Tractability properties of the weighted star discrepancy

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Abstract

Tractability properties of various notions of discrepancy have been intensively studied in the last decade. In this paper we consider the so-called weighted star discrepancy which was introduced by Sloan and Woźniakowski. We show that under a very mild condition on the weights one can obtain tractability with s-exponent zero (s is the dimension of the point set). In the case of product weights we give a condition such that the weighted star discrepancy is even strongly tractable. Furthermore, we give a lower bound for the weighted star discrepancy for a large class of weights. This bound shows that for such weights one cannot obtain strong tractability.

Keywords: Weighted star discrepancy, (strong) tractability, quasi-Monte Carlo.

1 Introduction

For quasi-Monte Carlo integration of functions over the s-dimensional unit cube $[0,1]^s$ one needs point sets which are very well distributed. In many cases the quality of the distribution of a point set is measured by the star discrepancy which is intimately linked to the worst-case error of quasi-Monte Carlo integration via the well known Koksma-Hlawka inequality (see, for example, [8, 9, 14]).

For a point set $\mathcal{P}_{N,s} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in the s-dimensional unit cube $[0,1)^s$ the *discrepancy function* Δ is defined by

$$\Delta(\alpha_1, \ldots, \alpha_s) := \frac{A_N\left(\prod_{i=1}^s [0, \alpha_i)\right)}{N} - \alpha_1 \cdots \alpha_s$$

for $0 < \alpha_1, \ldots, \alpha_s \leq 1$. Here $A_N(E)$ denotes the number of indices n, $0 \leq n \leq N-1$, such that $\boldsymbol{x}_n$ is contained in the set E. By taking the sup norm of this function, we obtain the *star discrepancy*

$$D_N^*(\mathcal{P}_{N,s}) = \sup_{\boldsymbol{z} \in (0,1]^s} |\Delta(\boldsymbol{z})|$$

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of the point set $\mathcal{P}_{N,s}$. We will often refer to the star discrepancy as the *classical* star discrepancy in contrary to the weighted star discrepancy defined below.

Sloan and Woźniakowski [17] (see also [2]) introduced the notion of weighted discrepancy and proved a "weighted" Koksma-Hlawka inequality. The idea is that in many applications some projections are more important than others and that this should also be reflected in the quality measure of the point set.

We start with some notation which goes back to the paper [17]: let $I_s = \{1, 2, \ldots, s\}$ denote the set of coordinate indices. For $\mathfrak{u} \subseteq I_s$, $\mathfrak{u} \neq \emptyset$, let $\gamma_{\mathfrak{u},s}$ be a nonnegative real number (the *weight*), $|\mathfrak{u}|$ the cardinality of $\mathfrak{u}$, and for a vector $\boldsymbol{z} \in [0,1]^s$ let $\boldsymbol{z}(\mathfrak{u})$ denote the vector from $[0,1]^{|\mathfrak{u}|}$ containing the components of $\boldsymbol{z}$ whose indices are in $\mathfrak{u}$. By $(\boldsymbol{z}_{\mathfrak{u}}, 1)$ we mean the vector $\boldsymbol{z}$ from $[0,1]^s$ with all components whose indices are not in $\mathfrak{u}$ replaced by 1.

Definition 1 For a point set $\mathcal{P}_{N,s} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ in $[0,1)^s$ and given weights $\boldsymbol{\gamma} = \{\gamma_{\mathfrak{u},s} : \mathfrak{u} \subseteq I_s, \mathfrak{u} \neq \emptyset\}$, the *weighted star discrepancy* $D^*_{N,\boldsymbol{\gamma}}$ is given by

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_{N,s}) = \sup_{\boldsymbol{z} \in (0,1]^s} \max_{\emptyset \neq \mathfrak{u} \subseteq I_s} \gamma_{\mathfrak{u},s} |\Delta(\boldsymbol{z}_{\mathfrak{u}}, 1)|.$$

We remark that also the notion of weighted L_p-discrepancy is well known and studied (mainly for the special case $p = 2$) in a multitude of papers (see, for example, [1, 5, 10, 17]). The following kind of weights are often studied in literature:

- *Product weights* which are weights of the form $\gamma_{\mathfrak{u},s} = \prod_{j \in \mathfrak{u}} \gamma_{j,s}$, for $\emptyset \neq \mathfrak{u} \subseteq I_s$, where $\gamma_{j,s}$ is the weight associated with the j-th component. Often the weights $\gamma_{j,s}$ have no dependence on s, i.e., $\gamma_{j,s} = \gamma_j$. See, for example, [17, 2].

