Clones containing all almost unary functions

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Abstract. Let X be an infinite set of regular cardinality. We determine all clones on X which contain all almost unary functions. It turns out that independently of the size of X, these clones form a countably infinite descending chain. Moreover, all such clones are finitely generated over the unary functions. In particular, we obtain an explicit description of the only maximal clone in this part of the clone lattice. This is especially interesting if X is countably infinite, in which case it is known that such a simple description cannot be obtained for the second maximal clone over the unary functions.

1. Introduction

1.1. Clones and almost unary functions. Let X be a set and denote by $\mathscr{O}^{(n)}$ the set of all n-ary functions on X. Then $\mathscr{O} = \bigcup_{n=1}^{\infty} \mathscr{O}^{(n)}$ is the set of all functions on X. A *clone* $\mathscr{C}$ over X is a subset of $\mathscr{O}$ which contains the projections and which is closed under compositions. The set of all clones over X forms a complete lattice $Clone(X)$ with respect to inclusion. This lattice is a subset of the power set of $\mathscr{O}$. The clone lattice is countably infinite if X has only two elements, but is of size $2^{\aleph_0}$ for cardinality of X finite and greater than two. For infinite X we have $|Clone(X)| = 2^{2^{|X|}}$.

Let X be of infinite regular cardinality from now on unless otherwise stated. We call a subset $S \subseteq X$ *large* iff $|S| = |X|$, and *small* otherwise. If X is itself a regular cardinal, then the small subsets are exactly the bounded subsets of X. A function $f(x_1, ..., x_n) \in \mathscr{O}^{(n)}$ is *almost unary* iff there exists a function $F : X \to \mathscr{P}(X)$ and $1 \leq k \leq n$ such that $F(x)$ is small for all $x \in X$ and such that for all $(x_1, ..., x_n) \in X^n$ we have $f(x_1, ..., x_n) \in F(x_k)$. If we assume X to be a regular cardinal itself, this is equivalent to the existence of a function $F \in \mathscr{O}^{(1)}$ and a $1 \leq k \leq n$ such that $f(x_1, ..., x_n) < F(x_k)$ for all $(x_1, ..., x_n) \in X^n$. Because this is much more convenient and does not influence the properties of the clone lattice, we shall assume X to be a regular cardinal throughout this paper. Let $\mathscr{U}$ be the

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set of all almost unary functions. It is readily verified that $\mathscr{U}$ is a clone. We will determine all clones which contain $\mathscr{U}$; in particular, such clones contain $\mathscr{O}^{(1)}$.

1.2. Maximal clones above $\mathscr{O}^{(1)}$. A clone is called *maximal* iff it is a dual atom in $Clone(X)$. For finite X, the number of maximal clones is finite and all of them are known (a result due to I. Rosenberg [9], see also [8]). Moreover, the lattice is dually atomic in that case, i.e. every clone is contained in a maximal one. If X is infinite, the situation is rather hopeless as another theorem by I. Rosenberg [10] states that there exist $2^{2^{|X|}}$ maximal clones, see also [5]. In addition, a recent result due to M. Goldstern and S. Shelah [4] shows that if the continuum hypothesis holds, then the clone lattice of a countable base set is not even dually atomic.

However, by Zorn's lemma, the sublattice of $Clone(X)$ of functions containing $\mathscr{O}^{(1)}$ is dually atomic since $\mathscr{O}$ is finitely generated over $\mathscr{O}^{(1)}$. G. Gavrilov proved in [2] that for countably infinite X there are only two maximal clones containing all unary functions. M. Goldstern and S. Shelah extended this result to clones on weakly compact cardinals in the article [5], where an uncountable cardinal X is called *weakly compact* iff whenever we colour the edges of a complete graph G of size X with two colours, then there exists a complete subgraph of G of size X on which the colouring is constant. In the same paper, the authors proved that on other regular cardinals X satisfying a certain partition relation there are even 2^{2^X} maximal clones above $\mathscr{O}^{(1)}$.

There exists exactly one maximal clone above $\mathscr{U}$. So far, this clone has been defined using the following concept: Let $\rho \subseteq X^J$ be a relation on X indexed by J and let $f \in \mathscr{O}^{(n)}$. We say that f *preserves* ρ iff for all $r^1 = (r^1_i : i \in J), \ldots, r^n = (r^n_i : i \in J)$ in ρ we have $(f(r^1_i, ..., r^n_i) : i \in J) \in \rho$. We define the set of *polymorphisms* $Pol(\rho)$ of ρ to be the set of all functions in $\mathscr{O}$ preserving ρ; $Pol(\rho)$ is easily seen to be a clone. In particular, if $\rho \subseteq X^{X^k}$ is a set of k-ary functions, then a function $f \in \mathscr{O}^{(n)}$ preserves ρ iff for all functions $g_1, ..., g_n$ in ρ the composite $f(g_1, ..., g_n)$ is a function in ρ.

