathbb{F}_q[x, y]$ for $f \in \mathbb{F}_q[x, y]$, have been investigated, in particular, all polynomials f with the property that each $r \in R$ admits a finite representation

$$r \equiv d_0 + d_1 x + \cdots + d_\ell x^\ell \mod f$$

with “digits” $d_i \in \mathbb{F}_q[y]$ satisfying $\deg_y(d_i) < \deg_y(f)$ have been characterized (see Section 2 for a formal definition of these digit systems). Moreover, eventually periodic representations have been investigated in this paper. In the first part of the present paper we extend the characterization result of [**18**] to arbitrary fields $\mathbb{F}$. Even in this more general case the characterization problem of all such digit systems turns out to be completely solvable.

In order to motivate the second aim of our paper we again go back to canonical number systems. Already KNUTH [**14**], who studied the special instance $P(x) = x^2 + 2x + 2$ observed that canonical number systems have interesting geometrical properties. Indeed, considering representations involving negative powers of the base and defining so-called “fundamental domains” yields connections to fractals and the theory of tilings (see for instance [**2**, **10**, **17**]).

In the present paper we would like to carry over representations with respect to negative powers as well as the definition of fundamental domains to our new notion of digit system. The fundamental domain turns out to be a union of “boxes”. If $\mathbb{F} = \mathbb{F}_q$ is a finite field, this union becomes finite which makes the fundamental domain easy to describe.

The paper is organized as follows. In Section 2 we give the formal definition of digit systems in residue class rings R of the polynomial ring $\mathbb{F}[x,y]$ over a field $\mathbb{F}$. Moreover, we give a new proof of the theorem by SCHEICHER and THUSWALDNER [**18**] which characterizes all polynomials that admit finite representations. Moreover, we enlarge the space of interest to the residue class ring $\mathbb{F}((x^{-1},y^{-1}))/f\mathbb{F}((x^{-1},y^{-1}))$ of the ring of Laurent series in two variables. We are able to prove that each element of this ring admits a unique "digital" representation. Section 3 shows that our digit systems are symmetric in the variables x and y and gives a way to switch between the "x-" and the "y-digit representation". The switching algorithm gives insights into the arithmetic structure of the digit systems. In Section 4 we investigate which elements allow periodic representations. It turns out that this question can be answered with help of the total ring of fractions of R. In this section we confine ourselves to the case where $\mathbb{F}$ is a finite field. Finally, in Section 5 we construct "fundamental domains" for our digit systems. Interestingly, it turns out that these sets are bounded, closed and open and can be written as unions of cylinders. If $\mathbb{F}$ is a finite field these unions become finite which makes the set totally bounded and therefore compact. The fundamental domains induce a tiling of S.

2. *Digit systems*

Let $\mathbb{F}$ be a field and $G_x = \mathbb{F}[x][y]$ the polynomial ring in two commuting variables with coefficients in $\mathbb{F}$. Let $f \in G_x$. We first define digit representations for the elements of the residue class ring $R_x := G_x/fG_x$. Set

$$\mathcal{N}_x = \{d \in \mathbb{F}[y] \mid \deg_y(d) < \deg_y(f)\}.$$

We say that $g \in G_x$ is an *x-digit representation* of $r \in R_x$ if and only if the following two conditions hold:

- g maps to r under the canonical projection map.
- $g = \sum_{i=0}^{\ell} d_i x^i$ for some $\ell \in \mathbb{Z}$ and $d_i \in \mathcal{N}_x$.

The polynomials d_i are called the *x-digits* of r. If each element of R_x has a unique x-digit representation we say that $(x, \mathcal{N}_x)$ is a *digit system* in R_x with *base x* and *set of digits $\mathcal{N}_x$*.

REMARK 2·1. *If $\mathbb{F} = \mathbb{F}_q$ is a finite field, then R_x is an $(\mathcal{N}_x, x)$-ring in the sense of* ALLOUCHE *et al.* [**3**]. *Moreover, in this case the associated digit systems coincide with the notion of digit systems studied in* [**18**].

It is easy to characterize x-digit systems.

THEOREM 2·2 (Representations I). *Let $f \in G_x$. The pair $(x, \mathcal{N}_x)$ is a digit system in R_x if and only if f is monic in y, i.e., the leading coefficient of f written as a polynomial in y with coefficients in $\mathbb{F}[x]$ is a nonzero element of $\mathbb{F}$.*

Proof. Assume first that f is monic in y. Let $r \in R_x$ and pick any representative $g' \in G_x$ of r. Since f is monic in y using division with remainder we can find $g, a \in G_x$ with $\deg_y(g) < \deg_y(f)$ such that $g = af + g'$. Writing g as a polynomial in x with coefficients in $\mathbb{F}[y]$ we see that it is an x-digit representation of r. To prove that g is unique assume on the contrary that $g'' \in G_x$ is a different x-digit representation of r. Then $0 \neq g'' - g$ maps to zero in R_x and satisfies $\deg_y(g'' - g) < \deg_y(f)$. Therefore

$f \nmid g'' - g$. On the other hand, since the kernel of the quotient map $G_x \to R_x$ is generated by f, it follows that $f \mid g'' - g$ which is a contradiction.

Assume now that $(x, \mathcal{N}_x)$ is a digit system and $\sum_{i=0}^{\ell} d_i x^i$ is the x-digit representation of $y^{\deg_y(f)}$. Then $y^{\deg_y(f)} - \sum_{i=0}^{\ell} d_i x^i$ is monic in y and divisible by f, therefore f is monic in y as well. □

REMARK 2·3. *Obviously, expanding an element $r \in R_x$ in base x with y-digits is the same as reducing the exponents of y.*

REMARK 2·4. *If $\mathbb{F} = \mathbb{F}_q$ is a finite field, we obtain a generalization of* [**18**, Theorem 2.5].

If we define further

$$G_y := \mathbb{F}[y][x] \quad \text{and} \quad G := \mathbb{F}[x, y],$$

then all three rings G, G_x and G_y are trivially $\mathbb{F}$-isomorphic by sending $x \mapsto x$ and $y \mapsto y$. Accordingly we get isomorphisms of the residue class rings

$$R := G/fG, \quad R_x, \quad R_y := G_y/fG_y.$$

REMARK 2·5. *Assume that we are given a polynomial $f \in G$ such that* both *R_x admits an x-digit system* and *R_y admits a y-digit system. In view of Theorem 2·2 this is equivalent to the fact that f is monic with respect to* both *x* and *y. If we think of the x-digit (resp. y-digit) representation as being the canonical representation for elements of R_x (resp. R_y) then we may consider the isomorphism $R_x \to R_y$ as switching between x- and y-digit representations. For this reason, later on we will confine ourselves to these choices of f.*

So far we have carried over the notion of canonical number systems to polynomial rings over a field. However, we also aim at an analogy for representations including negative powers of the base. With these representations we finally wish to define "fundamental domains" for our digit systems. Thus for our further constructions we make use of the fields $\mathbb{F}((x^{-1}))$ and $\mathbb{F}((y^{-1}))$. These fields are complete with respect to $|\cdot|_x = q^{\deg_x(\cdot)}$ and $|\cdot|_y = q^{\deg_y(\cdot)}$, respectively. First define the rings

$$H := \mathbb{F}((x^{-1}, y^{-1})), \quad H_x := \mathbb{F}((x^{-1}))[y], \quad H_y := \mathbb{F}((y^{-1}))[x]$$

and their quotients modulo f

$$S := H/fH, \quad S_x := H_x/fH_x, \quad S_y := H_y/fH_y.$$

Note that S_x is a vector space over the complete field $\mathbb{F}((x^{-1}))$ of dimension $\deg_y(f)$.

In what follows we will rarely mention symmetric statements obtained by interchanging the roles of x and y.

