CLONES FROM IDEALS

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Abstract. On an infinite base set X, every ideal of subsets of X can be associated with the clone of those operations on X which map small sets to small sets. We continue earlier investigations on the position of such clones in the clone lattice.

0. Introduction

0.1. **Clones.** Let X be an infinite set and denote the set of all n-ary operations on X by $\mathscr{O}^{(n)}$. Then $\mathscr{O} := \bigcup_{n\geq 1} \mathscr{O}^{(n)}$ is the set of all finitary operations on X. A subset $\mathscr{C}$ of $\mathscr{O}$ is called a *clone* iff it contains all projections, i.e. for all $1 \leq k \leq n$ the function $\pi^n_k \in \mathscr{O}^{(n)}$ satisfying $\pi^n_k(x_1, \ldots, x_n) = x_k$, and is closed under composition. The set of all clones on X, ordered by set-theoretical inclusion, forms a complete algebraic lattice $\mathrm{Cl}(X)$. The size of this lattice is $2^{2^{|X|}}$, and it is known to be too complicated to be ever fully described: For example, it contains all algebraic lattices with no more than $2^{|X|}$ compact elements as complete sublattices [Pin07]. So the approach to an unveiling of the structure of $\mathrm{Cl}(X)$ is to concentrate on more tractable parts of it, such as large clones (e.g. dual atoms of the lattice), or clones with specific properties (e.g. natural intervals, clones closed under conjugation, etc.). A survey of clones on infinite sets is [GPb].

0.2. **Ideal clones.** In [Ros76], and before in [Gav65] for countably infinite X, it was shown that there exist as many dual atoms (“*precomplete clones*”) in $\mathrm{Cl}(X)$ as there are clones, suggesting that it is impossible to describe all of them (as opposed to the clone lattice on finite X, where the dual atoms are finite in number and explicitly known [Ros70]). Much more recently, a new and short proof of this fact was given in [GS02]. It was observed that given an ideal I of subsets of X, one can associate with it a clone $\mathscr{C}_I$ consisting of those operations which map small sets (i.e., products of sets in I) to small sets. The authors then showed that prime ideals correspond to precomplete clones, and that moreover the clones induced by

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distinct prime ideals differ, implying that there exist as many precomplete clones as prime ideals on X; the latter are known to amount to $2^{2^{|X|}}$.

The study of clones that arise in this way from ideals was pursued in [CH01], for countably infinite X. The authors concentrated on the question of which ideals induce precomplete clones, and provided a criterion for precompleteness.

0.3. **Precompleteness criteria.** We will consider a countable base set X; all our ideals I will be proper and properly contain the ideal of finite subsets of X. (See 0.7 for an explanation.) Under those assumptions on I, which will be valid throughout this paper, it was shown in [CH01] that $\mathscr{C}_I$ is precomplete if and only if for all $A \notin I$ there exists an operation $f \in \mathscr{C}_I$ such that $f[A^n] = X$.

0.3.1. *Arity.* It is a drawback of this criterion that, given a set $A \notin I$, we do not know in advance the arity of the required function mapping A onto X. It would be of much help if one had to check only whether, say, a binary operation of $\mathscr{C}_I$ can map A onto X. Unfortunately this is not the case: we will see in this paper that for every $n \geq 1$, there exists an ideal I_n on a countably infinite base set X such that $\mathscr{C}_{I_n}$ is precomplete, and its precompleteness can be verified using $(n+1)$-ary operations, but not with n-ary operations.

0.3.2. *Regular ideals.* A better criterion would be one where precompleteness of $\mathscr{C}_I$ can be read off the ideal I directly, without the use of the functions in $\mathscr{C}_I$. We will obtain such a criterion for certain ideals I, on a countably infinite base set X: Define $\hat{I}$ to consist of all subsets A of X such that every infinite subset B of A contains an infinite set in I. We will prove that if $X \notin \hat{I}$, then $\mathscr{C}_I$ is precomplete iff $I = \hat{I}$; so the only ideals left to consider are those which satisfy $X \in \hat{I}$, i.e. those for which $\hat{I}$ equals the power set of X. Utilizing this new criterion, we will once again construct $2^{2^{\aleph_0}}$ precomplete clones on a countably infinite base set, but for the first time in ZF, i.e. explicitly and without the use of the Axiom of Choice.

0.3.3. *A unary criterion.* In [GPb, Problem D], it was asked whether there is a precompleteness criterion for $\mathscr{C}_I$ using unary operations only. The authors justified this question by observing that every clone $\mathscr{C}_I$ is actually determined by its unary operations: For a transformation monoid $\mathscr{G} \subseteq \mathscr{O}^{(1)}$, define $\mathrm{Pol}(\mathscr{G}) \subseteq \mathscr{O}$ to consist of those operations f satisfying $f(g_1, \ldots, g_n) \in \mathscr{G}$ for all $g_1, \ldots, g_n \in \mathscr{G}$. Then we have $\mathscr{C}_I = \mathrm{Pol}(\mathscr{C}_I^{(1)})$. We will provide a criterion using only unary functions in this paper.

0.3.4. *Extension to precomplete ideal clones.* In [CH01, Problem 1], the authors asked whether every ideal clone could be extended to a precomplete ideal clone. We will provide a positive answer to this question, and moreover show that every precomplete clone extending an ideal clone is itself an ideal clone. It follows that an ideal clone is precomplete iff it is not properly contained in a precomplete ideal clone.

0.4. **The mutual position of ideal clones.** Our investigations will not exclusively treat the precompleteness of ideal clones, but also their mutual position.

0.4.1. *Possible inclusions.* It will turn out that whenever I, J are ideals, then $\mathscr{C}_I \subseteq \mathscr{C}_J$ implies $I \subseteq J \subseteq \hat{I}$. However, we will also see that this implication cannot be reversed in general. Moreover, we will find ideal clones $\mathscr{C}_I, \mathscr{C}_J$ whose unary fragments are comparable, i.e., $\mathscr{C}_I \cap \mathscr{O}^{(1)} \subseteq \mathscr{C}_J \cap \mathscr{O}^{(1)}$, but which are incomparable, i.e., $\mathscr{C}_I \nsubseteq \mathscr{C}_J$.

0.4.2. *Chains.* We will prove that every ideal clone contains a maximal subclone which is also an ideal clone. This implies that there exist infinite descending chains of ideal clones. Using a theorem from combinatorial set-theory, we will also show that there exist infinite ascending chains of such clones.

0.5. **Overview.** Our paper is organized as follows: This introduction is followed by a section on the mutual position of ideal clones, introducing the important concept of $\hat{I}$ for an ideal I (Section 1). Precompleteness of ideal clones and ways of testing it are the topics of Sections 2 and 3. In Section 4, we move on to the study of an alternative way of defining clones by means of an ideal on X, and how this new concept relates to the old one. All this is done on countably infinite X, but we then discuss the possibility of a generalization to uncountable base sets in Section 5. Several problems that we had to leave open are listed in Section 6.

0.6. **Formal definition.** With the agenda at hand, we now officially introduce the main objects of our interest.

Definition 1. If I is an ideal on X we denote

$$\mathscr{C}_I := \bigcup_{n=1}^{\infty} \{f \in \mathscr{O}^{(n)} : f[A^n] \in I \text{ for all } A \in I\}$$

and speak of the *clone induced by* I. Clones of this form will be called *ideal clones.*

0.7. **Support.** Let I be an ideal, and call the set of those elements of X which are contained in some set of I the *support* of I:

$$\operatorname{supp}(I) = \bigcup I = \bigcup_{A \in I} A.$$

For a subset $Y \subseteq X$, write $\operatorname{Pol}(Y)$ for the set of all functions $f \in \mathscr{O}$ which satisfy $f(a_1, \ldots, a_n) \in Y$ for all $a_1, \ldots, a_n \in Y$. It is well-known and easy to see that $\operatorname{Pol}(Y)$ is always a precomplete clone in $\operatorname{Cl}(X)$. Also, if we set $\operatorname{Proj}(Y)$ to consist of all operations which behave like projections on Y, then the interval $[\operatorname{Proj}(Y), \operatorname{Pol}(Y)] \subseteq \operatorname{Cl}(X)$ is isomorphic to $\operatorname{Cl}(Y)$ via the mapping σ which sends every clone in the interval to the set of restrictions of its operations to Y. Now setting $Y := \mathrm{supp}(I)$, it is clear that $\mathscr{C}_I \in [\mathrm{Proj}(Y), \mathrm{Pol}(Y)]$, and that moreover σ maps $\mathscr{C}_I$ to $\mathscr{C}_J$, where J is the restriction of I to Y. Therefore, $\mathscr{C}_I$ can be imagined as an ideal clone on Y, and consequently it is enough to understand clones induced by ideals which have *full support*, i.e., which satisfy $\mathrm{supp}(I) = X$. This will be a permanent assumption from now on. Also, if I contains all subsets of X, or only the finite subsets of X, then $\mathscr{C}_I = \mathscr{O}$; therefore, we consider only ideals I satisfying $\mathscr{P}_{\mathrm{fin}}(X) \subsetneq I \subsetneq \mathscr{P}(X)$, where $\mathscr{P}(X)$ denotes the power set of X and $\mathscr{P}_{fin}(X)$ the set of all finite subsets of X.

0.8. **Notation.** For a set of operations $\mathscr{F} \subseteq \mathscr{O}$, we denote the smallest clone containing $\mathscr{F}$ by $\langle \mathscr{F} \rangle$. Given a clone $\mathscr{C}$ and a function f we let $\mathscr{C}(f)$ abbreviate $\langle \mathscr{C} \cup \{f\} \rangle$. If $n \geq 1$, then $\mathscr{F}^{(n)} := \mathscr{F} \cap \mathscr{O}^{(n)}$ is the set of all n-ary operations in $\mathscr{F}$. Given an operation $f \in \mathscr{O}$ and a subset $A \subseteq X$, we will often write $f[A^n]$ for the image of the appropriate power of A under f, thereby implicitly assigning the symbol n to the arity of f. Rather than writing π_1^1, we will use the symbol *id* for the (unary) identity operation on X.