- *Finite-order weights* of fixed order $k \in \mathbb{N}$ which are weights with $\gamma_{\mathfrak{u},s} = 0$ for all $\mathfrak{u} \subseteq I_s$ with $|\mathfrak{u}| > k$. See, for example, [6, 16].

We would like to have a point set in the s-dimensional unit cube with weighted star discrepancy of at most $\varepsilon \in (0,1)$ and we are looking for the smallest cardinality N of a point set such that this can be achieved. For $\varepsilon \in (0,1)$ and dimension $s \in \mathbb{N}$ we define the quantity

$$N_{\min}(\varepsilon, s) := \min\{N \in \mathbb{N} : \exists \mathcal{P}_{N,s} \subset [0,1)^s \text{ such that } D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_{N,s}) \leq \varepsilon\},$$

which is often called the *inverse of the weighted star discrepancy.*

Definition 2 1. We say that the weighted star discrepancy is *tractable*, if there exist non-negative C, α and β such that

$$N_{\min}(\varepsilon, s) \leq C s^{\alpha} \varepsilon^{-\beta} \tag{1}$$

holds for all dimensions $s = 1, 2, \ldots$ and for all $\varepsilon \in (0,1)$. The infima of α and β such that (1) holds are called the s-exponent and the ε-exponent of tractability.

2. We say that the weighted star discrepancy is *strongly tractable*, if inequality (1) holds with $\alpha = 0$.

Tractability means that there exists a point set whose cardinality is polynomial in s and ε^{-1} such that the weighted star discrepancy of this point set is bounded by ε.

An excellent survey on tractability of different notions of discrepancy can be found in the paper [15].

Tractability and strong tractability for the classical star discrepancy are defined in the same manner as in the weighted case. Here it is known that for any number of points N and dimension s there exists a point set $\mathcal{P}_{N,s} \subset [0,1)^s$, such that

$$D_N^*(\mathcal{P}_{N,s}) \leq C\sqrt{\frac{s}{N}}$$

for some constant $C > 0$. This was first proved by Heinrich, Novak, Wasilkowski, and Woźniakowski in [12]. For an extension of this result see [3]. Hence the classical star discrepancy is tractable with s-exponent at most one and ε-exponent at most two. It was further shown in [12] that the inverse of the classical star discrepancy is at least $cs \log \varepsilon^{-1}$ with an absolute constant $c > 0$ for all $\varepsilon \in (0, \varepsilon_0]$ and $s \in \mathbb{N}$. This lower bound was improved by Hinrichs [13] to $cs\varepsilon^{-1}$ with an absolute constant $c > 0$ for all $\varepsilon \in (0, \varepsilon_0]$ and $s \in \mathbb{N}$. From these results it follows, that the classical star discrepancy cannot be strongly tractable. We stress that all mentioned results are non-constructive. A first constructive approach is given in [7]. However here for given s and ε the authors can only ensure a running time for the construction algorithm of order $C^s s^s (\log s)^s \varepsilon^{-2(s+2)}$ which is too expensive for practical applications. An overview of many open questions concerning this topic can be found in [11].

Here we are interested in tractability properties of the weighted star discrepancy. Of course it follows from the above results for the classical star discrepancy, that the weighted star discrepancy is tractable with s-exponent at most one and ε-exponent at most two as long as the weights are bounded. Now one may ask for conditions on the weights such that a fewer dependence on the dimension s, i.e., a smaller s-exponent or even strong tractability, can be obtained.

In Section 2 of this paper we show that under a very mild condition on the weights we can indeed obtain an s-exponent equal to zero (Theorems 1 and 2). The proofs for these results are based on the results from [12] and [7] for the classical star discrepancy. Furthermore we consider the case of product weights (independent of s) and give conditions such that the weighted star discrepancy is strongly tractable (Theorem 3). The proof of this result is an extension of a result from [4]. Finally, in Section 3 we give a lower bound on the weighted star discrepancy for a large class of weights (Theorem 4). From this bound we conclude that for such weights we cannot have strong tractablity.

2 Upper Bounds

First we prove the existence of point sets in the s-dimensional unit cube whose star discrepancy satisfies a certain upper bound. From this result we deduce our tractability result for the weighted star discrepancy.