Write

$$T_1 = \mathscr{U}^{(2)} = \{f \in \mathscr{O}^{(2)} : f \text{ almost unary}\}.$$

The following was observed by G. Gavrilov [2] for countable base sets and extended to all regular X by R. Davies and I. Rosenberg [1]. Uniqueness on uncountable regular cardinals is due to M. Goldstern and S. Shelah [5].

Theorem 1. *Let X have infinite regular cardinality. Then $Pol(T_1)$ is a maximal clone containing all unary functions. Furthermore, $Pol(T_1)$ is the only maximal clone containing all almost unary functions.*

For S a subset of X we set

$$\Delta_S = \{(x, y) \in S^2 : y < x\}, \quad \nabla_S = \{(x, y) \in S^2 : x < y\}.$$

We will also write Δ and ∇ instead of Δ_X and ∇_X. Now define

$$T_2 = \{f \in \mathscr{O}^{(2)} : \forall S \subseteq X\,(S \text{ large} \to \text{ neither } f\restriction_{\Delta_S} \text{ nor } f\restriction_{\nabla_S} \text{ are 1-1})\}.$$

The next result is due to G. Gavrilov [2] for X a countable set and due to M. Goldstern and S. Shelah [5] for X weakly compact.

Theorem 2. *Let X be countably infinite or weakly compact. Then $Pol(T_2)$ is a maximal clone which contains $\mathscr{O}^{(1)}$. Moreover, $Pol(T_1)$ and $Pol(T_2)$ are the only maximal clones above $\mathscr{O}^{(1)}$.*

The definition of $Pol(T_2)$ not only looks more complicated than the one of $Pol(T_1)$. First of all, a result of R. Davies and I. Rosenberg in [1] shows that assuming the continuum hypothesis, T_2 is not closed under composition on $X = \aleph_1$ and so it is unclear what $Pol(T_2)$ is. Secondly, on countable X, if we equip $\mathscr{O}$ with the natural topology which we shall specify later, then T_2 is a non-analytic set in that space and so is $Pol(T_2)$; in particular, neither $\langle T_2\rangle$ nor $Pol(T_2)$ are countably generated over $\mathscr{O}^{(1)}$ (see [3]), where for a set of functions $\mathscr{F}$ we denote by $\langle \mathscr{F}\rangle$ the clone generated by $\mathscr{F}$. The clones $\langle T_1\rangle$ and $Pol(T_1)$ on the other hand turn out to be rather simple with respect to this topology, and both clones are finitely generated $\mathscr{O}^{(1)}$.

Fix any injection p from X^2 to X; for technical reasons we assume that $0 \in X$ is not in the range of p.

Fact 3. *$\langle \{p\} \cup \mathscr{O}^{(1)}\rangle = \mathscr{O}$, that is, the function p together with $\mathscr{O}^{(1)}$ generate $\mathscr{O}$.*

For a subset S of X^2 we write

$$p_S(x_1, x_2) = \begin{cases} p(x_1, x_2) & , (x_1, x_2) \in S \\ 0 & , \text{otherwise} \end{cases}$$

M. Goldstern observed the following [3]. Since the result has not yet been published, but is important for our investigations, we include a proof here.

Fact 4. *$\langle \{p_\Delta\} \cup \mathscr{O}^{(1)}\rangle = \langle T_1\rangle$.*

Proof. Set $\mathscr{C} = \langle \{p_\Delta\} \cup \mathscr{O}^{(1)}\rangle$. Since $p_\Delta(x_1, x_2)$ is obviously bounded by the unary function $\gamma(x_1) = \sup\{p_\Delta(x_1, x_2) : x_2 \in X\} + 1 = \sup\{p(x_1, x_2) : x_2 < x_1\} + 1$, where by $\alpha + 1$ we mean the successor of an ordinal α, we have $p_\Delta \in T_1$ and hence $\mathscr{C} \subseteq \langle T_1\rangle$.