By π_G, π_{G_x}, π_H and π_{H_x} we denote the respective quotient maps of the rings defined above. The elements of G, G_x, H and H_x are naturally represented as formal sums $h = \sum_{(i,j)\in\mathbb{Z}^2} h_{i,j} x^i y^j$ with certain support restrictions. The support of h, denoted by $\operatorname{supp}(h)$, is the set of lattice points $(i, j) \in \mathbb{Z}^2$ such that $h_{i,j} \neq 0$. For a binary relation $*$ on $\mathbb{Z}$ we use the shorthand notation $\mathbb{Z}_{*a} := \{i \in \mathbb{Z} \mid i * a\}$. We have the following assertions:

- $h \in G \iff \operatorname{supp}(h) \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is finite.
- $h \in G_x \iff \operatorname{supp}(h) \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is finite.
- $h \in H \iff$ there are $a, b \in \mathbb{Z}$ such that $\operatorname{supp}(h) \subseteq \mathbb{Z}_{\leq a} \times \mathbb{Z}_{\leq b}$.
- $h \in H_x \iff$ there are $a, b \in \mathbb{Z}$ such that $\operatorname{supp}(h) \subseteq \mathbb{Z}_{\leq a} \times (\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq b})$.

Setting

$$m := \deg_x(f) \quad \text{and} \quad n := \deg_y(f)$$

we will assume from now on that f is of the shape

$$f = \sum_{0 \leq i \leq m} b_i x^i \quad \text{with} \quad b_i \in \mathbb{F}[y], \quad b_m \neq 0.$$

Moreover, we define two standard areas in the exponent lattice,

$$A_x := \mathbb{Z} \times (\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{<n}) \subset \mathbb{Z}^2$$

and

$$A_y := (\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{<m}) \times \mathbb{Z} \subset \mathbb{Z}^2.$$

We are now able to extend the notion of digit system to the larger ring S_x.

DEFINITION 2·6 (Digit representation). *We say that $h \in H_x$ is an x-*digit representation *of $s \in S_x$ if and only if the following two conditions hold:*

- $\pi_{H_x}(h) = s$.
- $h = \sum_{i=-\infty}^{\ell} d_i x^i$ *for some $\ell \in \mathbb{Z}$ and $d_i \in \mathcal{N}_x$.*

The second condition is equivalent to $\deg_y(h) < n$ *or* $\operatorname{supp}(h) \subset A_x$. *For short we write*

$$s = (d_\ell \dots d_0.d_{-1}d_{-2}\dots)_x.$$

If there exist κ and μ such that $d_i = d_{i-\kappa}$ for all $i < -\mu$ we say that s admits an eventually periodic *x-digit representation. This will be written as*

$$s = (d_\ell \dots d_0.d_{-1}\dots d_{-\mu}\overline{d_{-\mu-1}\dots d_{-\mu-\kappa}})_x.$$

Moreover, we say that s admits a purely periodic *x-digit representation if*

$$s = (.\overline{d_{-1}\dots d_{-\kappa}})_x.$$

If each element of S_x has a unique x-digit representation we say that $(x, \mathcal{N}_x)$ is a digit system *in S_x with* base x *and* set of digits $\mathcal{N}_x$.

By copying the proof of Theorem 2·2, we immediately get the following Theorem 2·7. Note that we may drop the constraint of f being monic in y since we are now working in a polynomial ring over a field.

THEOREM 2·7 (Representations II). *Let $f \in G$. Then $(x, \mathcal{N}_x)$ is a digit system in S_x.*

REMARK 2·8. *By our definition the pair $(x, \mathcal{N}_x)$ forms a digit system in S_x for* arbitrary *$f \in G$. However, it forms a digit system in R_x only if f is monic in y. Indeed, if f is not monic in y then the elements of R_x (regarded as a subset of S_x) generally only admit x-digit representations in H_x but not in G_x.*

3. *The transformation between x- and y-digit representations*

In what follows we want to study the relation between x- and y-digit representations. Since we built our theory starting from $(x, \mathcal{N}_x)$ digit systems in R_x in view of Remarks 2·5 and 2·8 we assume that f is monic in x and y. Under this assumption we will prove that the identity

$$\varphi_{xy} : R_x \to R_y; \qquad x \mapsto x, \quad y \mapsto y \tag{3·1}$$

can be extended to an isomorphism between S_x and S_y. Recall that by Theorem 2·7 each element of S_x admits a unique x-digit representation. In the present section we want to describe an explicit transformation procedure that turns an x-digit representation of an element of S_x into a y-digit representation of an element of S_y (see Theorem 3·10).

REMARK 3·1. *Showing that the spaces S_x and S_y are isomorphic amounts (by symmetry) to showing that each element in H has an equivalent one in H_x modulo f. This is not hard to establish by the iteration of usual division with remainder by f which successively kills negative powers of y. However, we give the transformation procedure leading to this isomorphism in great detail in the following lemmas. The reason for this lengthy treatment can be seen in Section 5. Indeed, in the proof of Lemma* 5·2 *we need all the detailed information of the transformation procedure between S_x and S_y contained in Lemmas* 3·2 *to* 3·9 *below. On the other hand, the content of Lemma* 5·2 *is essential for many properties of fundamental domains established in Section 5.*

In what follows we will decompose a formal series $h = \sum_{(i,j)} h_{i,j} x^i y^j$ into its *y-fractional part* $\{h\}_y := \sum_{j<0} (\sum_i h_{i,j} x^i) y^j$ and its *y-integer part* $\lfloor h \rfloor_y := \sum_{j \geq 0} (\sum_i h_{i,j} x^i) y^j$.

The idea for establishing the transformation procedure is to show that elements of S – the ring not favoring one of its variables – are representable uniquely with respect to both standard areas, i.e., have a unique x- and a unique y-digit representation. This transformation process is done by applying so-called *atomic steps* which will be treated in the following lemmas.

First we define the atomic step of the first kind. It cuts off the negative powers of y for a single coefficient h_k in a representation $\sum_i h_i(y) x^i$.

LEMMA 3·2 (Atomic step of the first kind). *Let $h \in H$ be given by*

$$h = \sum_{i=-\infty}^{\ell} h_i x^i, \qquad h_i \in \mathbb{F}((y^{-1})).$$

Let $k \leq \ell$ be an integer and set $l_k := \{h_k\}_y$. Then we say that

$$h' = \sum_{i=-\infty}^{\ell} h'_i x^i, \qquad h'_i \in \mathbb{F}((y^{-1}))$$

emerges from h by an atomic step of the first kind at index k (notation: $h' = A_k(h)$) if

$$h' = h - \frac{l_k}{b_m} x^{k-m} f.$$

In this case we have $h'_k \in \mathbb{F}[y]$ *and the estimates*

$$\begin{aligned}\deg_y(h'_k) &\le \deg_y(h_k),\\ \deg_y(h'_{k-m+i}) &\le \max(\deg_y(l_k)+n-1, \deg_y(h_{k-m+i})) \qquad (0<i<m),\\ \deg_y(h'_{k-m}) &\le \max(\deg_y(l_k)+n, \deg_y(h_{k-m})).\end{aligned}$$

Moreover, $\pi_H(h) = \pi_H(h')$ *and* $h'_i = h_i$ *for* $i > k$ *or* $i < k-m$.

Proof. This follows immediately from the equality

$$h' = h - \frac{l_k}{b_m}\sum_{i=0}^{m} b_i x^{k-m+i}$$

because $\deg_y(b_i) < n$ for $0 < i < m$, $\deg_y(b_0) = n$ and $\deg_y(b_m) = 0$. □

The following lemma shows how a combination of several atomic steps of the first kind affects a given representation.