1. The mutual position of ideal clones

In this section we study order-theoretic properties of the mapping $I \mapsto \mathscr{C}_I$, and in particular examine the question of when $\mathscr{C}_I \subseteq \mathscr{C}_J$ holds for two distinct ideals I, J.

We emphasize once again that throughout this paper, except for the special section on uncountable base sets, the base set X is countably infinite. Moreover, all ideals are assumed to contain all finite sets, at least one infinite set, and do not contain all subsets of X. To stress these otherwise tacit assumptions we sometimes speak of 'ideals in our sense'.

Under these restrictions different ideals give rise to different clones. This follows from

Proposition 2. *For any ideals I and J, $\mathscr{C}_I^{(1)} \subseteq \mathscr{C}_J^{(1)}$ implies $I \subseteq J$.*

Proof. Assume $\mathscr{C}_I^{(1)} \subseteq \mathscr{C}_J^{(1)}$. In order to prove $I \subseteq J$, fix an infinite set $B \in J$ and let $A \in I$ be arbitrary. To show $A \in J$ choose a function $f : X \to A$ such that $f[B] = A$. As the range of f belongs to I, we have $f \in \mathscr{C}_I^{(1)}$. The assumption then yields $f \in \mathscr{C}_J^{(1)}$, which in turn forces $A = f[B]$ into J. □

Next we show that $\mathscr{C}_I^{(1)} \subseteq \mathscr{C}_J^{(1)}$ is possible for distinct ideals, but only if $J \supseteq I$ is rather close to I. To make this precise we need the notion of a *regular* ideal:

Definition 3. Given an ideal I, the *regularization* of I is defined by

$$\hat{I} := \{A \subseteq X : \text{ each infinite } B \subseteq A \text{ contains an infinite set } C \in I\}.$$

I is called

- *dense* (or *tall*) iff $\hat{I} = \mathscr{P}(X)$, and
- *regular* (or *nowhere tall*) iff $I = \hat{I}$.

Remark 4. $\hat{I}$ can be written as $(I^\perp)^\perp$, where

$$I^\perp = \{A \subseteq X : \forall B \in I \; |A \cap B| < \aleph_0\}.$$

The initiated reader might notice that under Stone duality,

- ideals (in our sense) correspond to open sets in $\beta\omega \setminus \omega$,
- $\cdot^\perp$ gives the interior of the complement,
- $I \mapsto \hat{I}$ corresponds to $U \mapsto \bar{U}^\circ$,
- regular ideals just correspond to regular open sets,
- and similarly dense ideals to (topologically) dense open sets.

Topological considerations will play no role in what follows but were of heuristic use in understanding the relationship between I and $\hat{I}$.

The following facts are easy to verify and will be used without further reference.

Observations 5. For all ideals I we have $I \subseteq \hat{I}$. If I is not dense, then $\hat{I}$ is again an ideal (in our restricted sense) and turns out to be regular, i.e. $\hat{\hat{I}} = \hat{I}$. If $I \subseteq J \subseteq \hat{I}$, then $\hat{J} = \hat{I}$. Non-principal prime ideals are dense. The intersection of two dense ideals is dense.

A little harder to prove is

Proposition 6. *All countably generated ideals are regular.*

Proof. Let I be a countably generated ideal. As $I \subseteq \hat{I}$ is trivial, we have to prove $\hat{I} \subseteq I$. To do so, we consider an arbitrary $A \notin I$ and find an infinite subset B with no infinite I-subset.

Let $(G_n)_{n=1}^\infty$ be an enumeration of some set of generators of I. $A \notin I$ means that $A \setminus \bigcup_{k=1}^n G_k$ is infinite for all n. So we can recursively build a sequence $(b_n)_{n=1}^\infty$ such that

$$b_{n+1} \in A \setminus \left(\bigcup_{k=1}^{n} G_n \cup \{b_1, \dots, b_n\} \right).$$

Then $B := \{b_n : n = 1, 2, \dots\}$ is the desired infinite subset of A, because by construction, no infinite subset of B is covered by finitely many Gs. $\square$

Proposition 7. *If* $\mathscr{C}_I^{(1)} \subseteq \mathscr{C}_J^{(1)}$, *then* $J \subseteq \hat{I}$. *If I is not dense, then* $\mathscr{C}_I \subseteq \mathscr{C}_{\hat{I}}$.

If I is dense, then $\hat{I} = \mathscr{P}(X)$ and the last implication is also true, except that we should not write $\mathscr{C}_{\hat{I}}$ in that case.

Proof. Assume $\mathscr{C}_I^{(1)} \subseteq \mathscr{C}_J^{(1)}$. To see that $J \subseteq \hat{I}$, let B in J be arbitrary. We must show that below each infinite $C \subseteq B$ there is an infinite I-set. Assume not and let C witness to this. Then any function f mapping C onto X and being constant on $X \setminus C$ would belong to $\mathscr{C}_I$, but not to $\mathscr{C}_J$ (as $f[B] = X \notin J$), a contradiction. Thus, there is no bad C and $B \in \hat{I}$, as demanded.

To prove that $\mathscr{C}_I \subseteq \mathscr{C}_{\hat{I}}$, let $f \in \mathscr{C}_I$ and $A \in \hat{I}$. Our aim is to show that $f[A^n] \in \hat{I}$. Given an infinite subset B of $f[A^n]$, pick $D \subseteq A^n$ such that $f[D] = B$ and such that f is injective on D. If $\pi_1^n[D] \in I$, then set $A_1 := \pi_1^n[D]$ and $D_1 := D$. If $\pi_1^n[D] \in \hat{I} \setminus I$, then pick an infinite I-set $A_1 \subseteq \pi_1^n[D]$ and let $D_1 := D \cap (A_1 \times X^{n-1})$. In the second step thin out D_1 to an infinite set D_2 whose projection on the second coordinate lies in I. After n such steps we arrive at an infinite set D_n which is contained in C^n for some $C \in I$. Thus $f[D_n]$ is an infinite I-set contained in B. Since B was arbitrary, we conclude $f[A^n] \in \hat{I}$. □

Example 8. Ideals I, J showing that the implication $\mathscr{C}_I^{(1)} \subseteq \mathscr{C}_J^{(1)} \Rightarrow I \subseteq J \subseteq \hat{I}$ cannot be reversed.

Let $X = \omega \times \omega$,

$$I := \{A \subseteq X : \exists\, c \in \omega\ \forall\, n \in \omega\ (|A \cap (\omega \times \{n\})| \leq c)\}$$

and

$$J := \{A \subseteq X : \exists\, c, d \in \omega\ \forall\, n \in \omega\ \ (|A \cap (\omega \times \{n\})| \leq c + d \cdot n)\}.$$

Then

$$\hat{I} = \hat{J} = \{A \subseteq X : \forall\, n \in \omega\ \ (A \cap (\omega \times \{n\}) \text{ is finite})\}.$$

Clearly $I \subseteq J \subseteq \hat{I}$. For every 1-1 map $f : \omega \to \omega$ the map $F : X \to X$, $(k, n) \mapsto (k, f(n))$ preserves I, but it might not preserve J. Take for example

$$f(n) = \begin{cases} 2m & \text{if } n = 2^m \\ 2n+1 & \text{else.} \end{cases}$$

The ideal I of this example was introduced in [CH01] in order to show that there are non-precomplete ideal clones. As we shall see later in Proposition 17, $\mathscr{C}_{\hat{I}}$ is the unique precomplete clone above $\mathscr{C}_I$. It is fairly easy to see that there are no other ideal clones between $\mathscr{C}_I$ and $\mathscr{C}_{\hat{I}}$. In fact, as was shown in [GPa], there is no clone at all between $\mathscr{C}_I$ and $\mathscr{C}_{\hat{I}}$, i.e. $(\mathscr{C}_I, \mathscr{C}_{\hat{I}})$ is a covering in the clone lattice.

Proposition 9. *Every ideal clone contains a maximal proper subclone which is itself an ideal clone.*

It follows that there are dense ideals whose clones are not precomplete. For, if I is dense and $\mathscr{C}_J$ a proper subclone of $\mathscr{C}_I$, then $X \in \hat{I} \subseteq \hat{J}$, by Proposition 7. So J is dense, but $\mathscr{C}_J$ cannot be precomplete.

This proposition will follow from Lemmas 10 and 11. We will need the following notation: For any ideal P on X and any function $f : X \to Y$ we define an ideal $\tilde{f}(P)$ on Y via $\tilde{f}(P) = \{B \subseteq Y : f^{-1}[B] \in P\}$. If P is prime, then $\tilde{f}(P)$ is easily seen to be a prime ideal again. The quasiorder $Q \leq P \Leftrightarrow \exists f : Q = \tilde{f}(P)$ is called the RUDIN-KEISLER-ordering on the prime ideals on X. It is, usually in the language of ultrafilters, extensively studied in the literature (see e.g. [CN74]).

Lemma 10. *Let $\mathcal{P}$ be a set of non-principal prime ideals, and Q a non-principal prime ideal such that $\tilde{f}(P) \neq Q$ for all $P \in \mathcal{P}$ and all $f \in \mathscr{O}^{(1)}$. Denoting $I := \bigcap \mathcal{P}$, we have $\mathscr{C}_{I \cap Q} \subseteq \mathscr{C}_I$.*

Proof. Let $f \in \mathscr{C}_{I \cap Q}$ be n-ary, $A \in I$. We derive a contradiction from the assumption $B := f[A^n] \notin I$. So assume that $B \notin P$, for some $P \in \mathcal{P}$. Find a tuple $\bar{g} = (g_1, \ldots, g_n)$ of unary functions such that $b = f(\bar{g}(b))$ for all $b \in B$, and $g_i[X] \subseteq A$ for $i = 1, \ldots, n$. From $\tilde{g}_i(P) \neq Q$ we get $C_i \in Q$ such that $g_i^{-1}[C_i] \notin P$. Let $D := B \cap g_1^{-1}[C_1] \cap \cdots \cap g_n^{-1}[C_n]$, then $D \notin P$ (as P is a prime ideal).