Theorem 1 *There exists a constant $C > 0$ with the following property: for given number of points N and dimension s there exists a point set $\mathcal{P}_{N,s}$ consisting of N points in the* *s-dimensional unit cube, such that*

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_{N,s}) \le C\, \frac{1+\sqrt{\log s}}{\sqrt{N}} \max_{\emptyset \ne \mathfrak{u} \subseteq I_s} \gamma_{\mathfrak{u},s} \sqrt{|\mathfrak{u}|}. \tag{2}$$

Proof. It was shown in [12, Theorem 3] that for given number of points N and dimension s the probability that an i.i.d. randomly chosen point set $\mathcal{P}_{N,s}$ has star discrepancy at most $\lambda\sqrt{s/N}$ is at least

$$1-\left(K\lambda^2 \mathrm{e}^{-2\lambda^2}\right)^s,$$

for some (unknown) constant K and for all $\lambda \ge \max\{1, K, \lambda_0\}$, where λ_0 is such that $K\lambda^2 \le \mathrm{e}^{2\lambda^2}$ for all $\lambda \ge \lambda_0$.

For given number of points N and dimension s we consider the set

$$A_s := \left\{\mathcal{P}_{N,s} \subset [0,1)^s \,:\, D^*_N(\mathcal{P}_{N,s}(\mathfrak{u})) \le \lambda \sqrt{\frac{|\mathfrak{u}|}{N}} \;\; \forall \mathfrak{u} \subseteq I_s, \mathfrak{u} \ne \emptyset\right\},$$

where $\mathcal{P}_{N,s}(\mathfrak{u}) := \{\boldsymbol{x}_0(\mathfrak{u}), \ldots, \boldsymbol{x}_{N-1}(\mathfrak{u})\}$ if $\mathcal{P}_{N,s} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$. Furthermore, for $\mathfrak{u} \subseteq I_s$, $\mathfrak{u} \ne \emptyset$, we define

$$A_{\mathfrak{u},s} := \left\{\mathcal{P}_{N,s} \subset [0,1)^s \,:\, D^*_N(\mathcal{P}_{N,s}(\mathfrak{u})) \le \lambda \sqrt{\frac{|\mathfrak{u}|}{N}}\right\}.$$

Then we have

$$A_s = \bigcap_{\emptyset \ne \mathfrak{u} \subseteq I_s} A_{\mathfrak{u},s}.$$

Hence

$$\begin{aligned}
\mathbb{P}[A_s] &= \mathbb{P}\left[\bigcap_{\emptyset \ne \mathfrak{u} \subseteq I_s} A_{\mathfrak{u},s}\right] = 1-\mathbb{P}\left[\bigcup_{\emptyset \ne \mathfrak{u} \subseteq I_s} A^c_{\mathfrak{u},s}\right] \ge 1-\sum_{\emptyset \ne \mathfrak{u} \subseteq I_s} \mathbb{P}\left[A^c_{\mathfrak{u},s}\right] \\
&> 1-\sum_{\emptyset \ne \mathfrak{u} \subseteq I_s} \left(K\lambda^2 \mathrm{e}^{-2\lambda^2}\right)^{|\mathfrak{u}|} = 1-\sum_{u=1}^{s} \binom{s}{u} \left(K\lambda^2 \mathrm{e}^{-2\lambda^2}\right)^u = 2-\left(1+K\lambda^2 \mathrm{e}^{-2\lambda^2}\right)^s.
\end{aligned}$$

Now we choose $\lambda := c \max\left\{1, \sqrt{(\log s)/(\log 2)}\right\}$ with $c := \max\{2, K, \lambda_0\}$. Then for $s=1$ we obtain

$$\mathbb{P}[A_1] > 1-Kc^2 \mathrm{e}^{-2c^2} \ge 0$$

as $c \ge \lambda_0$. For $s \ge 2$ and $x := c^2/\log 2 > 5$ we have $x^2 \le 2^x \le s^x$ and $\log s \le s^{x-1}$. Therefore it follows that $x^2 \log s \le s^{2x-1}$ and hence

$$\frac{c^3 \log s}{(\log 2) s^{2c^2/(\log 2)}} \le \frac{\log 2}{cs}.$$

From this inequality we obtain (for $s \ge 2$)

$$\begin{aligned}
\mathbb{P}[A_s] &> 2-\left(1+K\lambda^2 \mathrm{e}^{-2\lambda^2}\right)^s \ge 2-\left(1+\frac{c^3 \log s}{(\log 2) s^{2c^2/(\log 2)}}\right)^s \\
&\ge 2-\left(1+\frac{\log 2}{cs}\right)^s > 2-\mathrm{e}^{(\log 2)/c} = 2-2^{1/c} > 0.
\end{aligned}$$