Lemma 3·3 (Succession of atomic steps of the first kind). *Let* $h \in H$ *be given by*

$$h = \sum_{i=-\infty}^{\ell} h_i x^i, \qquad h_i \in \mathbb{F}((y^{-1})),$$

and

$$k_0 := \begin{cases} \min\Big(\ell, \ell + m\deg_y(h)\Big), & n = 1,\\ \min\Big(\ell, \ell - \Big\lceil \frac{-\deg_y(h)}{n-1}\Big\rceil\Big), & n \ge 2.\end{cases}$$

Then for $k \le k_0$ *the element*

$$A_k \circ \cdots \circ A_\ell(h) = \sum_{i=-\infty}^{k_0} h'_i x^i, \qquad h'_i \in \mathbb{F}((y^{-1})),$$

satisfies $h'_i \in \mathbb{F}[y]$ *for all* $k \le i \le k_0$*. In particular, its* x*-degree is bounded by* k_0*.*

Proof. Assume that $\deg_y(h) \ge 0$. Then $k_0 = \ell$ and the lemma is easily proved by $\ell - k + 1$ successive applications of Lemma 3·2. Now assume $\deg_y(h) < 0$. For $k_0 < j \le \ell$ set

$$h^{(j)} := A_j \circ \cdots \circ A_\ell(h) = \sum_{i=-\infty}^{\ell} h_i^{(j)} x^i, \qquad h_i^{(j)} \in \mathbb{F}((y^{-1})).$$

If $n = 1$ we claim that

$$\deg_y(h_i^{(j)}) \le \begin{cases} -\infty, & i \ge j,\\ \deg_y(h) + \big\lfloor \frac{\ell - i}{m}\big\rfloor, & j - m \le i < j,\\ \deg_y(h), & \text{otherwise},\end{cases} \tag{3·2}$$

and if $n \ge 2$ we claim that

$$\deg_y(h_i^{(j)}) \le \begin{cases} -\infty, & i \ge j,\\ \deg_y(h) + (\ell - j + 1)(n-1), & j - m < i < j,\\ \deg_y(h) + (\ell - j + 1)(n-1) + 1, & i = j - m,\\ \deg_y(h), & \text{otherwise}.\end{cases} \tag{3·3}$$

Both estimates, (3·2) and (3·3), are proved by induction. Since $h^{(j)} = A_\ell(h)$ for $j = \ell$, the induction start is an immediate consequence of Lemma 3·2 in both cases. Also in both cases, the induction step follows directly from this lemma and works as long as $\deg_y(h_j^{(j+1)}) < 0$, which holds because $j > k_0$.

With the choice $j = k_0 + 1$ relation (3·2) as well as relation (3·3) first imply that

$$\deg_x(h^{(k_0+1)}) \le k_0.$$

An application of A_j cannot increase the x-degree of the argument. So by another $k_0 - k + 1$ successive applications of Lemma 3·2 we have $\deg_x(h^{(k)}) \le k_0$ and $h_i^{(k)} \in \mathbb{F}[y]$ for $i \le k_0$. $\square$

The atomic step of the second kind contained in the next lemma cuts off a single coefficient h_k in a representation $\sum_i h_i(y)x^i$ in a way that its y-degree becomes less than n.

LEMMA 3·4 (Atomic step of the second kind). *Let $h \in H$ be given by*

$$h = \sum_{i=-\infty}^{\ell} h_i x^i, \qquad h_i \in \mathbb{F}((y^{-1})).$$

Let $k \le \ell$ be an integer and set $l_k := y^n \lfloor y^{-n} h_k \rfloor_y$. Then we say that

$$h' = \sum_{i=-\infty}^{\max(\ell,k+m)} h'_i x^i, \qquad h'_i \in \mathbb{F}((y^{-1}))$$

emerges from h by an atomic step of the second kind at index k (notation: $h' = B_k(h)$) if

$$h' = h - \frac{l_k}{b_0} x^k f.$$

In this case we have the estimates

$$\begin{aligned} \deg_y(h'_k) &< n, \\ \deg_y(h'_{k+i}) &\le \max(\deg_y(h_{k+i}), \deg_y(h_k) - 1) \qquad (0 < i < m), \\ \deg_y(h'_{k+m}) &\le \max(\deg_y(h_{k+m}), \deg_y(h_k) - n). \end{aligned}$$

Moreover, $\pi_H(h) = \pi_H(h')$ and $h'_i = h_i$ for $i < k$ or $i > k + m$.

Proof. This follows immediately from the equality

$$h' = h - \frac{l_k}{b_0} \sum_{i=0}^{m} b_i x^{k+i}$$

because $\deg_y(b_i) < n$ for $0 < i < m$, $\deg_y(b_0) = n$ and $\deg_y(b_m) = 0$. $\square$

Again we need to dwell upon the effect of a combination of atomic steps of the second kind to a given representation. This is done in the following lemma.

LEMMA 3·5 (Succession of atomic steps of the second kind). *Let $h \in H$ be given by*

$$h = \sum_{i=-\infty}^{\ell} h_i x^i, \qquad h_i \in \mathbb{F}((y^{-1})),$$

$k \le \ell$ and $\ell_0 := \max(\ell, \ell + (m-1)(\deg_y(h) - n))$. Then for $\ell' \ge \ell_0$ the element

$$B_{\ell'} \circ \cdots \circ B_k(h) = \sum_{i=-\infty}^{\ell_0+m} h'_i x^i$$

satisfies $\deg_y(h'_i) < n$ for all $i \ge k$. In particular, its x-degree is bounded by $\ell_0 + m$.

Proof. Assume that $\deg_y(h) < n$. Then $\ell_0 = \ell$ and by the definition of the operators B_t in Lemma 3·4, the operator $B_{\ell'} \circ \cdots \circ B_k$ does not change h. Now assume $\deg_y(h) \ge n$. For $0 \le j \le \deg_y(h) - n$ set

$$h^{(j)} := B_{\ell+(m-1)j} \circ \cdots \circ B_k(h) := \sum_{i=-\infty}^{\ell+(m-1)j+m} h_i^{(j)} x^i.$$

We claim that

$$\deg_y(h_i^{(j)}) < \begin{cases} n, & k \le i \le \ell + (m-1)j, \\ \deg_y(h) - j, & \ell + (m-1)j < i < \ell + (m-1)j + m, \\ \deg_y(h) - j - (n-1), & i = \ell + (m-1)j + m. \end{cases} \tag{3·4}$$

This is shown by induction on j. For $j = 0$ this is easily seen by $\ell - k + 1$ successive applications of Lemma 3·4. Each induction step follows by another $m - 1$ applications. After $\deg_y(h) - n$ steps we arrive at the element

$$h^{(\deg_y(h)-n)} = B_{\ell_0} \circ \cdots \circ B_k(h)$$

which, in view of (3·4), has the desired properties. Again the operator $B_{\ell'} \circ \cdots \circ B_{\ell_0+1}$ does not change $h^{(\deg_y(h)-n)}$. $\square$

COROLLARY 3·6 (A Cauchy property). *Let $h \in H$ be given by*

$$h = \sum_{i=-\infty}^{\ell} h_i x^i, \qquad h_i \in \mathbb{F}((y^{-1})),$$

let $k < \ell$ and

$$\ell'_1 := \max(k + m, k + m + (m-1)(\deg_y(h) - n)),$$
$$\ell'_2 := \max(\ell'_1, \ell, \ell + (m-1)(\deg_y(h) - n)).$$

Then

$$h' := B_{\ell'_2} \circ \cdots \circ B_k(h) = \sum_{i=-\infty}^{\ell'_2+m} h'_i x^i \qquad \textit{and}$$

$$h'' := B_{\ell'_2} \circ \cdots \circ B_{k+1}(h) = \sum_{i=-\infty}^{\ell'_2+m} h''_i x^i$$

satisfy

$$\max\Big(\deg_y(h'_i), \deg_y(h''_i)\Big) < n \qquad \textit{for } i > k \qquad \textit{and}$$
$$h'_i = h''_i \qquad \textit{for } i > \ell'_1 + m.$$

Proof. The claim about the maximum of the y-degrees and the bounds on the x-degrees of h' and h'' follow from direct applications of Lemma 3·5 with $\ell' = \ell'_2$, so it remains to show the last claim or equivalently that $\deg_x(h'' - h') \leq \ell'_1 + m$ holds. Note that by the definition of l_k in Lemma 3·4 it is clear that the operators B_t are $\mathbb{F}$-linear. Hence, setting $g := B_k(h)$ we may write

$$h'' - h' = B_{\ell'_2} \circ \cdots \circ B_{k+1}(h - g).$$

By the definition of B_k we have

$$h - g = \sum_{i=k}^{k+m} g_i x^i$$

for certain g_i with $\deg_y(g_i) \leq \deg_y(h)$. Another application of Lemma 3·5 with $\ell' = \ell'_1$ shows that

$$B_{\ell'_1} \circ \cdots \circ B_{k+1}(h - g)$$

has x-degree bounded by $\ell'_1 + m$ and y-degree less than n. Finally,

$$B_{\ell'_2} \circ \cdots \circ B_{k+1}(h - g) = B_{\ell'_1} \circ \cdots \circ B_{k+1}(h - g)$$

holds because applying $B_{\ell'_2} \circ \cdots \circ B_{\ell'_1+1}$ to an element of y-degree smaller than n is the identity. This yields the result. $\square$

We can define a norm $|h| := e^{\deg(h)}$ for $h \in H$ (where $\deg(\cdot)$ denotes the total degree). Then H becomes a topological ring, i.e., addition and multiplication are continuous with respect to the induced metric. Multiplication is even an open mapping since the degree of a product is equal to the sum of the degrees of the factors. Note that H is not complete, for example, the sequence $(\sum_{0\leq i\leq j} x^i y^{-2i})_j$ is Cauchy but has no limit in H.