On the other hand we have

$$g_i[D] \subseteq g_i[g_i^{-1}[C_i]] \subseteq C_i \in Q \text{ and } g_i[D] \subseteq g_i[X] \subseteq A \in I.$$

So $g_i[D] \in I \cap Q$, hence

$$D = \{f(\bar{g}(d)) : d \in D\} \subseteq f[g_1[D] \times \cdots \times g_n[D]] \in I \cap Q,$$

because f preserves $I \cap Q$. But then $D \in P$, contradiction. □

Lemma 11. *Let $I \nsubseteq Q$ be ideals, Q prime.*
If $f \in \mathscr{C}_I \setminus \mathscr{C}_{I \cap Q}$, then $\mathscr{C}_I \subseteq \mathscr{C}_{I \cap Q}(f)$.

Proof. Assume first that the given $f \in \mathscr{C}_I \setminus \mathscr{C}_{I \cap Q}$ is unary. Fix $A \in I \cap Q$ such that $B := f[A] \notin I \cap Q$. As $f \in \mathscr{C}_I$, we must have $B \in I$, so $B \notin Q$, hence $X \setminus B \in Q$. Note that B and A must be infinite. By shrinking A we may additionally assume

(1) $f \restriction A$ is a bijection from A onto B.
(2) $A \cap B = \emptyset$
(3) There is some infinite $C \in I$ disjoint from $A \cup B$.
    As $C \subseteq X \setminus B$, we have $C \in Q$.
(4) Summarizing: A, B, C are disjoint, $A, C \in I \cap Q$, $B \in I \setminus Q$.

Now let $g : X \to X$ be constant outside C and map C bijectively onto $A \cup C$. Clearly $g \in \mathscr{C}_{I \cap Q}$. The function

$$s(x, y, z) := \begin{cases} y & \text{if } x \in A \\ z & \text{if } x \in C \\ x & \text{otherwise} \end{cases}$$

belongs to every ideal clone. It follows that $p(x) := s(x, f(x), g(x))$ belongs to $\mathscr{C}_{I \cap Q}(f)$. Notice that p maps $X \setminus B$ bijectively onto X; let q denote its inverse. q is the identity outside $A \cup B \cup C$ and maps $A \cup B \cup C$ to $A \cup C \in I$. Hence $q \in \mathscr{C}_I$.

We now show that for arbitrary $g \in \mathscr{C}_I$ we have $q \circ g \in \mathscr{C}_{I \cap Q}$: Let $D \in I \cap Q$. As both q and g are in $\mathscr{C}_I$, we have $q \circ g[D^m] \in I$. But we also have $q \circ g[D^m] \subseteq q[X] \subseteq X \setminus B \in Q$.

Hence we have for all $g \in \mathscr{C}_I$: $g = p \circ (q \circ g) \in \mathscr{C}_{I \cap Q}(f)$, so $\mathscr{C}_I \subseteq \mathscr{C}_{I \cap Q}(f)$, as desired.

Now consider the case that the given $f \in \mathscr{C}_I \setminus \mathscr{C}_{I \cap Q}$ is n-ary for some $n > 1$. We show that f can be replaced by a unary function f'. For some $A \in I \cap Q$ we must have $f[A^n] \notin I \cap Q$. Choose $g_1, \ldots, g_n : X \to A$ such that $\{(g_1(a), \ldots, g_n(a)) : a \in A\} = A^n$. Having range A, all g_i belong to $\mathscr{C}_I$ and to $\mathscr{C}_{I \cap Q}$. It follows that the unary function $f' = f(g_1, \ldots, g_n)$ belongs to $\mathscr{C}_I$ and to $\mathscr{C}_{I \cap Q}(f)$ but not to $\mathscr{C}_{I \cap Q}$ because $f'[A] = f[A^n] \notin I \cap Q$. From the first case it follows that $\mathscr{C}_I \subseteq \mathscr{C}_{I \cap Q}(f') \subseteq \mathscr{C}_{I \cap Q}(f)$. $\square$

Proof of Proposition 9 from the lemmas. Let I be the given ideal. It is well known that I is an intersection of prime ideals. Therefore, we can choose for each $A \notin I$ some prime ideal P_A such that $I \subseteq P_A \not\ni A$. Then $I = \bigcap_{A \notin I} P_A$ and the set $\mathcal{P} := \{P_A : A \notin I\}$ has power at most $2^{\aleph_0}$. As there are only $2^{\aleph_0}$ functions $X \to X$, the set $\{\tilde{g}(P) : P \in \mathcal{P} \text{ and } g \in \mathscr{O}^{(1)}\}$ has power $2^{\aleph_0}$, too.

As I is an ideal ‘in our sense’, there exists some infinite $B \in I$ such that $C := X \setminus B$ is also infinite. Again it is well-known that there are $2^{2^{\aleph_0}}$ non-principal prime ideals containing C. So letting Q be one of them, which is not of the form $\tilde{g}(P)$ for any $P \in \mathcal{P}$, we have $I \not\subseteq Q$ (because of B) and the two lemmas now show that $J := I \cap Q$ is as desired. $\square$

Corollary 12. *Below each ideal clone there is an infinite descending chain of ideal clones. There are also infinite ascending chains of ideal clones.*

We do not know what other types of chains of ideal clones exist.

To get the descending chain, start with an arbitrary ideal clone and successively apply Proposition 9.

To get the ascending chain is much harder because a deeper result of combinatorial set-theory is needed:

Theorem 13 (Kunen [Kun72], see also [CN74, Theorem 10.4])**.** *There are $2^{\aleph_0}$ many maximal ideals on $\mathbb{N}$ which are pairwise incomparable in the* Rudin-Keisler-*order.*

We only need countably many incomparable prime ideals. So let $(P_k)_{k=1}^{\infty}$ be a sequence of prime ideals such that for any $g \in \mathscr{O}^{(1)}$ and any $m \neq n$ we have $\tilde{g}(P_m) \neq P_n$.

Now we may simply put $I_n := \bigcap_{k \geq n} P_k$. Then $I_n = I_{n+1} \cap P_n$ and Lemma 10 says $\mathscr{C}_{I_n} \subsetneq \mathscr{C}_{I_{n+1}}$.

2. Precompleteness and regularity

Problem 1 of [CH01] asks whether every ideal clone lies below a precomplete ideal clone. The following theorem gives a positive answer to this question and shows that an even stronger statement holds.

Theorem 14. *Every ideal clone is contained in a precomplete clone. Moreover, every precomplete clone containing an ideal clone is an ideal clone itself.*

To prove Theorem 14 we employ the following lemmas which will also be useful later on.

Lemma 15. *Let I be an ideal and $f \in \mathscr{O}$. If $f[B^n] = X$ for some $B \in I$, then $\mathscr{C}_I(f) = \mathscr{O}$.*

Proof. Choose $g_1, \dots, g_n : X \to B$ such that $(g_1, \dots, g_n) : X \to X^n$ maps B onto B^n. Then the function $h := f(g_1, \dots, g_n)$ maps B onto X. Pick $C \subseteq B$ such that h maps C bijectively onto X and fix functions $h_k \in \mathscr{O}^{(k)}$ mapping X^k bijectively onto C. Let g be any k-ary operation on X. Its action on $\bar{x}$ can be channeled through C as follows: For every $\bar{x}$, there exists precisely one $d \in C$ such that $h(d) = g(\bar{x})$. Moreover, there is precisely one $c \in C$ such that $h_k(\bar{x}) = c$. We define a unary operation $\tilde{g} : C \to C$ by setting $\tilde{g}(c) = d$ for all $c, d \in C$ obtained that way. Now extend $\tilde{g}$ anyhow to a unary operation on X, but in such a way that its range is still contained in C. We have: $\bar{x} \mapsto h_k(\bar{x}) =: c \mapsto \tilde{g}(c) := d \mapsto h(d) = g(\bar{x})$. From $g(\bar{x}) = h(\tilde{g}(h_k(\bar{x})))$ we conclude that g is a composition of f and $g_1, \dots, g_n, h_k, \tilde{g}$. As the latter functions all have ranges in I, namely B or C, they automatically belong to $\mathscr{C}_I$. So $g \in \mathscr{C}_I(f)$. As g was arbitrary, $\mathscr{C}_I(f) = \mathscr{O}$, as claimed. ☐

Lemma 16. *Let $A \subseteq X$ be an infinite set and $\mathscr{A}$ a clone containing an ideal clone. Set $J := \{B \subseteq f[A^n] : f \in \mathscr{A}\}$. Then either $X \in J$, or: J is an ideal (in our sense), $A \in J$, and $\mathscr{A} \subseteq \mathscr{C}_J$.*

Proof. We start by showing that J is an ideal: By definition J is closed under the formation of subsets. To see that J is closed under finite unions, consider the switching function

$$s(x, y, u, v) = \begin{cases} x, & u = v \\ y, & u \neq v. \end{cases}$$

One easily checks that every ideal clone contains s and thus $s \in \mathscr{A}$. Assume that $B, C \in J$, say $B \subseteq f[A^m], C \subseteq g[A^n]$ for some $f, g \in \mathscr{A}$. Then $B \cup C \subseteq f[A^m] \cup g[A^n] = s[f[A^m] \times g[A^n] \times A \times A]$. Therefore, $B \cup C \in J$ as the rightmost set is the image of A under the operation $s(f(x_1, \dots, x_m), g(y_1, \dots, y_n), u, v) \in \mathscr{A}$. Since every ideal clone contains all functions with finite image, so does $\mathscr{A}$. Therefore J contains all finite sets. As $A = \mathrm{id}[A] \in J$ there is an infinite set in J. So J is an ideal unless $X \in J$ (in which case $J = \mathscr{P}(X)$).

Finally we have to prove that $\mathscr{A} \subseteq \mathscr{C}_J$. Consider any $f \in \mathscr{A}^{(n)}$ and $B \in J$. Choose $g \in \mathscr{A}$ such that $B \subseteq g[A^m]$. Then $f[B^n] \subseteq f[g[A^m] \times \ldots \times g[A^m]] \in J$ since the latter set is the image of A under the operation $f(g(x_1^1, \ldots, x_m^1), \ldots, g(x_1^n, \ldots, x_m^n)) \in \mathscr{A}$. □

Proof of Theorem 14. Pick an infinite $A \in I$ and choose a function $g \in \mathscr{O}^{(1)}$ which maps A onto X and is constant on $X \setminus A$. By Lemma 15, $\mathscr{O} = \mathscr{C}_I(g)$. Consider the set S of all clones above $\mathscr{C}_I$ which do not contain g. Zorn's Lemma yields the existence of a maximal element $\mathscr{A}$ in S. Every clone which is strictly larger than $\mathscr{A}$ contains g and is thus equal to $\mathscr{O}$. Hence $\mathscr{A}$ is precomplete.