Hence for all $s \in \mathbb{N}$ we have $\mathbb{P}[A_s] > 0$. Thus we have shown that there exists a point set $\mathcal{P}_{N,s} \subset [0,1)^s$ such that for each $\emptyset \neq \mathfrak{u} \subseteq I_s$ we have

$$\begin{aligned} D_N^*(\mathcal{P}_{N,s}(\mathfrak{u})) &\leq c \max\left\{1, \sqrt{\frac{\log s}{\log 2}}\right\} \sqrt{\frac{|\mathfrak{u}|}{N}} \\ &\leq C\left(1+\sqrt{\log s}\right)\sqrt{\frac{|\mathfrak{u}|}{N}}. \end{aligned}$$

For the weighted star discrepancy of this point set we obtain

$$D_{N,\boldsymbol{\gamma}}^*(\mathcal{P}_{N,s}) \leq C\,\frac{1+\sqrt{\log s}}{\sqrt{N}} \max_{\emptyset \neq \mathfrak{u} \subseteq I_s} \gamma_{\mathfrak{u},s}\sqrt{|\mathfrak{u}|},$$

which is the desired result. □

Remark 1 One can see from the proof above that, by increasing c (and therefore C), the probability such that (2) holds tends to one.

From Theorem 1 we obtain the following conclusion.

Corollary 1 *If*

$$C_{\boldsymbol{\gamma}} := \sup_{s=1,2,\ldots} \max_{\emptyset \neq \mathfrak{u} \subseteq I_s} \gamma_{\mathfrak{u},s}\sqrt{|\mathfrak{u}|} < \infty, \tag{3}$$

then for the weighted star discrepancy of the point set from Theorem 1 we have

$$D_{N,\boldsymbol{\gamma}}^*(\mathcal{P}_{N,s}) \leq C \cdot C_{\boldsymbol{\gamma}}\,\frac{1+\sqrt{\log s}}{\sqrt{N}}, \tag{4}$$

where $C > 0$ is the (unknown) constant from Theorem 1. Hence we have

$$N_{\min}(\varepsilon, s) \leq \left\lceil \frac{C^2 \cdot C_{\boldsymbol{\gamma}}^2 \left(1+\sqrt{\log s}\right)^2}{\varepsilon^2} \right\rceil. \tag{5}$$

From (5) we obtain, that if condition (3) holds, then the weighted star discrepancy is tractable with s-exponent zero and with ε-exponent at most 2. We stress that we do not obtain strong tractability in this case as we still have the logarithmic dependence on the dimension s.

We note that condition (3) is a very mild condition on the weights. For example for finite order weights it is always fulfilled. In the case of product weights it is enough that the weights γ_j are decreasing and that $\gamma_j < 1$ for an index $j \in \mathbb{N}$. In fact, we have

$$\max_{\emptyset \neq \mathfrak{u} \subseteq I_s} \gamma_{\mathfrak{u},s}\sqrt{|\mathfrak{u}|} = \max_{u=1,\ldots,s} \sqrt{u} \prod_{j=1}^{u} \gamma_j$$

and hence $C_{\boldsymbol{\gamma}} = \sup_{s=1,2,\ldots} \sqrt{s} \prod_{j=1}^{s} \gamma_j$. We have

$$\frac{\sqrt{s}\prod_{j=1}^{s}\gamma_j}{\sqrt{s+1}\prod_{j=1}^{s+1}\gamma_j} = \sqrt{\frac{s}{s+1}}\frac{1}{\gamma_{s+1}} > 1$$

for s large enough and therefore it follows that $C_{\boldsymbol{\gamma}} < \infty$.

For example, if $\gamma_j = 1/\log(j+1)$, then $C_{\boldsymbol{\gamma}} = \frac{\sqrt{2}}{\log 2 \log 3}$.