LEMMA 3·7. *If $(f_j)_j$ is a sequence in H which is Cauchy and* $\operatorname{supp}(f_j) \subseteq \mathbb{Z}_{\leq a} \times \mathbb{Z}_{\leq b}$ *for certain fixed $a, b \in \mathbb{Z}$ and j sufficiently large, then* $\lim_{j\to\infty} f_j$ *exists in H.*

Proof. Trivial. $\square$

The following two lemmas form the basis of the switching process between x- and y-digit representations.

LEMMA 3·8 (Uniformization Lemma I). *Let $h \in H$. Then there is some $h' \in H$ and $\ell \in \mathbb{Z}$ with $\pi_H(h') = \pi_H(h)$ and* $\operatorname{supp}(h') \subseteq \mathbb{Z}_{\leq \ell} \times \mathbb{Z}_{<n}$.

Proof. Let ℓ' be defined as in Lemma 3·5 and set

$$h^{(k)} = B_{\ell'} \circ \cdots \circ B_k(h)$$

for k sufficiently small. Then the sequence $(h^{(k)})_{-k}$ is Cauchy by Corollary 3·6 and $\operatorname{supp}(h^{(k)}) \subseteq \mathbb{Z}_{\leq \ell'+m} \times \mathbb{Z}_{\leq \deg_y(h)}$. Hence, there is some h' with

$$h' = \lim_{k\to-\infty} h^{(k)}.$$

Moreover, it is easy to see that $\operatorname{supp}(h') \subseteq \mathbb{Z}_{\leq \ell'+m} \times \mathbb{Z}_{<n}$. Setting $\ell := \ell' + m$ it remains to show that $\pi_H(h') = \pi_H(h)$, or in other words that $h' - h \in fH$.

By construction we have $h^{(k)} - h = e^{(k)} f$ for certain $e^{(k)} \in H$. Since $(h^{(k)} - h)_{-k}$ is a Cauchy sequence and multiplication by f is an open mapping, we conclude that also $(e^{(k)})_{-k}$ is Cauchy and the support of its elements is suitably bounded. Hence,

$$h' - h = \lim_{k\to-\infty}(h^{(k)} - h) = \lim_{k\to-\infty} e^{(k)} f = \left(\lim_{k\to-\infty} e^{(k)}\right) f \in fH.$$

□

LEMMA 3·9 (Uniformization Lemma II). *Let $h \in H$ and assume that* $\mathrm{supp}\,(h) \subseteq \mathbb{Z}_{\le\ell}\times \mathbb{Z}_{<n}$ *for some $\ell \in \mathbb{Z}$. Then there exists an $h' \in H$ with $\pi_H(h') = \pi_H(h)$ and* $\mathrm{supp}\,(h') \subseteq \mathbb{Z}_{\le\ell} \times (\mathbb{Z}_{\ge0} \cap \mathbb{Z}_{<n}) \subset A_x$.

Proof. Set

$$h^{(k)} = A_k \circ \cdots \circ A_\ell(h) = \sum_{i=-\infty}^{\ell} h_i^{(k)} x^i, \quad h_i^{(k)} \in \mathbb{F}((y^{-1})),$$

for k sufficiently small. Then $h_i^{(k)} = h_i^{(k')}$ for $i \ge \max(k, k')$ from the definition of the operators A_t in Lemma 3·2. It follows immediately that $(h^{(k)})_{-k}$ is Cauchy. Also $\mathrm{supp}\,(h^{(k)}) \subseteq \mathbb{Z}_{\le\ell} \times \mathbb{Z}_{<n}$ and hence $h' := \lim_{k\to-\infty} h^{(k)}$ exists. Moreover, it is easy to see that in fact $\mathrm{supp}\,(h') \subseteq \mathbb{Z}_{\le\ell} \times (\mathbb{Z}_{\ge0} \cap \mathbb{Z}_{<n})$. The fact that $\pi_H(h') = \pi_H(h)$ is shown exactly in the same way as in the proof of Lemma 3·8. □

The following theorem forms the main result of the present section.

THEOREM 3·10 (Representations III). *Assume that f is monic in x and y. Then for each $s \in S$ there is a unique $h \in H$ with* $\mathrm{supp}\,(h) \subset A_x$ *and $\pi_H(h) = s$. In other words each $s \in S$ admits a unique x-digit representation.*

Proof. Choose $h'' \in H$ arbitrary such that $\pi_H(h'') = s$. Apply Lemma 3·8 to h'' in order to produce an element $h' \in H$ with $\mathrm{supp}\,(h') \subseteq \mathbb{Z}_{\le\ell} \times \mathbb{Z}_{<n}$. In a second step apply Lemma 3·9 to h' to get $h \in H$ with $\mathrm{supp}\,(h) \subset A_x$ and note that still $\pi_H(h) = s$. It remains to show that such an h is unique.

If we had two distinct representations, their difference would be an element $g \in H\setminus\{0\}$ with $\mathrm{supp}\,(g) \subset A_x$. It will be sufficient to prove that $\pi_H(g) \ne 0$. Assume on the contrary that $g = af$ for some $a \in H$. Let $\Pi(g)$ be the Minkowski sum of the convex hull of $\mathrm{supp}\,(g)$ and the third quadrant. Furthermore, let $(i_1, j_1) \in \mathbb{Z}^2$ be the vertex on the horizontal face of $\Pi(g)$ and $(i_2, j_2) \in \mathbb{Z}^2$ be the vertex on the vertical face of $\Pi(g)$ and set $d(g) := j_1 - j_2 \in \mathbb{Z}_{\ge0}$. Define $\Pi(a)$, $\Pi(f)$, $d(a)$ and $d(f)$ analogously. Then $\Pi(g) = \Pi(a) + \Pi(f)$ and $d(g) = d(a) + d(f)$. But $d(a) \ge 0$ and $d(f) = n$ because f is monic in x and $d(g) \le n - 1$ because $\mathrm{supp}\,(g) \subset A_x$, contradiction. □

REMARK 3·11. *The argument of the proof makes essential use of the fact that f is monic in x and y.*

In what we did so far we also constructed an explicit isomorphism between the sets S_x, S and S_y. This is emphasized in the following corollary.

COROLLARY 3·12 (The isomorphism). *Assume that f is monic in x and y. Then we have $S \cong S_x \cong S_y$. In fact, the isomorphism φ_{xy}, see* (3·1)*, extends to $S_x \to S_y$.*

Proof. We show $S_x \cong S$. The natural inclusion $H_x \hookrightarrow H$ induces a homomorphism $\psi : S_x \to S$. Let $s \in S_x \setminus \{0\}$. Then s has a unique representative $h \in H_x \setminus \{0\}$ with

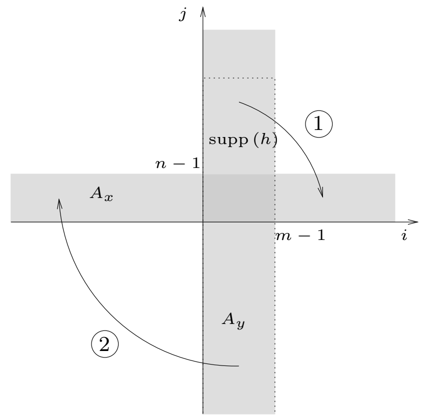

Fig. 1. The isomorphism $\varphi_{yx} : S_y \to S_x$

$\operatorname{supp}(h) \subset A_x$ by Theorem 2·7. But then $\psi(s)$ is represented by the same h regarded as an element of H, and by Theorem 3·10 we find $\psi(s) \neq 0$, hence ψ is injective.