It remains to show that each precomplete $\mathscr{A}$ above $\mathscr{C}_I$ is an ideal clone. Set $J = \{B \subseteq f[A^n] : f \in \mathscr{A}\}$.

X cannot be in J, for, otherwise $X = f[A^n]$ for some $f \in \mathscr{A}$. By Lemma 15 this implies $\mathscr{O} = \mathscr{C}_I(f) \subseteq \mathscr{A}$ which is impossible since $\mathscr{A}$ is a proper subclone of $\mathscr{O}$. So, by Lemma 16, J is an ideal and $\mathscr{A} \subseteq \mathscr{C}_J \neq \mathscr{O}$ Since $\mathscr{A}$ is precomplete, we must have $\mathscr{A} = \mathscr{C}_J$. □

The following proposition reveals the significance of regular ideals for our purposes.

Proposition 17. *Assume that I is not dense. Then $\mathscr{C}_{\hat{I}}$ is precomplete and in fact the only precomplete clone above $\mathscr{C}_I$. So $\mathscr{C}_I$ is precomplete iff I is regular.*

In the case of non-dense ideals we therefore have a very satisfactory criterion: the precompleteness of $\mathscr{C}_I$ can be read off the ideal I directly without even looking at the operations in $\mathscr{C}_I$. If I is dense, then no general statement can be made: Prime ideals are obviously dense and give rise to precomplete clones (Example 21). Proposition 9 and the remark following it show that there are many dense ideals which lead to a non-precomplete clone.

Notice that the precompleteness of $\mathscr{C}_{\hat{I}}$ is proved in ZF, i.e. without the use of the Axiom of Choice.

Proof. From $X \notin \hat{I}$ we infer that $\hat{I}$ is an ideal (in our sense) to which the former machinery applies. We start by proving the precompleteness of $\mathscr{C}_{\hat{I}}$. Consider any $f \in \mathscr{O}^{(n)} \setminus \mathscr{C}_{\hat{I}}$. Pick $B \in \hat{I}$ such that $f[B^n]$ contains an infinite set C which contains no infinite I-set. Choose any $g \in \mathscr{O}^{(1)}$ such that $g[C] = X$ and such that g is constant on $X \setminus C$. Then $g \in \mathscr{C}_I \subseteq \mathscr{C}_{\hat{I}}$ (as images of I-sets are finite) and $g \circ f$ maps B^n onto X. Thus $\mathscr{C}_{\hat{I}}(f) \supseteq \mathscr{C}_{\hat{I}}(g \circ f) = \mathscr{O}$, by Lemma 15.

Next we consider any precomplete $\mathscr{A} \supseteq \mathscr{C}_I$ and show that $\mathscr{A} = \mathscr{C}_{\hat{I}}$. From Theorem 14 we know that $\mathscr{A} = \mathscr{C}_J$ for some ideal J. But then $I \subseteq J \subseteq \hat{I}$, by Propositions 2 and 7. Thus, $\hat{J} = \hat{I}$ (confer Observations 5). So, $\mathscr{A} = \mathscr{C}_J \subseteq \mathscr{C}_{\hat{J}} = \mathscr{C}_{\hat{I}}$. By maximality, $\mathscr{A} = \mathscr{C}_{\hat{I}}$, as desired. □

It was already mentioned that there are $2^{2^{\aleph_0}}$ precomplete clones on a countable base set. The hitherto known constructions of that many clones all used the Axiom of Choice in one way or the other. Next we set out to produce the maximal number of precomplete clones without using AC.

Theorem 18 (ZF)**.** *There exists an injective mapping from 2^{2^ω} into the set of precomplete clones over a countable set.*

From the above, it is clear that we only have to construct many regular ideals.

We begin with the following

Lemma 19. *Let $\mathscr{B}$ be a collection of subsets of X and denote by $I_{\mathscr{B}}$ the ideal generated by $\mathscr{B}$ and all finite sets. If A is infinite and $A \in \hat{I}_{\mathscr{B}}$, then A has infinite intersection with some $B \in \mathscr{B}$.*

Proof. By definition, there is some infinite $C \subseteq A$, such that $C \in I_{\mathscr{B}}$, i.e. $C \subseteq F \cup B_1 \cup \cdots \cup B_k$ for some finite set F and finitely many $B_1, \ldots, B_k \in \mathscr{B}$. As C is infinite, $C \cap B_i$ must be infinite for one B_i. Then $A \cap B_i$ is all the more infinite. □

For the actual construction we consider an infinite family $\mathscr{A}$ of what are called almost disjoint subsets of X, i.e., the sets in $\mathscr{A}$ are infinite but have finite pairwise intersections. If $\mathscr{B}$ is a subcollection of $\mathscr{A}$, the above lemma shows $\hat{I}_{\mathscr{B}} \cap \mathscr{A} = \mathscr{B}$. It follows that different $\mathscr{B}$s give rise to different regular ideals, hence to different precomplete clones (in fact for $\mathscr{B} = \emptyset$ or $\mathscr{B} = \mathscr{A}$ we may not get ideals in our sense). In other words, from $\mathscr{A}$ we have constructed $2^{|\mathscr{A}|}$ distinct precomplete clones.

We are left with constructing an almost disjoint family of power $2^{\aleph_0}$ without using the Axiom of Choice. There are a number of such constructions in the textbooks of set-theory. The most popular one (due to Sierpiński) is based on $X := 2^{<\omega}$, the countable set of all finite 0-1-sequences. If ξ is an infinite 0-1-sequence, let A_ξ be the set of its initial segments. Then $\{A_\xi : \xi \in 2^\omega\}$ is an almost disjoint family. Clearly, each infinite sequence has infinitely many initial segments. So each A_ξ is infinite. Moreover, if $\xi \neq \eta$ then $\xi(n) \neq \eta(n)$ for some (first) n. Then A_ξ and A_η have no initial segments of length $\geq n$ in common. Hence $A_\xi \cap A_\eta$ is finite.

3. The complexity of testing precompleteness

The following precompleteness test was already proved in [CH01]:

> *$\mathscr{C}_I$ is precomplete iff for all $A \notin I$ there exists $f \in \mathscr{C}_I$ such that $f[A^n] = X$.*

With the tools from the previous section the proof becomes very short. If $\mathscr{C}_I$ is not precomplete, then $\mathscr{C}_I \subseteq \mathscr{C}_J$ for some ideal $J \supsetneq I$. Taking $A \in J \setminus I$ and any $f \in \mathscr{C}_I$ we cannot have $X = f[A^n]$, for, f would belong to $\mathscr{C}_J$, too, and $A \in J$ implies $f[A^n] \in J \not\ni X$.

In the other direction we let $A \notin I$ be as stated. Applying Lemma 16 to $\mathscr{C}_I$ and A we get $J \supseteq I$. The assumption on A just says $X \notin J$. So J is an ideal and $\mathscr{C}_J \supsetneq \mathscr{C}_I$, disproving maximality of $\mathscr{C}_I$.

Applications of the test can be simplified by the observation that instead of $f[A^n] = X$ it is sufficient to demand that $f[A^n]$ be big, i.e. has a complement in I, or, more formally, belong to $F := \{X \setminus B : B \in I\}$, the dual filter of I. This is justified by the following

Lemma 20. *Each set in F can be mapped onto the whole of X by a unary $\mathscr{C}_I$-function.*

Proof. Denote the set in question by B. Belonging to F, B must be infinite. If B does not contain any infinite subset belonging to I, then any function mapping B onto X and constant on $X \setminus B$ will do, because it is automatically in $\mathscr{C}_I$. Otherwise, there is an infinite $C \subseteq B$ such that $C \in I$. Let us split it: $C = C_1 \cup C_2$ with both parts infinite. Then any function f such that $f[C_1] = C$, $f[C_2] = X \setminus B$, and $f(x) = x$ on $X \setminus C$ maps $B = (B \setminus C) \cup C_1 \cup C_2$ to $(B \setminus C) \cup C \cup (X \setminus B) = X$. Any such f also belongs to $\mathscr{C}_I$, because from $D \in I$ we have

$$f[D] = f[C_1 \cap D] \cup f[C_2 \cap D] \cup f[D \setminus C] \subseteq C \cup (X \setminus B) \cup (D \setminus C) \in I.$$

□

Example 21. Taking this simplification into account, precompleteness for $\mathscr{C}_I$ with I prime becomes a triviality: If $A \notin I$, then $\mathrm{id}[A]$ is in the dual (ultra)filter.

There is another way to liberalize the condition in the test. Instead of one $\mathscr{C}_I$-function one can allow a finite (and with some care even an infinite) number of functions. In the rest of this section we shall be mainly concerned with the question of how many functions of which arities are needed to test the precompleteness of a given $\mathscr{C}_I$. It will turn out that binary functions do not suffice, as some of us hoped to prove before we knew the counterexample. To make things more precise, we introduce some notation. For $A \subseteq X, n \geq 1$ and $1 \leq p \leq \infty$ we write $T(A, n, p)$ iff there is a sequence $(f_k)_{k=1}^p$ of n-ary functions such that

$$(i) \bigcup_{k=1}^{p} f_k[A^n] = X \qquad \text{and} \qquad (ii) \bigcup_{k=1}^{p} f_k[B^n] \in I \text{ for all } B \in I.$$

Obviously, $T(A, n, p)$ also depends on I. But the ideal will usually be fixed in the context. Therefore, we can safely suppress it in the notation.

If p is finite, then (ii) just says that all f_k belong to $\mathscr{C}_I$. By the above lemma, condition (i) can be replaced by $\bigcup_{k=1}^{p} f_k[A^n] \in F$.