For a much stronger condition on the weights we could even obtain strong tractability, namely if

$$\sup_{s=1,2,\ldots} \left(\sqrt{\log s} \max_{\emptyset \neq \mathfrak{u} \subseteq I_s} \gamma_{\mathfrak{u},s} \sqrt{|\mathfrak{u}|} \right) < \infty.$$

However, this condition is a very restrictive one. For example, it cannot hold for weights which are independent of the dimension.

In the same way as above, one can show a little bit weaker result, but with explicit constants.

Theorem 2 *For given number of points N and dimension s there exists a point set $\mathcal{P}_{N,s}$ consisting of N points in the s-dimensional unit cube, such that*

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_{N,s}) \leq \frac{1}{\sqrt{N}} \sqrt{2} \left(\log \left(\left\lceil \rho \sqrt{N} \right\rceil + 1 \right) + \log\left(2(\mathrm{e}-1)s\right) \right)^{1/2} \max_{\emptyset \neq \mathfrak{u} \subseteq I_s} \gamma_{\mathfrak{u},s} \sqrt{|\mathfrak{u}|},$$

where $\rho = \frac{3 \log 3}{\sqrt{2(3\log 3 + \log 2)}}$.

Proof. The proof of Theorem 2 follows exactly the lines of the proof of Theorem 1. The only difference is that here we use the fact that for given number of points N and dimension s the probability that an i.i.d. randomly chosen point set $\mathcal{P}_{N,s}$ has star discrepancy at most

$$\frac{\sqrt{2}}{\sqrt{N}} \left(s \log \left(\left\lceil \rho \sqrt{N} \right\rceil + 1 \right) + \log \left(\frac{2}{c} \right) \right)^{1/2},$$

where $\rho = \frac{3 \log 3}{\sqrt{2(3\log 3 + \log 2)}}$, is at least $1 - c$, where $0 < c \leq 1$ is a real. This result follows from a slight extension of the proof of [7, Theorem 3.2] (see also [12, Theorem 1]). $\square$

We obtain the following corollary.

Corollary 2 *If*

$$C_{\boldsymbol{\gamma}} := \sup_{s=1,2,\ldots} \max_{\emptyset \neq \mathfrak{u} \subseteq I_s} \gamma_{\mathfrak{u},s} \sqrt{|\mathfrak{u}|} < \infty, \tag{6}$$

then for the weighted star discrepancy of the point set from Theorem 2 we have

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_{N,s}) \leq \frac{C_{\boldsymbol{\gamma}}}{\sqrt{N}} \sqrt{2} \left(\log \left(\left\lceil \rho \sqrt{N} \right\rceil + 1 \right) + \log\left(2(\mathrm{e}-1)s\right) \right)^{1/2}. \tag{7}$$

Again, from the bound (7) we do not obtain that the weighted star discrepancy is strongly tractable. We only obtain that it is tractable with s-exponent equal to zero (and ε-exponent at most 2).

Now we turn to the case of product weights (independent of the dimension s) and give a condition under which the weighted star discrepancy is strongly tractable. The following result is an extension of [4, Corollary 8].

Theorem 3 *Let $N, s \in \mathbb{N}$. For product weights, if*

$$\sum_{j=1}^{\infty} \gamma_j < \infty,$$

then there exists a point set $\mathcal{P}_{N,s} \subset [0,1)^s$ such that for any $\delta > 0$ we have

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_{N,s}) \leq \frac{C_{\delta,\boldsymbol{\gamma}}}{N^{1-\delta}},$$

where $C_{\delta,\boldsymbol{\gamma}} > 0$ is independent of s and N. Hence the weighted star discrepancy is strongly tractable with ε-exponent equal to one.

Remark 2 We remark that the point set $\mathcal{P}_{N,s}$ considered in Theorem 3 is a superposition of digital nets over $\mathbb{Z}_2$. This will follow from the proof below. However, the result is still not constructive as the result which we will use was proved by averaging over all digital nets, see [4]. We remark that strong tractability results for the weighted star discrepancy can also be obtained from the results of Wang in [18] and [19]. Wang's results are constructive, but one needs much more restrictive conditions on the weights.

For the proof of Theorem 3 we need the subsequent lemma.

Lemma 1 *Let $\mathcal{P}_{N_1,s}, \ldots, \mathcal{P}_{N_m,s}$ be point sets with cardinality $N_1, \ldots, N_m$ respectively. Further let $\mathcal{P}_{N,s} = \mathcal{P}_{N_1,s} \cup \ldots \cup \mathcal{P}_{N_m,s}$ (here we mean a superposition where the multiplicity of elements matters) and $N = N_1 + \cdots + N_m$. Then we have*

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_{N,s}) \leq \sum_{i=1}^{m} \frac{N_i}{N} D^*_{N_i,\boldsymbol{\gamma}}(\mathcal{P}_{N_i,s}).$$

We omit the easy proof of this result. (See [9] for a proof of this result in the unweighted case.)