On the other hand, let $s \in S$ and $h \in H$ be a representative with $\operatorname{supp}(h) \subset A_x$. Then obviously h is in the image of H_x, hence ψ is also surjective. By the same reasoning we have $S \cong S_y$ and by composition we get an isomorphism $\psi_{xy} : S_x \to S_y$.

Now let $r \in R_x$, i.e., $r = \pi_{G_x}(h)$ for some $h \in G_x$ with $\operatorname{supp}(h) \subset A_x$. Also there is $h' \in G_y$ with $\operatorname{supp}(h') \subset A_y$ and $\varphi_{xy}(r) = \pi_{G_y}(h')$ by Theorem 2·2. Now consider h and h' as elements of H, then $h - h' \in fH$ and hence $\pi_H(h) = \pi_H(h')$. Now the construction of ψ_{xy} and the uniqueness statement in Theorem 3·10 imply that $\psi_{xy}|_{R_x} = \varphi_{xy}$. $\square$

We write again $\varphi_{xy} : S_x \to S_y$ for the extended isomorphism. For the representation Theorem 3·10 we had to show that we can transform an arbitrary support to fit into the region A_x. For the isomorphism φ_{yx} we have to transform a support in A_y into a support in A_x. This is illustrated in Figure 1 as follows: one application of Lemma 3·8 moves the upper part of h into A_x (indicated by arrow ①). A subsequent application of Lemma 3·9 moves the lower part of h into A_x (indicated by arrow ②). The only part which is affected by the transformations of both lemmas is the overlapping region $A_y \cap A_x$, which has been shaded in dark grey.

4. *The total ring of fractions and periodic digit representations*

Our next objective is to investigate under what condition the x-digit representation of an element of S_x is periodic. It turns out that periodic representations are related to the *total ring of fractions* $\mathcal{Q}(R)$ of R. Recall that $\mathcal{Q}(R) := \mathcal{S}^{-1}R$ where $\mathcal{S} \subset R$ denotes the multiplicative set of non-zero divisors. Let $F := \mathcal{Q}(R)$ and set

$$F_x := \mathbb{F}(x)[y]/f\mathbb{F}(x)[y] \quad \text{and} \quad F_y := \mathbb{F}(y)[x]/f\mathbb{F}(y)[x].$$

LEMMA 4·1 (Total rings of fractions). *We have $F \cong F_x \cong F_y$. In fact, the isomorphism φ_{xy}, see* (3·1), *extends uniquely to $F_x \to F_y$.*

Proof. Being monic in y, the polynomial f is in particular primitive as a polynomial with coefficients in $\mathbb{F}[x]$. Hence, the set $\mathcal{T} := \pi_G(\mathbb{F}[x] \setminus \{0\})$ consists of non-zero divisors and we have embeddings $R \hookrightarrow \mathcal{T}^{-1}R \cong F_x \hookrightarrow F$. This implies that $\mathcal{Q}(F_x) = F$.

Now let $h \in F_x$ be a non-zero divisor. In other words, h is represented by a polynomial $h' \in \mathbb{F}(x)[y]$ such that h' and f are coprime in the principal ideal domain $\mathbb{F}(x)[y]$. Then

$$\begin{tikzcd}
S_x \arrow[r] \arrow[rr, bend left, "\varphi_{xy}"] & S & S_y \arrow[l] \\
F_x \arrow[u, hook] \arrow[r] & F \arrow[u, hook] & F_y \arrow[u, hook] \arrow[l] \\
R_x \arrow[u, hook] \arrow[r] \arrow[rr, bend right, "\varphi_{xy}"'] & R \arrow[u, hook] & R_y \arrow[u, hook] \arrow[l]
\end{tikzcd}$$

Fig. 2. The commuting diagram

the image of h' is invertible in F_x, i.e., $h^{-1} \in F_x$. Hence, F_x is its own total ring of fractions and the last embedding above is already an isomorphism.

By composing $F_x \to F$ and the inverse of $F_y \to F$ we find an isomorphism $\psi_{xy} : F_x \to F_y$ clearly extending φ_{xy}. Let $a/b \in F_x$, then $b(a/b) = a$ and for any such isomorphism we have $\varphi_{xy}(b)\psi_{xy}(a/b) = \varphi_{xy}(a)$. Since $\varphi_{xy}(b)$ is a non-zero divisor this already determines $\psi_{xy}(a/b)$ uniquely. $\square$

Note that it is easy to compute the isomorphic images effectively, using the Extended Euclidean Algorithm. We write again $\varphi_{xy} : F_x \to F_y$ for the extended isomorphism. This is justified; indeed, there holds the commuting diagram in Figure 2, where all the maps (except the φ_{xy}) are induced by the respective maps of representatives. We have to argue, why the middle row fits into this diagram. The reason is that the vertical arrows are inclusions, the top and bottom horizontal arrows are all isomorphisms. The objects in the middle are the total rings of fractions that obviously have to be mapped isomorphically onto each other. In view of the above diagram we will from now on consider digit representations of elements in the rings R, F and S.

LEMMA 4·2 (Rational Laurent series).

(i) *Let $\mathbb{F}_q$ be the finite field with q elements and $z \in \mathbb{F}_q((x^{-1}))$. Then z is eventually periodic if and only if $z \in \mathbb{F}_q(x)$.*

(ii) *Let $z \in \mathbb{F}_q(x)$. Write $z = a + x^{-\kappa}(b + c/d)$ with $\kappa \in \mathbb{Z}_{\geq 0}$, $a, b, c, d \in \mathbb{F}_q[x]$ with $c \neq 0$, $d \neq 0$, $\deg_x(c) < \deg_x(d)$, $\deg_x(b) < \kappa$, $\gcd(c, d) = 1$ and $x \nmid d$. Let μ be minimal such that $x^\mu \equiv 1 \bmod d$. Then we have a representation*

$$z = (u_\rho \dots u_0.v_{\kappa-1} \dots v_0\overline{p_{\mu-1} \dots p_0})_x$$

with $\rho = \deg_x(a)$ and certain $u_i, v_i, p_i \in \mathbb{F}_q$. Furthermore μ is the minimal length of the period.

REMARK 4·3. *Representing elements $z \in \mathbb{F}(x)$ as above is possible even if $\mathbb{F}$ is infinite. However, choosing μ is only possible because $\mathbb{F}_q$ is finite and, hence, x maps to an element of the finite group of units modulo d.*

Proof. First assume $z' = c/d$ with the same conditions. Then $x^\mu = 1 + ed$ and $x^\mu c/d = c/d + p$ with $p := ec$. The first summand is purely non-integral, whereas the second summand is integral. Moreover, $\deg_x(p) = \deg_x(x^\mu c/d) < \mu$. So we have a representation $p = (p_{\mu-1} \dots p_0.)_x$. By repeating the argument we see that z' is purely periodic, in particular, we have $z' = (.\overline{p_{\mu-1} \dots p_0})_x$.

On the other hand assume that z' is purely periodic of this form. Then an easy calculation shows

$$z' = \frac{\sum_{i=0}^{\mu-1} p_i x^i}{x^\mu - 1}, \tag{4·1}$$

so, reducing this fraction, we get a representation $z' = c/d$ as above.

Now any element $z \in \mathbb{F}_q(x)$ can be written as in the claim and any eventually periodic $z \in \mathbb{F}_q((x^{-1}))$ can be written as $z = a + x^{-\kappa}(b + z')$ with $a, b \in \mathbb{F}_q[x]$ and $z' \in \mathbb{F}_q((x^{-1}))$ purely periodic.