We now let p enter the game and call the result

Theorem 22. *$\mathscr{C}_I$ is precomplete iff for each $A \notin I$ there are $n \geq 1$ and $1 \leq p \leq \infty$ such that $T(A, n, p)$ holds.*

Remark 23. This criterion strongly depends on our assumption that the base set is countable. Example 36 in Section 5 shows that this criterion fails on all uncountable sets.

In all examples we know, a stronger form holds: there are n and p such that $T(A, n, p)$ for all $A \notin I$. It is an open problem if that is always the case. If it is, we say in a somewhat sloppy way, that $\mathscr{C}_I$ is precomplete via p n-ary functions.

Proof of Theorem 22. Using the new notation the precompleteness test mentioned at the beginning of this section reads: $\mathscr{C}_I$ is precomplete iff for all $A \notin I$ there is some n such that $T(A, n, 1)$ holds. The rest follows from the observation that for all infinite A, $n \geq 1$ and $1 < p < q < \infty$

$$T(A, n, 1) \Rightarrow T(A, n, p) \Rightarrow T(A, n, q) \Rightarrow T(A, n, \infty) \Rightarrow T(A, n+1, 1).$$

Below we give examples showing that none of these implication can be reversed.

The others being obvious, only the last implication needs proof. Let $(f_k)_{k=1}^{\infty}$ witness $T(A, n, \infty)$. As A is infinite, we can choose a sequence $(a_k)_{k=1}^{\infty}$ of pairwise distinct elements of A. It allows us to define

$$f(x_1, \dots, x_n, y) := \begin{cases} f_k(x_1, \dots, x_n), & y = a_k \\ y, & \text{otherwise.} \end{cases}$$

An easy verification then shows that this function witnesses $T(A, n+1, 1)$. $\square$

[GPb, Problem D] asks for a precompleteness test using unary functions only. The following proposition gives such a test, which seems however of little practical use. The case $n = 1$ will turn out to be important in the next section, though.

Proposition 24. *$\mathscr{C}_I$ is precomplete iff for each $A \notin I$ there exist $n \geq 1$ and unary functions $g_1, \dots, g_n : X \to A$ such that $g_1^{-1}[B] \cap \dots \cap g_n^{-1}[B] \in I$ for all $B \in I$.*

The proposition will follow from Theorem 22 and the following lemma, by assembling the g_i to a vector function $\bar{g} = (g_1, \dots, g_n)$.

Lemma 25. *If $A \subseteq X$ is infinite, then condition $T(A, n, \infty)$ is equivalent to the existence of a vector function $\bar{g} : X \to A^n$ such that $\bar{g}^{-1}[B^n] \in I$ for all $B \in I$.*

Proof. Let $\bar{g} : X \to A^n$ be as stated. Choose $c \in X$ arbitrarily. Define a sequence $(f_k)_{k=1}^{\infty}$ of n-ary functions in such a way that

$$\{f_k(\bar{y}) : k = 1, 2, \dots\} = \{c\} \cup \{x \in X : \bar{g}(x) = \bar{y}\}$$

for each tuple $\bar{y} \in X^n$. This is possible because the set of the right-hand side is non-empty and countable.

These functions are as demanded by $T(A, n, \infty)$. Indeed, for each $x \in X$, the tuple $\bar{g}(x)$ belongs to A^n and has x in its preimage. So $x = f_k(\bar{g}(x)) \in$

$f_k[A^n]$ for some k. Hence $X \subseteq \bigcup_{k=1}^{\infty} f_k[A^n]$. Moreover, for any $B \in I$ we have $\bigcup_{k=1}^{\infty} f_k[B^n] \subseteq \{c\} \cup \bar{g}^{-1}[B^n] \in I$, as demanded.

In the other direction we start from a sequence $(f_k)_{k=1}^{\infty}$ as in $T(A, n, \infty)$. As $\bigcup_{k=1}^{\infty} f_k[A^n] = X$, for each $x \in X$ there exists some (smallest) k and some $\bar{a} \in A^n$ such that $f_k(\bar{a}) = x$. Let $\bar{g}(x)$ pick such tuple $\bar{a}$. Then $\bar{g}$ is a vector function $X \to A^n$ and for each $B \in I$ we have $\bar{g}^{-1}[B^n] \subseteq \bigcup_{k=1}^{\infty} f_k[B^n] \in I$. □

In the rest of this section we give examples showing that there is no general bound on the number and arity of the functions that are needed to establish precompleteness according to Theorem 22.

Example 26. For every $1 < p \leq \infty$, there exists an ideal I such that $\mathscr{C}_I$ is precomplete via p unary functions but fewer unary functions do not suffice.

Let Y be any countable set and put $X := \bigcup_{k=1}^{p} Y \times \{k\}$. Let P be any prime ideal on Y. Then the ideal I we aim at, consists of all sets $A = \bigcup_{k=1}^{p} A_k \times \{k\} \subseteq X$ for which all A_k belong to P.

First we show that $T(A, 1, p)$ holds for all $A \notin I$. Write $A = \bigcup_{k=1}^{p} A_k \times \{k\}$ and notice that for some k we must have $A_k \notin P$. For notational convenience we assume that $A_1 \notin P$. By Lemma 20 there is a $\mathscr{C}_P$-function $f : Y \to Y$ that maps A_1 to Y (not being in the prime ideal P means being in its dual (ultra)filter).

Using this f we define $f_k : X \to X$ by setting

$$f_k(y, m) = \begin{cases} (f(y), k), & m = 1 \\ (y, m), & m \neq 1 \end{cases}.$$

Then

$$\bigcup_{k=1}^{p} f_k[A] \supseteq \bigcup_{k=1}^{p} f_k[A_1 \times \{1\}] = \bigcup_{k=1}^{p} f[A_1] \times \{k\} = \bigcup_{k=1}^{p} Y \times \{k\} = X$$

and for any $B = \bigcup_{m=1}^{p} B_m \times \{m\} \in I$

$$\bigcup_{k=1}^{p} f_k[B] \subseteq B \cup \bigcup_{k=1}^{p} f_k[B_1 \times \{1\}] = B \cup \bigcup_{k=1}^{p} f[B_1] \times \{k\} \in I.$$

Next we show that $T(Y \times \{1\}, 1, q)$ does not hold for any $q < p$ (notice that q is finite, even if $p = \infty$). Consider functions $g_1, \ldots, g_q : X \to X$ such that $\bigcup_{m=1}^{q} g_m[Y \times \{1\}] = X$. We show that $g_m \notin \mathscr{C}_I$ for at least one of the functions.

For each $m \leq q$ we write $g_m[Y \times \{1\}]$ as $\bigcup_{k=1}^{p} D_{mk} \times \{k\}$. Then for each k we have $\bigcup_{m=1}^{q} D_{mk} = Y$. It follows that for each k there must be some m such that $D_{mk} \notin P$. As there are more ks than ms, one m must serve two ks. In other words: there is some $m \leq q$ such that $D_{mi} \notin P$ and $D_{mj} \notin P$ for some $i \neq j$.

The sets $D_{mi} \times \{i\}$ and $D_{mj} \times \{j\}$ do not belong to I then. As they are disjoint, so are their preimages under g_m and even more so

$$g_m^{-1}[D_{mi} \times \{i\}] \cap (Y \times \{1\}) \quad \text{and} \quad g_m^{-1}[D_{mj} \times \{j\}] \cap (Y \times \{1\})$$

As (the trace of) I is prime on $Y \times \{1\}$ one of these disjoint sets is in I, but none of their g_m images is. So $g_m \notin \mathscr{C}_I$, as claimed.

As a byproduct of the above, we get

Example 27. Ideals I, J such that $\mathscr{C}_I^{(1)} \subseteq \mathscr{C}_J^{(1)}$ but $\mathscr{C}_I \not\subseteq \mathscr{C}_J$.

Let I be the ideal constructed in the previous example for $p = \infty$ and let $B \notin I$ be such that $T(B, 1, q)$ fails for all finite q. Then $\mathscr{C}_I$ is precomplete and $J = \{A \subseteq \bigcup_{i=1}^n f_i[B] : f_1, \ldots, f_n \in \mathscr{C}_I^{(1)}\}$ is an ideal in our sense. The definition of J immediately yields $\mathscr{C}_I^{(1)} \subseteq \mathscr{C}_J^{(1)}$ and, since $B \in J \setminus I$, this inclusion is proper, by Proposition 2. The precompleteness of $\mathscr{C}_I$ shows that we cannot have $\mathscr{C}_I \subseteq \mathscr{C}_J$.

Example 28. For each $n \geq 1$ there is an ideal I such that $\mathscr{C}_I$ is precomplete via one $n+1$-ary function but not via (even infinitely many) n-ary functions.

We let P be a non-principal prime ideal on the countably infinite set Y and put

$$X := Y^{n+1} \quad \text{and} \quad I = P^{n+1} = \{B \subseteq X : B \subseteq C^{n+1} \text{ for some } C \in P\}.$$

Regardless of the choice of P we then have $T(A, n+1, 1)$ for all $A \notin I$. To see this, let $A \notin I$ be given. By definition, there is a projection $p : X = Y^{n+1} \to Y$ on one of the coordinate axes such that $B := p[A]$ does not belong to P. By Lemma 21, we may choose a function $g : Y \to Y$ that maps P-sets to P-sets and B onto the whole of Y. Then

$$f(x_1, \ldots, x_{n+1}) = \Big(g(p(x_1)), \ldots, g(p(x_{n+1}))\Big)$$

belongs to $\mathscr{C}_I$ and maps A^{n+1} to $X = Y^{n+1}$.

We do not know if there are prime ideals P such that the above construction would yield an ideal I such that $\mathscr{C}_I$ was precomplete via infinitely many n-ary functions. By properly choosing P, we show that, at least, this is not always the case.

We first explain what property of P yields the desired result. We let 0 denote any fixed element of Y and consider $A := \{(0, 0, \ldots, 0, y) : y \in Y\}$. Then A has full projection onto the last coordinate, hence $A \notin I$.