To see the other inclusion, note first that the function

$$q(x_1, x_2) = \begin{cases} p_\Delta(x_1, x_2) & , (x_1, x_2) \in \Delta \\ x_1 & , \text{otherwise} \end{cases}$$

is in $\mathscr{C}$. Indeed, choose $\epsilon \in \mathscr{O}^{(1)}$ strictly increasing such that $p_\Delta(x_1, x_2) < \epsilon(x_1)$ for all $x_1, x_2 \in X$ and consider $t(x_1, x_2) = p_\Delta(\epsilon(x_1), p_\Delta(x_1, x_2))$. On Δ, t is still one-one, and outside Δ, the term is a one-one function of the first component x_1.

Moreover, the ranges $t[\Delta]$ and $t[X^2 \setminus \Delta]$ are disjoint. Hence, we can write $q = u \circ t$ for some unary u. By the same argument we see that for arbitrary unary functions $a, b \in \mathscr{O}^{(1)}$ the function

$$q_{a,b}(x_1, x_2) = \begin{cases} a(p_\Delta(x_1, x_2)) & , (x_1, x_2) \in \Delta \\ b(x_1) & , \text{otherwise} \end{cases}$$

is an element of $\mathscr{C}$.

Now let $f \in T_1$ be given and say $f(x_1, x_2) < \delta(x_1)$ for all $x_1, x_2 \in X$, where $\delta \in \mathscr{O}^{(1)}$ is strictly increasing. Choose $a \in \mathscr{O}^{(1)}$ such that $a(p_\Delta(x_1, x_2)) = f(x_1, x_2) + 1$ for all $(x_1, x_2) \in \Delta$. Then set

$$f_1(x_1, x_2) = q_{a,\delta+1}(x_1, x_2) = \begin{cases} f(x_1, x_2) + 1 & , (x_1, x_2) \in \Delta \\ \delta(x_1) + 1 & , \text{otherwise} \end{cases}$$

We construct a second function

$$f_2(x_1, x_2) = \begin{cases} 0 & , (x_1, x_2) \in \Delta \\ f(x_1, x_2) + 1 & , \text{otherwise} \end{cases}$$

It is readily verified that $f_2(x_1, x_2) = u(p_\Delta(x_2 + 1, x_1))$ for some unary u. Now $f_2(x_1, x_2) < f_1(x_1, x_2)$ and $f_1, f_2 \in \mathscr{C}$. Clearly

$$f(x_1, x_2) = u(p_\Delta(f_1(x_1, x_2), f_2(x_1, x_2)))$$

for some unary u. This shows $f \in \mathscr{C}$ and so $\langle T_1 \rangle \subseteq \mathscr{C}$ as $f \in T_1$ was arbitrary. $\square$

We shall see that $Pol(T_1)$ is also finitely generated over $\mathscr{O}^{(1)}$. Moreover, for countable X it is a Borel set in the topology yet to be defined. Our explicit description $Pol(T_1)$ holds for all infinite X of regular cardinality, but is interesting only if there are not too many other maximal clones containing $\mathscr{O}^{(1)}$. By Theorem 2, this is at least the case for X countably infinite or weakly compact.

Throughout this paper, the assumption that the base set X has regular cardinality is essential. To give an example, we prove now that $\mathscr{U}$ is a clone. Let $f \in \mathscr{U}^{(n)}$ and $g_1, ..., g_n \in \mathscr{U}^{(m)}$. By definition, there exists $F \in \mathscr{O}^{(1)}$ and some $1 \leq k \leq n$ such that $f(x) < F(x_k)$ for all $x \in X^n$. Because $g_k \in \mathscr{U}$, we obtain $G_k \in \mathscr{O}^{(1)}$ and $1 \leq i \leq m$ such that $g_k(x) < G_k(x_i)$ for all $x \in X^m$. Therefore $f(g_1, ..., g_n)(x) < H(x_i)$, where we define $H(x_i) = \sup_{y<G_k(x_i)}\{F(y)\}$. Now since $F(y) < X$ for all $y \in X$, and since the supremum ranges over a set of size $G_k(x_i) < X$, the regularity of X implies that $H(x_i) < X$, so that the composite $f(g_1, ..., g_n)$ is bounded by a unary function and hence an element of $\mathscr{U}$. It is easy to see that on singular X, neither of the definition of an almost unary function by means of small sets nor the one via boundedness by a unary function yield a clone. Also, the two definitions differ on singulars, whereas on regulars they coincide.