It remains to prove the assertion on the period length. Assume that $\mu' \leq \mu$ is the length of the minimal period. Then Equation (4·1) with μ replaced by μ' implies that $d \mid x^{\mu'} - 1$, in other words $x^{\mu'} \equiv 1 \bmod d$ and hence also $\mu \leq \mu'$. $\square$

COROLLARY 4·4 (Periodicity). *Let $\mathbb{F} = \mathbb{F}_q$ be a finite field and let $s \in S$. The following assertions are equivalent:*

- *$s \in F$.*
- *s has an eventually periodic x-digit representation.*
- *s has an eventually periodic y-digit representation.*

Proof. By symmetry, it suffices to show the equivalence of the first two statements. By Lemma 4·1, $s \in F$ if and only if s can be represented by an element $h = \sum_{i=0}^{n-1} h_i y^i \in \mathbb{F}_q(x)[y] \subset H_x$. By Lemma 4·2 this is the case if and only if all the coefficients have eventually periodic representations in $\mathbb{F}_q((x^{-1}))$. Hence, $s \in F$ if and only if h is eventually periodic when written as a sum of x-digits. $\square$

The difficulty of making statements about the periodic representations now depends heavily on the representation of an element $s \in F$. The easy case is when s is represented by an element of $\mathbb{F}_q(x)[y]$ and we want to study the x-digit representation.

THEOREM 4·5 (Shape of the period). *Let $\mathbb{F} = \mathbb{F}_q$ be a finite field and let $s \in F$ be represented by*

$$h = \sum_{i=0}^{n-1} h_i y^i$$

with $h_i = a_i + x^{-\kappa_i}(b_i + c_i/d_i) \in \mathbb{F}_q(x)$ such that $\kappa_i \in \mathbb{Z}_{\geq 0}$, $a_i, b_i, c_i, d_i \in \mathbb{F}_q[x]$ with $c_i \neq 0$, $\deg_x(c_i) < \deg_x(d_i)$, $\deg_x(b_i) < \kappa_i$, $\gcd(c_i, d_i) = 1$ and $x \nmid d_i$. Let μ_i be minimal such that $x^{\mu_i} \equiv 1 \bmod d_i$. Then we have a representation

$$s = (u_\rho \dots u_0.v_{\kappa-1} \dots v_0\overline{p_{\mu-1} \cdots p_0})_x$$

for certain x-digits u_i, v_i, p_i. Here $\rho = \max_i(\deg_x(a_i))$, $\kappa = \max_i(\kappa_i)$ and $\mu = \operatorname{lcm}_i(\mu_i)$ is the minimal length of the period.

Proof. The x-digit representation is inferred directly from the set of Laurent series representations of the h_i. So this theorem is an immediate consequence of Lemma 4·2. $\square$

The more difficult case is when s is represented by an element of $\mathbb{F}_q(y)[x]$ in the same situation. In this case, we first use the Extended Euclidean Algorithm to convert the representation appropriately and then apply Theorem 4·5.

Example 4·6. *Let* $f := x^3 + (3y+1)x + 2y + y^2 \in \mathbb{F}_5[x,y]$ *and consider the element*

$$s := \frac{y^5 + xy + 2y + 4x}{y^3 + 2y^2x + y^2 + y} \in \mathcal{Q}(\mathbb{F}_5[x,y]/f\mathbb{F}_5[x,y]).$$

We want to compute, say, its y*-digit representation. Using the Extended Euclidean Algorithm, we first compute a nicer representation by*

$$\begin{aligned} h &= h_0 + h_1x + h_2x^2 \\ &= \left(\frac{y^8 + 4y^7 + 2y^6 + 2y^5 + 4y^4 + 4y^3 + 2y^2 + 4y + 2}{y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1}\right) + \\ &\quad \left(\frac{3y^8 + 3y^7 + 3y^6 + y^5 + 2y^4 + 2y^3 + 4y + 4}{y^7 + 2y^6 + y^5 + 3y^4 + 3y^2 + y}\right)x + \\ &\quad \left(\frac{4y^6 + 3y^3 + 3y^2 + 2}{y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1}\right)x^2. \end{aligned}$$

Decomposing coefficients as in Theorem 4·5 *yields*

$$\begin{aligned} h_0 &= (y^2 + 2y + 2) + y^0\left(0 + \frac{3y^5 + y^4 + y}{y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1}\right), \\ h_1 &= (3y + 2) + y^{-1}\left(1 + \frac{3y^5 + y^2 + 4y + 3}{y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1}\right), \\ h_2 &= (4) + y^0\left(0 + \frac{2y^5 + y^4 + y^3 + 3y^2 + 3y + 3}{y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1}\right). \end{aligned}$$

One computes that y *has order* 208 *modulo* $y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1$*, indeed*

$$y^{208} - 1 \equiv (y^{202} + 3y^{201} + \cdots + 3y + 4)(y^6 + 2y^5 + y^4 + 3y^3 + 3y + 1) \quad \mod 5.$$

In this example $\rho = \max_i(\deg_x(a_i)) = 2$, $\kappa = \max_i(\kappa_i) = 1$ *and* $\mu = \mathrm{lcm}_i(\mu_i) = 208$. *And indeed, if we develop the coefficients separately and combine them to* y*-digits, we get the representation*

$$s = ((1)(2+3x)(2x+2+4x^2).(x+3+2x^2)\underbrace{\overline{(2x^2+3x)(4x+2)\ldots(2x^2)(2x+3+2x^2)}}_{208\ y\text{-digits}})_y.$$

5. *The fundamental domain associated to a digit system*

Now we want to give a more detailed study of the isomorphism $\varphi_{xy} : S_x \to S_y$. Instead of considering the map φ_{xy}, we compare the x- and y-digit representations of elements of s.

Definition 5·1 (Height of elements). *Let* $s \in S$ *and* $h \in H$ *be the* x*-digit representation of* s*. Then we define*

$$\mathrm{hgt}_x(s) := \begin{cases} \deg_x(h), & s \in S \setminus \{0\}, \\ -\infty, & s = 0. \end{cases}$$

In other words, if $s \neq 0$ is represented by $h = \sum_{j=-\infty}^{\ell} h_j x^j$ where the h_j are x-digits and $h_\ell \neq 0$, then $\mathrm{hgt}_x(s) = \ell$. By Theorem 3·10, we can convert from one representation into the other, so we may ask how to bound one height in terms of the other.

LEMMA 5·2 (Mutual bounds on height). *Let $s \in S$. Then we have the implications:*

$$\begin{array}{llll} \text{(i)} & \operatorname{hgt}_y(s) < 0 & \Rightarrow & \operatorname{hgt}_x(s) \le \begin{cases} m-1+m\operatorname{hgt}_y(s), & n=1, \\ m-1-\left\lceil \frac{-\operatorname{hgt}_y(s)}{n-1} \right\rceil, & n \ge 2. \end{cases} \\ \text{(ii)} & 0 \le \operatorname{hgt}_y(s) \le n-1 & \Rightarrow & \operatorname{hgt}_x(s) \le m-1. \\ \text{(iii)} & n-1 < \operatorname{hgt}_y(s) & \Rightarrow & \operatorname{hgt}_x(s) \le (\operatorname{hgt}_y(s)-n+2)(m-1)+1. \end{array}$$

Proof. Let

$$h = \sum_{i=0}^{m-1} h_i x^i, \qquad h_i \in \mathbb{F}((y^{-1}))$$

be the y-digit representation of an element $s \in S$.

(i) Applying Lemma 3·3 with $\ell = m-1$ and $\deg_y(h) = \operatorname{hgt}_y(s)$ yields (with $h^{(k)}$ defined as in this lemma)

$$\deg_x(h^{(k)}) \le \begin{cases} m-1+m\operatorname{hgt}_y(s), & n=1, \\ m-1-\left\lceil \frac{-\operatorname{hgt}_y(s)}{n-1} \right\rceil, & n \ge 2 \end{cases}$$

for k sufficiently small. Setting $h' := \lim_{k\to-\infty} h^{(k)}$ this implies that

$$\deg_x(h') \le \begin{cases} m-1+m\operatorname{hgt}_y(s), & n=1, \\ m-1-\left\lceil \frac{-\operatorname{hgt}_y(s)}{n-1} \right\rceil, & n \ge 2. \end{cases}$$

However, since h' is the x-digit representation of s this implies that $\deg_x(h') = \operatorname{hgt}_x(s)$ and we are done.