If $T(A, n, \infty)$ were true, then Lemma 25 would yield a vector function $\bar{g} = (g_1, \ldots, g_n) : X \to A^n$ such that $\bar{g}^{-1}[B^n] \in I$ for all $B \in I$. Let p denote the projection $Y^{n+1} \to Y$ onto the last coordinate (so $p[A] = Y$) and put $h_i = p(g_i)$. Then $\bar{h} = (h_1, \ldots, h_n)$ maps X to Y^n. Moreover, for each $B \in P$ we have $\tilde{B} := \{(0, 0, \ldots, 0, b) : b \in B\} \in I$ and, therefore,

$$\bar{h}^{-1}[B^n] = \bigcap_{i=1}^n h_i^{-1}[B] = \bigcap_{i=1}^n g_i^{-1}[\tilde{B}] = \bar{g}^{-1}[\tilde{B}^n] \in I.$$

So, in order to guarantee that $T(A, n, \infty)$ fails, we can choose P in such a way that

($*$) for each vector function $\bar{h} : X = Y^{n+1} \to Y^n$ there exists some $C \in P$ such that $\bar{h}^{-1}[C^n] = h_1^{-1}[C] \cap \cdots \cap h_n^{-1}[C]$ does not belong to I (e.g. because this set has full projection onto some coordinate).

Let H denote the set of all $\bar{h} : Y^{n+1} \to Y^n$ to be dealt with. For each $\bar{h} \in H$ we are going to choose a set $C_{\bar{h}}$ such that $\bar{h}^{-1}[C_{\bar{h}}{}^n]$ has full projection onto one coordinate. Some extra care is needed to ensure that all the chosen sets peacefully live together in one prime ideal.

To achieve this, we need large independent families: A family $\mathbb{F}$ of subsets of Y is called *independent* iff for all finite disjoint $\mathbb{F}_1, \mathbb{F}_2 \subseteq \mathbb{F}$

$$\bigcap\{F : F \in \mathbb{F}_1\} \cap \bigcap\{Y \setminus F : F \in \mathbb{F}_2\} \neq \emptyset.$$

The following classical result shows that large independent families exist:

Lemma 29 (Fichtenholz, Kantorovich and Hausdorff, see [Jec02, Lemma 7.7])**.** *There is an independent family $\mathbb{F}$ of subsets of Y which has power $2^{\aleph_0}$.*

A particularly concrete example of such a family was given by Menachem Kojman in [GK99]: on the countable set $\mathbb{Z}[x]$ of polynomials with integer coefficients define, for any real r, the set Z_r as

$$Z_r := \{p(x) \in \mathbb{Z}[x] : p(r) > 0\}.$$

Then the collection of all Z_r with $r \in \mathbb{R}$ is independent.

As $|\mathbb{F}| = |H| = 2^{\aleph_0}$ we can split $\mathbb{F}$ as $\bigcup_{\bar{h} \in H} \mathbb{F}_{\bar{h}}$, where the $\mathbb{F}_{\bar{h}}$ are pairwise disjoint and infinite. If we choose the $C_{\bar{h}}$ as Boolean combinations of sets in $\mathbb{F}_{\bar{h}}$ (and different from X), then no finite union of them covers Y. In other words, $\{C_{\bar{h}} : \bar{h} \in H\}$ extends to a prime ideal.

It remains to explain, how an individual $C_{\bar{h}}$ can be found. So let $\bar{h} = (h_1, \dots, h_n) : Y^{n+1} \to Y^n$ be given and take a decomposition $Y = D_1 \cup D_2 \cdots \cup D_{n+1}$ into mutually disjoint non-empty sets (Boolean combinations from sets in $\mathbb{F}_{\bar{h}}$). We claim that for one index $m \leq n + 1$ the preimage $\bar{h}^{-1}[(Y \setminus D_m)^n] \subseteq Y^{n+1}$ has full projection onto one coordinate. Then the corresponding set $C_m := Y \setminus D_m$ does the required job.

Consider any $x \in X$. By the Pigeonhole Principle, there must be some $i \leq n + 1$ such that D_i contains none of $h_1(x), \dots h_n(x)$, so C_i contains all of them, or, in other words, $\bar{h}(x) \in C_i^n$. This shows

$$Y^{n+1} = X = \bigcup_{i=1}^{n+1} \bar{h}^{-1}[C_i^n].$$

But then one set in the union has full projection. For, if for each i there were some y_i that does not occur as i-th coordinate of an element of $\bar{h}^{-1}[C_i^n]$, then $(y_1, y_2, \dots, y_{n+1})$ would not occur in the whole union.

4. The dual construction: clones from filters

There are other ways to construct clones from ideals. In this section we discuss one approach which is, in a sense, dual to the former one. Functions have been put in $\mathscr{C}_I$ if images of small sets are small. Now we consider those functions for which preimages of small sets are small. Taken literally, this idea would lead to the set

$$\bigcup_{n\geq 1}\{f \in \mathscr{O}^{(n)} : f^{-1}[A] \in I^n \text{ for all } A \in I\},$$

which is not a clone, however (e.g. because it does not include the projections). The generated clone, denote it by $\mathscr{D}_I$, consists of all functions that are essentially in the set, i.e. up to fictitious variables. We did not study this construction in detail because there is another more promising way to make the above idea precise. The first step is to pass to complements: preimages of big sets have to be big. In the context of a given ideal, the small subsets of X are those in I and the big ones those in the dual filter $F := \{X \setminus A : A \in I\}$. If we take subsets of X^n to be big if they belong to F^n (the filter generated by all B^n for $B \in F$), then we arrive at the following set of functions

$$\mathscr{S}_F := \bigcup_{n=1}^{\infty}\{f \in \mathscr{O}^{(n)} : f^{-1}[B] \in F^n \text{ for all } B \in F\}.$$

A straight-forward verification shows $\mathscr{S}_F$ to be a clone. The functions in $\mathscr{S}_F$ will be called F-continuous. An alternative description of F-continuity demands that for all $B \in F$ some $C \in F$ can be found such that $f[C^n] \subseteq B$.

An easy verification shows that $\mathscr{D}_I \subseteq \mathscr{S}_F$ while $\mathscr{D}_I^{(1)} = \mathscr{S}_F^{(1)}$. Later on we have several times occasion to test wether a given unary function belongs to $\mathscr{S}_F^{(1)} = \mathscr{D}_I^{(1)}$. According to which is more convenient we can either check if $A \in I \Rightarrow f^{-1}[A] \in I$ or if $B \in F \Rightarrow f^{-1}[B] \in F$.

From now on we consider the clones $\mathscr{S}_F$ on their own right, not mentioning ideals for a while.

For the two extreme filters $\{X\}$ and $\mathscr{P}(X)$ all operations are continuous. Call a filter *proper* iff it is distinct from the two extreme ones. In contrast to the ideal case (where the ideal of all finite sets was an exception) no proper filter yields the full clone. To see this, denote the filter in question by F and choose $c \notin A \in F$. Then the constant function with value c is not F-continuous, for $c^{-1}[A] = \emptyset \notin F$. Hence, $\mathscr{S}_F^{(1)} \neq \mathscr{O}^{(1)}$.

Different proper filters yield clones with different unary parts. To see this, consider $F \neq G$ and choose (wlog) $B \in F, B \notin G$ and $c \in X$ such that $X \setminus \{c\} \in G$. Define

$$f(x) = \begin{cases} x, & x \in B \\ c, & x \notin B \end{cases}.$$

Then f belongs to $\mathscr{S}_F$, for, $A \in F$ implies $f^{-1}[A] \supseteq A \cap B \in F$. But f is not G-continuous, for $f^{-1}[X \setminus \{c\}] \subseteq B \notin G$.

Some clones $\mathscr{S}_F$ come with a handicap that prevents them outright from being maximal: they are not maximal in their *monoidal interval*, that is, in the set of clones $\mathscr{C}$ which have the same unary fragment $\mathscr{C}^{(1)}$ as $\mathscr{S}_F$. To remedy that, we also consider the biggest clone having $\mathscr{S}_F^{(1)}$ as its unary part. It will be denoted by $\mathscr{U}_F$ and can be described as follows.

$$\mathscr{U}_F = \mathrm{Pol}\left(\mathscr{S}_F^{(1)}\right) := \bigcup_{n\geq 1} \left\{f \in \mathscr{O}^{(n)} : f(g_1, \ldots, g_n) \in \mathscr{S}_F^{(1)} \text{ for all } g_1, \ldots g_n \in \mathscr{S}_F^{(1)}\right\}.$$

Then $\mathscr{U}_F^{(1)} = \mathscr{S}_F^{(1)}$ and $\mathscr{S}_F \subseteq \mathscr{U}_F$. In the dual case the same procedure does not lead anywhere; as already mentioned in 0.3.3: $\mathrm{Pol}(\mathscr{C}_I^{(1)}) = \mathscr{C}_I$ (exercise).

By now, it is not well-understood under what conditions $\mathscr{S}_F = \mathscr{U}_F$. Leaving the proof as an exercise we mention that $\mathscr{S}_F = \mathscr{U}_F$ holds for countably generated filters and admit that we do not know if this also holds in the general regular case. In contrast, we have the following

Proposition 30. *Let F be a non-principal ultrafilter.*

(1) $\mathscr{U}_F = \{f \in \mathscr{O} : fix(f) \in F\}$, *where $fix(f)$ denotes the fixed-point-set of f, i.e., $fix(f) := \{x \in X : f(x, x, \ldots, x) = x\}$.*
(2) $\mathscr{S}_F$ *is a proper subset of* $\mathscr{U}_F$.

The proof of (1) is based on the following result:

Lemma 31 (Katetov)**.** *A unary function is F-continuous, iff its fixed-point-set belongs to F.*

Proof of Lemma 31. The implication $fix(f) \in F \Rightarrow f \in \mathscr{U}_F$ is clear.

Assume that the unary function f is F-continuous. We want to show that $fix(f) \in F$. Let $C := \{x : \exists k\, f^k(x) = x\}$. Any unary function f defines an undirected graph on X with edges $(x, f(x))$. In each connected component of X pick a representative — if possible, in C. Let B be the set of those representatives. Notice that $B \cap f^{-1}[B] \subseteq fix(f)$.