Proof of Theorem 3. Under the assumption $\sum_{i=1}^{\infty} \gamma_i < \infty$ it was shown in [4, Corollary] by averaging over all digital nets, that for each $\delta > 0$ there exists for each prime p and each $m \in \mathbb{N}$ a digital net over $\mathbb{Z}_p$ with p^m points, say $\mathcal{P}_{p^m,s} \subset [0,1)^s$, such that

$$D^*_{p^m,\boldsymbol{\gamma}}(\mathcal{P}_{p^m,s}) \leq \frac{C_{\delta,\boldsymbol{\gamma}}}{p^{m(1-\delta)}},$$

where $C_{\delta,\boldsymbol{\gamma}} > 0$ is independent of s and m.

Now for simplicity we consider the case $p = 2$ only. Let $\delta > 0$ and let $N \in \mathbb{N}$ with binary representation $N = 2^{r_1} + \cdots + 2^{r_m}$, where $0 \leq r_1 < r_2 < \ldots < r_m$, i.e., $r_m = \lfloor \log_2 N \rfloor$, where $\log_2$ denotes the logarithm in base 2. For each $1 \leq i \leq m$ there exists a point set $\mathcal{P}_{2^{r_i},s} \subset [0,1)^s$, such that

$$D^*_{2^{r_i},\boldsymbol{\gamma}}(\mathcal{P}_{2^{r_i},s}) \leq \frac{C_{\delta,\boldsymbol{\gamma}}}{2^{r_i(1-\delta)}}.$$

Let $\mathcal{P}_{N,s} = \mathcal{P}_{2^{r_1},s} \cup \ldots \cup \mathcal{P}_{2^{r_m},s}$ (here we mean a superposition where the multiplicity of elements matters). Then it follows from Lemma 1, that

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_{N,s}) \leq \sum_{i=1}^{m} \frac{2^{r_i}}{N} D^*_{2^{r_i},\boldsymbol{\gamma}}(\mathcal{P}_{2^{r_i},s}) \leq \frac{C_{\delta,\boldsymbol{\gamma}}}{N} \sum_{i=1}^{m} 2^{r_i \delta} \leq \frac{C_{\delta,\boldsymbol{\gamma}}}{N} \sum_{j=0}^{\lfloor \log_2 N \rfloor} 2^{j\delta} \leq \frac{\widetilde{C}_{\delta,\boldsymbol{\gamma}}}{N^{1-\delta}}.$$

Hence for each $s, N \in \mathbb{N}$ there exists a point set $\mathcal{P}_{N,s}$ with $D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_{N,s}) \leq \frac{\widetilde{C}_{\delta,\gamma}}{N^{1-\delta}}$ which is the desired result. This point set is a superposition of digital nets over $\mathbb{Z}_2$. □

3 Lower Bounds

The aim of this section is to show that the logarithmic factor in the dimension in the tractability results is indeed necessary for a large class of weigths. That implies that the star discrepancy is not strongly tractable for such weights. In particular, this includes finite order weights of order $k \geq 2$ if all the weights of order 2 are bounded below by a constant $c > 0$.

To prove these lower bounds we start with an elementary lemma. For $\mathfrak{u} \subseteq I_s$ and $k \in \{0, 1\}$ let

$$B_k(\mathfrak{u}) = \left\{ \boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1)^s \,:\, x_i \in \left[\frac{k}{2}, \frac{k+1}{2}\right) \text{ for } i \in \mathfrak{u} \right\}.$$

Lemma 2 *Let $\mathcal{P}_{N,s} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\} \subset [0,1)^s$. Then there exists $\mathfrak{u} \subseteq I_s$ with cardinality at least $s/2^N$ such that one of the sets $B_0(\mathfrak{u})$ and $B_1(\mathfrak{u})$ contains at least half of the points of $\mathcal{P}_{N,s}$.*