(ii) This follows directly from Lemma 3·9.

(iii) If $\operatorname{hgt}_y(s) = \deg_y(h)$ satisfies the bounds in (iii) we need to apply Lemma 3·8 first. Since $\deg_x(h) \le m-1$ by Lemma 3·5 (setting $\ell = m-1$ and $\deg_y(h) = \operatorname{hgt}_y(s)$) this yields h' with $\pi_H(h) = \pi_H(h')$ and

$$\deg_x(h') \le (m-1)(\operatorname{hgt}_y(s)-n+2)+1.$$

By Lemma 3·9 we obtain an x-representation h'' of s satisfying the same bounds on $\deg_x$ as h'. This yields (iii).

□

The following example illustrates the fact that the bounds are actually sharp.

EXAMPLE 5·3. *Let $f := x^3 - y^3x^2 + y^4 \in \mathbb{F}_5[x,y]$, $a := x^2y^{-10}$, $b := x^2y^3$ and $c := x^2y^{10}$. Here $m=3$, $n=4$, $\operatorname{hgt}_y(a) = -10$, $\operatorname{hgt}_y(b) = 3$ and $\operatorname{hgt}_y(c) = 10$. In these cases one checks:*

$$\begin{aligned} &a \equiv 4y^2x^{-7} + 3yx^{-5} + 4y^3x^{-4} + 2x^{-3} + y^2x^{-2} \mod f, \\ &b \equiv y^3x^2 \mod f, \\ &c \equiv y^2x^8 + 2yx^{10} + y^3x^{11} + 3x^{12} + 4y^2x^{13} + 4yx^{15} + y^3x^{16} + 4x^{17} \mod f. \end{aligned}$$

Hence, $\operatorname{hgt}_x(a) = -2$, $\operatorname{hgt}_x(b) = 2$ and $\operatorname{hgt}_x(c) = 17$. These are exactly the bounds of the above lemma.

We may consider S_x (and also S) as an $\mathbb{F}((x^{-1}))$-vector space or S_y (and also S) as an $\mathbb{F}((y^{-1}))$-vector space. In both cases we are dealing with normed topological vector spaces; we just set $|s|_x := q^{\mathrm{hgt}_x(s)}$ and $|s|_y := q^{\mathrm{hgt}_y(s)}$ (with q being the cardinality of the finite field or e in case of an infinite ground field). The two vector spaces are obviously complete with respect to to the respective metric.

Definition 5·4 (Fundamental domain). *We define*

$$\mathcal{F}_x := \{s \in S \mid \ \mathrm{hgt}_x(s) < 0\} = \{s \in S \mid |s|_x < 1\}$$

and call it the fundamental domain with respect to x.

In other words $\mathcal{F}_x \subset S$ is the set of those elements that have a purely fractional representation in x-digits. We are interested in the structure of $\mathcal{F}_x$ when written in terms of y-digits. A similar question is how the fundamental domains $\mathcal{F}_x$ and $\mathcal{F}_y$ are related or how the two different norms on S compare. This yields information about the arithmetic properties of the isomorphism φ_{xy}. From the above lemma, we immediately get the following corollary.

Corollary 5·5 (Continuity of φ_{xy}). *Let $s \in S$. We have the following assertions:*

(i) *The isomorphism φ_{xy} is a continuous map.*
(ii) $\mathrm{hgt}_y(s) < -(n-1)(m-1)$ *then* $s \in \mathcal{F}_x$.
(iii) *If* $s \in \mathcal{F}_x$ *then* $\mathrm{hgt}_y(s) \leq n-1$.

Proof. All items are straight-forward consequences of Lemma 5·2.

(i) We may restrict our attention to neighborhoods of 0 because φ_{xy} can be viewed as a homomorphism between topological groups. Lemma 5·2 now becomes a very explicit statement of the ϵ-δ-definition of continuity at 0.
(ii) In particular, we have $\mathrm{hgt}_y(s) < 0$, so we can apply Lemma 5·2(i). If $n = 1$, the claim follows immediately from this part of Lemma 5·2. If $n \geq 2$, we get

$$\begin{aligned} & & -\mathrm{hgt}_y(s) &> (n-1)(m-1) \\ &\Rightarrow & (n-1)\left\lceil \tfrac{-\mathrm{hgt}_y(s)}{n-1} \right\rceil &> (n-1)(m-1) \\ &\Rightarrow & \left\lceil \tfrac{-\mathrm{hgt}_y(s)}{n-1} \right\rceil &> m-1 \\ &\Rightarrow & (m-1) - \left\lceil \tfrac{-\mathrm{hgt}_y(s)}{n-1} \right\rceil &< 0. \end{aligned}$$

Hence, by Lemma 5·2(ii) we have $\mathrm{hgt}_x(s) < 0$ and thus $s \in \mathcal{F}_x$.
(iii) This follows directly from the symmetric statement of Lemma 5·2(iii).

□

Corollary 5·6. *If $n = 1$ then $\mathcal{F}_x = \mathcal{F}_y$.*

Proof. This follows directly from Corollary 5·5(ii) and its symmetric statement. □

Corollary 5·7 (Mutual composition of fundamental domains). *Let*

$$\rho = (n-1)(m-1)$$

and

$$V_x = \Big\{ s \in \mathcal{F}_x \mid \; s = \pi_H(h) \text{ for some } h \in H \text{ with } \operatorname{supp}(h) \subseteq \{0, \ldots, m-1\} \times \{-\rho, \ldots, n-1\} \Big\}.$$

Then V_x is an $\mathbb{F}$-vector space and

$$\mathcal{F}_x = \bigcup_{s \in V_x} (s + y^{-\rho} \mathcal{F}_y). \tag{5·1}$$

In particular, $\mathcal{F}_x$ is a clopen, bounded *subset of the topological $\mathbb{F}((y^{-1}))$-vector space S. If $\mathbb{F} = \mathbb{F}_q$ is a finite field then V_x is finite and $\mathcal{F}_x$ is even* compact.

Proof. By definition, V_x is trivially an $\mathbb{F}$-vector space. By Corollary 5·5(iii) we can write any element $s \in \mathcal{F}_x$ as $s = \pi_H(h')$ for some $h' \in H$ with $\operatorname{supp}(h') \subset A_y$ and $\deg_y(h') < n$. Write now $h' = h + h''$ with $\operatorname{supp}(h) \subseteq \{0, \ldots, m-1\} \times \{-\rho, \ldots, n-1\}$ and $\deg_y(h'') < -\rho$. Then by Corollary 5·5(ii) we always have $\pi_H(h'') \in \mathcal{F}_x$. Therefore $\pi_H(h') \in \mathcal{F}_x$ if and only if $\pi_H(h) \in \mathcal{F}_x$. But elements of the form $\pi_H(h'')$ are exactly the elements in the set $y^{-\rho}\mathcal{F}_y$. This proves (5·1).

Set

$$W = \Big\{ s \in S \mid \; s = \pi_H(h) \text{ for some } h \in H \text{ with } \operatorname{supp}(h) \subseteq \{0, \ldots, m-1\} \times \{-\rho, \ldots, \infty\} \Big\}.$$

Now S is the disjoint union of the open sets of the form $s + y^{-\rho}\mathcal{F}_y$ where $s \in W$. V_x is a subset of W, hence $\mathcal{F}_x$ and $S \setminus \mathcal{F}_x$ can both be written as infinite unions of open sets. Therefore $\mathcal{F}_x$ is open, closed and bounded (again by Corollary 5·5(iii)). If $\mathbb{F} = \mathbb{F}_q$ is a finite field, it is easily seen that $\mathcal{F}_x$ is *totally* bounded and therefore compact. □

REMARK 5·8.

(i) The interest of this and the following corollaries lies in the fact that we view $\mathcal{F}_x$ (which corresponds to the open unit ball of S as an $\mathbb{F}((x^{-1}))$-vector space) as a subset of S considered as an $\mathbb{F}((y^{-1}))$-vector space.