For each $x \in X$ let $n(x)$ be minimal such that $f^n(x) \in B$; if this is not defined, let $n(x) := \min\{k + j : \exists b \in B\ f^k(x) = f^j(b)\}$. In other words, $n(x)$ is

– the length of the unique path from x to an element of B, if x is in a component without fixed points or cycles
– the smallest n with $f^n(x) \in B$, otherwise.

Let $X_i := \{x \in X : n(x) \equiv i \pmod 2\}$ for $i = 0, 1$. It is easy to see that $f^{-1}[X_i] \subseteq X_{1-i} \cup B$. Clearly, one of the X_i is in the ultrafilter F and the other is not. Assume $X_i \in F$. By F-continuity, $f^{-1}[X_i] \in F$, hence $X_{1-i} \cup B \in F$. Now, $X_{1-i} \notin F$ yields $B \in F$. But then also $f^{-1}[B] \in F$, so $F \ni B \cap f^{-1}[B] \subseteq fix(f)$.

□

We now prove Proposition 30 for functions of arbitrary arity:

Proof of Proposition 30. Let $f \in \mathscr{U}_F$ be given. Then $f(\mathrm{id}, \mathrm{id}, \dots, \mathrm{id})$ is unary and F-continuous, so its fixed-point-set, which is $fix(f)$ belongs to F.

The other way round. Assuming $fix(f) \in F$ and $g_1, \dots, g_n \in \mathscr{S}_F^{(1)}$, we must prove $f(g_1, \dots, g_n) \in \mathscr{S}_F^{(1)}$, i.e. $fix\Big(f(g_1, \dots, g_n)\Big) \in F$. But this follows from the obvious

$$F \ni fix(f) \cap fix(g_1) \cap \dots \cap fix(g_n) \subseteq fix\Big(f(g_1, \dots, g_n)\Big).$$

To exhibit $f \in \mathscr{U}_F \setminus \mathscr{S}_F$, and thus proving (2), we let 0 be any point in X and define

$$f(x, y) := \begin{cases} x, & x = y \\ 0, & \text{otherwise} \end{cases}.$$

Then $fix(f) = X \in F$. But for any infinite B we have $f[B^2] \ni 0$, hence $f[B^2] \subseteq X \setminus \{0\}$ is impossible, disproving F-continuity. □

Remark. With the description (1) the clones $\mathscr{U}_F$ were used in [Mar81] by Marchenkov, who showed them maximal and distinct for different ultrafilters. These were the first easy examples of $2^{2^{\aleph_0}}$ maximal clones.

Next we characterize the precomplete clones of type $\mathscr{U}_F$. Notice that the following theorem is true without the countability assumption on X neither do we assume that all cofinite sets are in F. The filter has just to be proper, i.e. different from $\{X\}$ and $\mathscr{P}(X)$.

Theorem 32. *If F is a proper filter on X, then each of the following conditions is equivalent to the precompleteness of $\mathscr{U}_F$.*

1. *There is no proper filter $G \supsetneq F$ with $\mathscr{S}_F^{(1)} \subsetneq \mathscr{S}_G^{(1)}$.*
2. *For each $A \notin F$ there exists $f \in \mathscr{S}_F^{(1)}$ such that $f^{-1}[A] = \emptyset$.*
3. *$\mathscr{S}_F(h) = \mathscr{O}$ for each unary $h \notin \mathscr{S}_F$.*

Remarks. Condition (1) is formally weaker than the maximality of $\mathscr{U}_F$ among filter clones. These conditions may be equivalent, however.

From (3) it does not follow that $\mathscr{S}_F$ is maximal, because there can be a binary function in $\mathscr{U}_F \setminus \mathscr{S}_F$. This is the case for ultrafilters. We do not know if $\mathscr{U}_F$ is generated by $\mathscr{S}_F$ and some binary non-continuous function.

Proof. The necessity of (1) will be established by constructing a clone above $\mathscr{U}_F$ from a proper filter $G \supsetneq F$ such that $\mathscr{S}_G^{(1)} \supsetneq \mathscr{S}_F^{(1)}$. The first idea is, of course, trying $\mathscr{U}_G$. But this need not work, so we have to come up with something more tricky. We consider

$$\mathscr{M} := \Big\{ f \in \mathscr{O}^{(1)} : [\![\, f = \tilde{f} \,]\!] \in G \text{ for some } \tilde{f} \in \mathscr{S}_F^{(1)} \Big\},$$

where $[\![\, f = \tilde{f} \,]\!]$ denotes the so-called *equalizer* $\{x \in X : f(x) = \tilde{f}(x)\}$. A number of easy verifications then yields that $\mathscr{M}$ is a monoid and $\mathscr{S}_F^{(1)} \subseteq \mathscr{M} \subseteq \mathscr{S}_G^{(1)}$. To see, for example, that $\mathscr{M}$ is closed under composition, consider $f, g \in \mathscr{M}$ witnessed by $\tilde{f}, \tilde{g} \in \mathscr{S}_F^{(1)}$. Then $\tilde{f} \circ \tilde{g}$ witnesses $f \circ g \in \mathscr{M}$, because

$$[\![\, f \circ g = \tilde{f} \circ \tilde{g} \,]\!] \supseteq [\![\, g = \tilde{g} \,]\!] \cap \tilde{g}^{-1} \left[[\![\, f = \tilde{f} \,]\!] \right]$$

belongs to G (for the preimage $\mathscr{S}_F^{(1)} \subseteq \mathscr{S}_G^{(1)}$ is used). To see that $\mathscr{M} \subseteq \mathscr{S}_G^{(1)}$, let $f \in \mathscr{M}$ be witnessed by $\tilde{f}$. Because $\mathscr{S}_F^{(1)} \subseteq \mathscr{S}_G^{(1)}$, we have $\tilde{f} \in \mathscr{S}_G^{(1)}$, therefore, $f^{-1}[A] \supseteq \tilde{f}^{-1}[A] \cap [\![\, f = \tilde{f} \,]\!]$ belongs to G, whenever $A \in G$.

Now we prove $\mathscr{U}_F = \operatorname{Pol}\left(\mathscr{S}_F^{(1)}\right) \subseteq \operatorname{Pol}(\mathscr{M})$. Consider $h(x_1, \ldots, x_n) \in \mathscr{U}_F$ and $f_1, \ldots f_n \in \mathscr{M}$, witnessed by $\tilde{f}_1, \ldots, \tilde{f}_n \in \mathscr{S}_F^{(1)}$. Then the function $h(\tilde{f}_1, \ldots, \tilde{f}_n)$ witnesses $h(f_1, \ldots, f_n) \in \mathscr{M}$. For, $h \in \operatorname{Pol}\left(\mathscr{S}_F^{(1)}\right)$ implies $h(\tilde{f}_1, \ldots, \tilde{f}_n) \in \mathscr{S}_F^{(1)}$, and

$$[\![\, h(f_1, \ldots, f_n) = h(\tilde{f}_1, \ldots, \tilde{f}_n) \,]\!] \supseteq [\![\, f_1 = \tilde{f}_1 \,]\!] \cap \cdots \cap [\![\, f_n = \tilde{f}_n \,]\!] \in G.$$

As G is proper and $\mathscr{M} \subseteq \mathscr{S}_G^{(1)} \neq \mathscr{O}^{(1)}$ we cannot have $\operatorname{Pol}(\mathscr{M}) = \mathscr{O}$. It remains to exhibit a function in $\operatorname{Pol}(\mathscr{M})$ which is not in $\mathscr{U}_F$. In fact, we find one in $\mathscr{M} \setminus \mathscr{S}_F^{(1)}$. Take $A \in G \setminus F$ and $c \in X$ such that $X \setminus \{c\} \in F$. The former is possible because G strictly includes F; the latter because F is proper. Then the function $p(x) := \begin{cases} x, & x \in A \\ c, & x \notin A \end{cases}$ is in $\mathscr{M}$, because $[\![\, p = \mathrm{id} \,]\!] \supseteq A \in G$, but not F-continuous, because $p^{-1}[X \setminus \{c\}] \subseteq A \notin F$.

Next we prove (1) $\Rightarrow$ (2). Actually, we assume that $A \notin F$ is a counterexample to (2), i.e. $f^{-1}[A] \neq \emptyset$ for all $f \in \mathscr{S}_F^{(1)}$, and show that

$$G := \{B \subseteq X : f^{-1}[A] \subseteq B \text{ for some } f \in \mathscr{S}_F^{(1)}\}$$

is a filter contradicting (1). It is clear that G is upward-closed. To see that G is closed under unions, observe that $h^{-1}[A] = f_1^{-1}[A] \cup f_2^{-1}[A]$ for the function

$$h(x) = \begin{cases} f_1(x), & f_1(x) \in A \\ f_2(x), & \text{otherwise} \end{cases}.$$

If $f_1, f_2 \in \mathscr{S}_F^{(1)}$, then so is h, for $h^{-1}[B] \supseteq f_1^{-1}[B] \cap f_2^{-1}[B] \in F$ for any $B \in F$.

The choice of A just means that $\emptyset \notin G$. From $A \in G$ we conclude that G is proper and not equal to F. As $\mathscr{S}_F^{(1)} \subseteq \mathscr{S}_G^{(1)}$ is obvious, it just remains to show $F \subseteq G$. Let $B \in F$ be given and let c be some element outside A (if there were none, we had the impossible $X = A \notin F$). Then the function

$$q(x) = \begin{cases} x, & x \in B \\ c, & x \notin B \end{cases}$$

belongs to $\mathscr{S}_F^{(1)}$ and $q^{-1}[A] \subseteq B$ proves $B \in G$.

Next we prove (2) $\Rightarrow$ (3). Let $h \notin \mathscr{S}_F^{(1)}$ be given. Then $h^{-1}[B] \notin F$ for some $B \in F$. Using (2) we take $f \in \mathscr{S}_F^{(1)}$ such that $f^{-1}[h^{-1}[B]] = \emptyset$. In other words $h \circ f$ does not take any value in B.