Proof. There exists $\mathfrak{u}_0 \subseteq I_s$ with cardinality at least $s/2$ and $k_0 \in \{0, 1\}$ such that $\boldsymbol{x}_0 \in B_{k_0}(\mathfrak{u}_0)$. Inductively, for $h = 1, \ldots, N-1$, we can choose $\mathfrak{u}_h \subseteq \mathfrak{u}_{h-1}$ with cardinality at least $s/2^{h+1}$ and $k_h \in \{0, 1\}$ such that $\boldsymbol{x}_h \in B_{k_h}(\mathfrak{u})$. Set $\mathfrak{u} = \mathfrak{u}_{N-1}$ and let $k \in \{0, 1\}$ be such that at least half of the $k_h, h = 0, \ldots, N-1$ are equal to k. Then the cardinality of $\mathfrak{u}$ is at least $s/2^N$ and at least half of the points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}$ are in $B_k(\mathfrak{u})$. □

We now give the announced lower bound for the weighted star discrepancy.

Theorem 4 *If the weights $\boldsymbol{\gamma} = \{\gamma_{\mathfrak{u},s} : \mathfrak{u} \subseteq I_s, \mathfrak{u} \neq \emptyset\}$ are such that there exists a constant $c > 0$ with $\gamma_{\mathfrak{u},s} \geq c$ for all $\mathfrak{u} \subseteq I_s$ with cardinality 2 then*

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_{N,s}) \geq \frac{c}{12}$$

for any point set $\mathcal{P}_{N,s}$ consisting of N points in the s-dimensional unit cube with $s \geq 2^{N+1}$. In particular, the weighted star discrepancy is not strongly tractable for such weights.

Proof. With the preceding lemma we find $\mathfrak{u}_0 \subseteq I_s$ with cardinality 2 such that one of the sets $B_0(\mathfrak{u}_0)$ or $B_1(\mathfrak{u}_0)$ contains at least $N/2$ points of $\mathcal{P}_{N,s}$. Without loss of generality we assume that $\mathfrak{u}_0 = \{1, 2\}$. Let $\boldsymbol{z}^{(0)} = (1/2, 1/2, 1/2, \ldots, 1/2)$, $\boldsymbol{z}^{(1)} = (1, 1/2, 1/2, \ldots, 1/2)$ and $\boldsymbol{z}^{(2)} = (1/2, 1, 1/2, \ldots, 1/2)$. Furthermore, let n_0, n_1, n_2 be the number of points in the point set $\mathcal{P}_{N,s}$ which are contained in the boxes $I_1 \times I_2 \times [0,1)^{s-2}$ for $I_1 = I_2 = [0, 1/2)$, $I_1 = [1/2, 1), I_2 = [0, 1/2)$ and $I_1 = [0, 1/2), I_2 = [1/2, 1)$, respectively.

Let us first assume that the set $B_0(\mathfrak{u}_0)$ contains at least $N/2$ points. Then

$$\Delta(\boldsymbol{z}^{(0)}_{\mathfrak{u}_0}, 1) = \frac{A_N(B_0(\mathfrak{u}_0))}{N} - \frac{1}{4} \geq \frac{1}{4}$$

which implies

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_{N,s}) \geq \frac{c}{4}.$$

Finally, we treat the case that the set $B_1(\mathfrak{u}_0)$ contains at least $N/2$ points so that its complement contains at most $N/2$ points, i.e.

$$n_0 + n_1 + n_2 \leq N/2.$$

Then at least one of the following three inequalities holds

$$n_0 + n_1 \leq \frac{5N}{12}, \quad n_0 + n_2 \leq \frac{5N}{12}, \quad n_0 \geq \frac{N}{3}.$$

If the first inequality holds then it follows that

$$\Delta(\boldsymbol{z}^{(1)}_{\mathfrak{u}_0}, 1) = \frac{n_0 + n_1}{N} - \frac{1}{2} \leq -\frac{1}{12}.$$

If the second inequality holds, we have

$$\Delta(\boldsymbol{z}^{(2)}_{\mathfrak{u}_0}, 1) = \frac{n_0 + n_2}{N} - \frac{1}{2} \leq -\frac{1}{12}.$$

If the third inequality is true then

$$\Delta(\boldsymbol{z}^{(0)}_{\mathfrak{u}_0}, 1) = \frac{n_0}{N} - \frac{1}{4} \geq \frac{1}{12}.$$

In any case,

$$D^*_{N,\boldsymbol{\gamma}}(\mathcal{P}_{N,s}) \geq \frac{c}{12}.$$

□

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