(ii) The statement about compactness becomes wrong in case of an infinite ground field $\mathbb{F}$. Then $\mathcal{F}_x$ is not even compact in S as an $\mathbb{F}((x^{-1}))$-vector space. For example let $(a_i)_{i \in \mathbb{N}}$ be an infinite sequence of pairwise distinct x-digits. Then $b_i := (.a_i)$ is a sequence in $\mathcal{F}_x$ without accumulation point.

Corollary 5·7 gives a way for computing $\mathcal{F}_x$ in terms of $\mathcal{F}_y$. All we need to do is to determine V_x. We will now sketch an algorithm for doing this fastly.

We have to decide for each $h \in H$ with $\operatorname{supp}(h) \subseteq \{0, \ldots, m-1\} \times \{-\rho, \ldots, n-1\}$ whether $\pi_H(h) \in \mathcal{F}_x$ or not. Since the set A_x of x-digit representations is closed under addition, we first compute the x-digit representation $h_{i,j}$ of $x^i y^j$ for each pair $i, j \in \{0, \ldots, m-1\} \times \{-\rho, \ldots, n-1\}$. Observe that h is of the form

$$h = \sum_{i=0}^{m-1} \sum_{j=-\rho}^{n-1} a_{i,j} x^i y^j.$$

Note that $\pi_H(h) \in \mathcal{F}_x$ if and only if $\mathrm{hgt}_x(h) < 0$. By the definition of $h_{i,j}$ and because A_x is closed under addition this is equivalent to

$$\deg_x \left(\sum_{i=0}^{m-1} \sum_{j=-\rho}^{n-1} a_{i,j} h_{i,j} \right) < 0.$$

Setting $h'_{i,j} := \lfloor h_{i,j} \rfloor_x$ this can be rewritten as

$$\sum_{i=0}^{m-1} \sum_{j=-\rho}^{n-1} a_{i,j} h'_{i,j} = 0. \tag{5·2}$$

Thus $\pi_H(h)$ is in V_x if and only if (5·2) holds. Hence, 5·2 gives a set of linear equations for the elements $a_{i,j} \in \mathbb{F}$ that characterizes V_x.

One can view a fundamental domain as the self affine-solution of an *iterated function system* (for the definition, we refer to HUTCHINSON [**9**]) in case $\mathbb{F} = \mathbb{F}_q$ is a finite field.

COROLLARY 5·9 (Fundamental domains viewed as self-affine sets). *Let $\mathbb{F} = \mathbb{F}_q$ be a finite field. Then the fundamental domain $\mathcal{F}_x$ is the unique non-empty compact subset of S (seen as $\mathbb{F}((y^{-1}))$-vector space) satisfying the set equation*

$$\mathcal{F}_x = \bigcup_{d \in \mathcal{N}_x} x^{-1}(\mathcal{F}_x + d).$$

Proof. This is an easy consequence of the general theory of self-affine sets and iterated function systems (see for instance HUTCHINSON [**9**]). □

REMARK 5·10. *If $\mathbb{F}$ is infinite, $\mathcal{F}_x$ cannot be seen as a solution of an infinite iterated function system in S (in the sense of* FERNAU [**5**]*, for instance), because the contraction ratios of the mappings $\chi_d(z) := x^{-1}(z + d)$, where $d \in \mathcal{N}_x$, do not converge to zero. We could only regard it as a non-compact solution of*

$$T = \bigcup_{d \in \mathcal{N}_x} x^{-1}(T + d).$$

However, this set equation has more than one non-empty non-compact solutions.

THEOREM 5·11 (A tiling induced by the fundamental domain). *The collection*

$$\{\mathcal{F}_x + r \mid r \in R\}$$

forms a tiling of the space S (seen as an $\mathbb{F}((y^{-1}))$-vector space) in the sense that

$$\bigcup_{r \in R} (\mathcal{F}_x + r) = S$$

and

$$(\mathcal{F}_x + r_1) \cap (\mathcal{F}_x + r_2) = \emptyset$$

for $r_1, r_2 \in R$ distinct.

Proof. Any $s \in S$ has a unique x-digit representation h by Theorem 3·10. Setting $s_0 := \pi_H(\{h\}_x)$ and $s_1 := \pi_H(\lfloor h \rfloor_x)$ we find that s decomposes uniquely as $s = s_0 + s_1$ with $s_0 \in \mathcal{F}_x$ and $s_1 \in R \subset S$, hence $S = \mathcal{F}_x \oplus R$ is a direct sum of $\mathbb{F}$-vector spaces. The claim of the theorem is just another formulation of the same fact. □

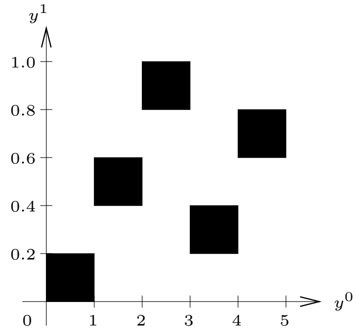

Fig. 3. The fundamental domain $\mathcal{F}_y$

This picture shows the fundamental domain $\mathcal{F}_y$ in S (when viewed as a vector space over $\mathbb{F}_5((x^{-1})))$ induced by the polynomial $x^3 + (3y + 1)x + 2y + y^2 \in \mathbb{F}_5[x, y]$. For illustration purposes elements of $\mathbb{F}_5((x^{-1}))$ were mapped to $\mathbb{R}$ by mapping x to 5 and the field elements to their canonical representatives in $\{0, \ldots, 4\}$.

Finally, we want to present a way how to visualize a fundamental domain in the real vector space if $\mathbb{F} = \mathbb{F}_q$ is a finite field. Let $\nu : \mathbb{F}_q \to \{0, \ldots, q-1\}$ be an enumeration of the elements of the finite field. We define the mapping

$$\mu : \mathbb{F}_q((y^{-1})) \to \mathbb{R}, \quad \sum_{i=-\infty}^{\ell} c_i y^i \mapsto \sum_{i=-\infty}^{\ell} \nu(c_i) q^i.$$

Note that the right sum converges since it can be bounded by a geometric sum.

In order to keep the notation simple, we will assume that $n = 2$. Then we may consider elements $s \in S$ as vectors of the plane $\mathbb{F}_q((y^{-1})) \times \mathbb{F}_q((y^{-1}))$ and visualize them via the map $\mu \times \mu$ in $\mathbb{R}^2$. If we use the algorithm sketched above in order to express $\mathcal{F}_x$ in terms of $\mathcal{F}_y$ we see immediately that $\mathcal{F}_x$ corresponds to a finite union of closed boxes of side length $q^{-\rho}$ where ρ is defined as in Corollary 5·7. Since elements of R have integral coordinates in $\mathbb{R}^2$ under the map $\mu \times \mu$ it is clear that the overall volume of the image of the fundamental domain $\mathcal{F}_x$ is 1. This gives an intuitive picture of the fundamental domain.

EXAMPLE 5·12. *Let $f := x^3 + (3y + 1)x + 2y + y^2 \in \mathbb{F}_5[x, y]$, hence $q = 5$. We want to express, say, $\mathcal{F}_y$ in terms of $\mathcal{F}_x$. To this matter we compute the vector space of Corollary* 5·7 *where $\rho = (3-1)(2-1) = 2$ and get*

$$V_y = \langle 1 + 2x^{-1}y, x^{-2}y, x^{-1}, x^{-2} \rangle_{\mathbb{F}_5} \subset \mathbb{F}_5((x^{-1}, y^{-1}))/f\mathbb{F}_5((x^{-1}, y^{-1}))$$

which reads in vector notation as

$$V_y = \langle (1, 2x^{-1}), (0, x^{-2}), (x^{-1}, 0), (x^{-2}, 0) \rangle_{\mathbb{F}_5} \subset \mathbb{F}((x^{-1})) \times \mathbb{F}((x^{-1})).$$

Since $\mathbb{F}_5$ is finite, V_y is finite too, and one computes exactly 625 *different $\mathbb{F}_5$-linear combinations of the basis elements in V_y. Applying the map $\mu \times \mu$ to each of these elements one gets the lower left vertex of a box in $\mathbb{R}^2$ of side length $5^{-\rho} = 0.04$. The union of these boxes visualizes the fundamental domain and has volume* 1*, see Figure* 3*.*

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