Our aim is to show $\mathscr{S}_F(h) = \mathscr{O}$. So let any $g(x_1, \ldots, x_n)$ be given. Put

$$\tilde{g}(x_1, \ldots, x_n, y) := \begin{cases} g(x_1, \ldots, x_n), & y \notin B \\ y, & y \in B. \end{cases}.$$

Then $\tilde{g}$ is F-continuous. For, if $C \in F$ then $B \cap C \in F$ and $\tilde{g}[(B \cap C)^{n+1}] = B \cap C \subseteq C$.

As, obviously, $g(x_1, \ldots, x_n) = \tilde{g}(x_1, \ldots, x_n, h(f(x_1)))$, we have $g \in \mathscr{S}_F(h)$.

It remains to see that (3) is sufficient for the precompleteness of $\mathscr{U}_F$. For an arbitrary operation $g(x_1, \ldots, x_n)$ outside $\mathscr{U}_F$ we have to show that $\mathscr{U}_F(g) = \mathscr{O}$.

From $g \notin \mathscr{U}_F = \mathrm{Pol}(\mathscr{S}_F^{(1)})$ we get $f_1, \ldots, f_n \in \mathscr{S}_F^{(1)}$ such that $h := g(f_1, \ldots, f_n) \notin \mathscr{S}_F^{(1)}$. But then, by (3),

$$\mathscr{O} = \mathscr{S}_F(h) \subseteq \mathscr{S}_F(g) \subseteq \mathscr{U}_F(g).$$

□

The theorem is now completely proved. If true, condition (2) is usually easy to verify. It yields the precompleteness of $\mathscr{U}_F$ for, e.g., ultrafilters and countably generated filters.

No new considerations are, however, needed in these examples, because it is possible to relate the precompleteness of $\mathscr{U}_F$ to that of $\mathscr{C}_I$ for the dual ideal. This is at first sight surprising because these clones sit in rather different parts of the lattice. The operations of $\mathscr{U}_F$ and $\mathscr{C}_I$ are very different: For example, $\mathscr{C}_I$ contains all constant operations, whereas $\mathscr{U}_F$ will never contain any constant operation. In a very free interpretation, one could say that the unary operations in $\mathscr{U}_F$ are in a way close to injective, since the preimages of small sets are small, whereas an operation is more likely to belong to $\mathscr{C}_I$ the less injective it is.

In the following the countability of X is essential again, and I must be an ideal 'in our sense'.

Corollary 33. *Let I be any ideal and F its dual filter. $\mathscr{U}_F$ is maximal iff $T(A, 1, \infty)$ holds for all $A \notin I$, i.e. iff $\mathscr{C}_I$ is maximal via (possibly infinitely many) unary functions.*

This follows easily from what we have already proved.

If we switch to complements, condition (2) of the last theorem says that for all $A \notin I$ there is some $f \in \mathscr{S}_F^{(1)}$ such that $f^{-1}[A] = X$. The latter amounts to $f : X \to A$ and $f \in \mathscr{S}_F^{(1)}$ can be read as $f^{-1}[B] \in I$ for all $B \in I$.

In other words condition (2) says the same as the case $n = 1$ of Lemma 25, where it was shown to be equivalent to $T(A, 1, \infty)$. □

As an immediate consequence of Corollary 33 we have that there exists an ideal I such that $\mathscr{C}_I$ is maximal while $\mathscr{U}_F$ is not maximal. Just choose I (using Example 28 with $n = 1$) such that binary functions are required to check the precompleteness of $\mathscr{C}_I$.

5. Uncountable base sets

We briefly discuss the possibility of generalizing the results of this paper to uncountable base sets. As in the countable case, we may assume that all ideals have full support, that is, they contain all finite subsets of X. For countable X, the assumption that an ideal contains at least one infinite set and does not contain some infinite set implies that the induced ideal clone is proper, i.e. it does not contain all operations on X. This is no longer the case for uncountable X. Define for all infinite $\lambda \leq |X|$ an ideal I_λ consisting of all sets $S \subseteq X$ with $|S| < \lambda$. Then we have

Lemma 34. *Let X be infinite and let I be a proper ideal with full support. Then $\mathscr{C}_I = \mathscr{O}$ iff $I = I_\lambda$ for some infinite $\lambda \leq |X|$.*

We skip the easy proof.

The preceding lemma immediately makes it clear that things will be more complicated for uncountable X; in particular, the basic Proposition 2 does not hold anymore. It can be replaced by

Proposition 35. *Let I, J be ideals such that $\mathscr{C}_I^{(1)} \subseteq \mathscr{C}_J^{(1)}$, and let $\lambda \leq |X|$. If J contains a set of size λ, then all sets in I of size at most λ are contained in J.*
In particular, if I and J are ideals such that

$$\sup\{|A| : A \in I\} = \sup\{|A| : A \in J\} =: \lambda$$

and

$$\exists A \in I\,(|A| = \lambda) \leftrightarrow \exists A \in J\,(|A| = \lambda),$$

then $\mathscr{C}_I = \mathscr{C}_J$ iff $I = J$.

Again the proof is straightforward, so we skip it, too.

If we demand ideals to contain at least one *large* set, i.e. a set of size $|X|$, then Proposition 2 holds, but still the deeper results of this paper do not generalize, e.g. the maximality criterion from [CH01] fails in the following

Example 36. Let X be uncountable. Then there is an ideal I with the following properties: it has full support and contains a large set; the induced clone $\mathscr{C}_I$ is precomplete; but there is some $A \notin I$ such that $T(A, n, p)$ fails for all n and p.

We can assume that $X = Y \times \omega$, where Y is uncountable. This allows us to define the 'below'-relation on X via $(y_1, n_1) \prec (y_2, n_2) :\Longleftrightarrow n_1 < n_2$.

Let I denote the ideal of bounded subsets of X, i.e. $A \in I$ iff there is some (b, n) such that $(a, m) \prec (b, n)$ for all $(a, m) \in A$.

There are, of course, countable unbounded sets, i.e. $\{y_0\} \times \omega$. But these cannot be mapped onto X by any finitary function. So the test fails.

But $\mathscr{C}_I$ is precomplete, anyway. To see this, consider some $f \notin \mathscr{C}_I$ (wlog unary). Then f maps a bounded set S to an unbounded set U. Now let $g \in \mathscr{O}^{(n)}$ be arbitrary. Define an operation $h \in \mathscr{O}^{(n+1)}$ as follows:

$$h(x_1, \dots, x_n, y) = \begin{cases} g(x_1, \dots, x_n), & g(x_1, \dots, x_n) \prec y \\ y, & \text{otherwise.} \end{cases}$$

Since $h(x_1, \dots, x_n, y) \prec y$ for all $x_1, \dots, x_n, y \in X$, we have $f \in \mathscr{C}_I$. Now define another operation $t \in \mathscr{O}^{(n)}$ such that

$$t(x_1, \dots, x_n) \in \{s \in S : g(x_1, \dots, x_n) \prec f(s)\}.$$

Since t has bounded range, we have $t \in \mathscr{C}_I$. But now clearly, $g(x_1, \dots, x_n) = h(x_1, \dots, x_n, f(t(x_1, \dots, x_n))) \in \mathscr{C}_I(f)$, finishing the proof of precompleteness.

In order to generalize our results to uncountable base sets, the following restriction on ideals proves convenient: Call an ideal *suitable* iff it contains all small (small = non-large = of cardinality smaller than X) sets, and contains at least one large set but not all sets. When working with suitable ideals, all results and proofs of this paper generalize in a straightforward way, except for the construction of many precomplete clones without the Axiom of Choice (Theorem 18). The necessary big almost disjoint families exist only under additional assumptions on cardinal arithmetic. To carry out the generalization of the other results, the definition of $\hat{I}$ must be adjusted as follows: $\hat{I} = \{A \subseteq X :$ for all large $B \subseteq A$ there exists some large $C \subseteq B$ with $C \in I\}$. The corresponding operator $\cdot^{\perp}$ is the following: For a family $A \subseteq \mathscr{P}(X)$, $A^{\perp} := \{B \subseteq X : \forall C \in A\,(C \cap B \text{ small})\}$.

Observe that whereas the restriction to ideals having full support is natural and can easily be argued, there is no obvious reason to consider only suitable ideals, except for them being ... suitable.

6. Open problems

Problem 37. In Example 27, we exhibited two incomparable ideal clones with comparable unary fragments. Do there exist incomparable ideal clones with comparable n-ary fragments, where $n > 1$?

Problem 38. Is every ideal clone generated by its binary fragment? (A positive answer would yield a negative answer to the previous problem for all $n > 1$).

This is even open for many particular ideals. For example, let $I_{d=0}$ be the ideal of all subsets $A \subseteq \mathbb{N}$ with upper density 0. The upper density $\overline{d}(A)$ is defined as $\overline{d}(A) = \overline{\lim}_{n\to\infty}|A \cap \{0,\ldots,n\}|/(n+1)$. This ideal is well known and plays an important role in analysis and number theory.

Problem 39. Is the ideal clone $\mathscr{C}_{I_{d=0}}$ generated by its binary fragment?

Problem 40. Is $\mathscr{C}_{I_{d=0}}$ a precomplete clone?

Problem 41. Is there a prime ideal P on an infinite set Y such for the ideal $I = P \times P$ on $X = Y \times Y$ the clone $\mathscr{C}_I$ is precomplete via infinitely many unary functions?

Problem 42. Find an ideal I such that $\mathscr{C}_I$ is precomplete but for all n there is $A \notin I$ such that $T(A, n, 1)$ fails.

Problem 43. Which implications hold between

$$\mathscr{U}_F \subseteq \mathscr{U}_G, \qquad \mathscr{S}_F \subseteq \mathscr{S}_G \quad \text{and} \quad \mathscr{C}_I \subseteq \mathscr{C}_J,$$

where F and G are the dual filters of the ideals I and J, respectively.

Problem 44. Under what conditions $\mathscr{U}_F = \mathscr{S}_F$ holds? Is this true for regular filters (i.e. duals of regular ideals)?

Finally, we repeat a problem from [CH01]:

Problem 45. Let I be an ideal such that $\mathscr{C}_I$ is not precomplete. Is there an ideal J such that the clone $\mathscr{C}_J$ is an upper cover of $\mathscr{C}_I$ in the clone lattice?

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