1.3. Notation. For a set of functions $\mathscr{F}$ we shall denote the smallest clone containing $\mathscr{F}$ by $\langle\mathscr{F}\rangle$. By $\mathscr{F}^{(n)}$ we refer to the set of n-ary functions in $\mathscr{F}$.
We call the projections which every clone contains π_i^n where $n \geq 1$ and $1 \leq i \leq n$. If $f \in \mathscr{O}^{(n)}$ is an n-ary function, it sends n-tuples of elements of X to X and we write $(x_1, ..., x_n)$ for these tuples unless otherwise stated as in $f(x, y, z)$; this is the only place where we do not stick to set-theoretical notation (according to which we would have to write $(x_0, ..., x_{n-1})$). The set $\{1, ..., n\}$ of indices of n-tuples will play an important role and we write N for it. We denote the set-theoretical complement of a subset $A \subseteq N$ in N by $-A$. We identify the set X^n of n-tuples with the set of functions from N to X, so that if $A \subseteq N$ and $a : A \to X$ and $b : -A \to X$ are partial functions, then $a \cup b$ is an n-tuple. Sometimes, if the arity of $f \in \mathscr{O}$ has not yet been given a name, we refer to that arity by n_f.
If $a \in X^n$ is an n-tuple and $1 \leq k \leq n$ we write $(a)_k^n$ or only a_k for the k-th component of a. For $c \in X$ and J an index set we write c^J for the J-tuple with constant value c. The order relation $\leq$ on X induces the pointwise partial order on the set of J-tuples of elements of X for any index set J: For $x, y \in X^J$ we write $x \leq y$ iff $x_j \leq y_j$ for all $j \in J$. Consequently we also denote the induced pointwise partial order of $\mathscr{O}^{(n)}$ by $\leq$, so that for $f, g \in \mathscr{O}^{(n)}$ we have $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X^n$. Whenever we state that a function $f \in \mathscr{O}^{(n)}$ is monotone, we mean it is monotone with respect to $\leq$: $f(x) \leq f(y)$ whenever $x \leq y$. We denote the power set of X by $\mathscr{P}(X)$. The element $0 \in X$ is the smallest element of X.

2. Properties of clones above $\mathscr{U}$ and the clone $Pol(T_1)$

2.1. What $\langle T_1\rangle$ is. We start by proving that the almost unary clone $\mathscr{U}$ is a so-called *binary clone*, that is, it is generated by its binary part. Thus, when investigating $[\mathscr{U}, Pol(T_1)]$, we are in fact dealing with an interval of the form $[\langle\mathscr{C}^{(2)}\rangle, Pol(\mathscr{C}^{(2)})]$ for $\mathscr{C}$ a clone.

Lemma 5. *The binary almost unary functions generate all almost unary functions. That is, $\langle T_1\rangle = \mathscr{U}$.*

Proof. Trivially, $\langle T_1\rangle \subseteq \mathscr{U}$. Now we prove by induction that $\mathscr{U}^{(n)} \subseteq \langle T_1\rangle$ for all $n \geq 1$. This is obvious for $n = 1, 2$. Assume we have $\mathscr{U}^{(k)} \subseteq \langle T_1\rangle$ for all $k < n$ and take any function $f \in \mathscr{U}^{(n)}$. Say without loss of generality that $f(x_1, ..., x_n) \leq \gamma(x_1)$ for some $\gamma \in \mathscr{O}^{(1)}$. We will use the function $p_\Delta \in T_1$ to code two variables into one and then use the induction hypothesis. Define

$$g_1(x_1, ..., x_{n-2}, z) = \begin{cases} f(x_1, ..., x_{n-2}, (p_\Delta^{-1}(z))_1^2, (p_\Delta^{-1}(z))_2^2) & , z \in p_\Delta[X^2] \setminus \{0\} \\ x_1 & , \text{otherwise} \end{cases}$$

The function is an element of $\mathscr{U}^{(n-1)}$ as it is bounded by $\max(x_1, \gamma(x_1))$. Intuitively, g_1 does the following: If $z \neq 0$ and in the range of p_Δ, then g_1 imagines a pair (x_{n-1}, x_n) to be coded into z via p_Δ. It reconstructs the pair (x_{n-1}, x_n) and calculates $f(x_1, ..., x_n)$. If $z = 0$ or not in the range of p_Δ, then g knows there is no information in z; it simply forgets about the tuple $(x_2, ..., x_n)$ and returns x_1, relying on the following similar function to do the job: Set $\Delta' = \Delta \cup \{(x, x) : x \in X\}$ and define

$$g_2(x_1, ..., x_{n-2}, z) = \begin{cases} f(x_1, ..., x_{n-2}, (p_{\Delta'}^{-1}(z))^2_2, (p_{\Delta'}^{-1}(z))^2_1) & , z \in p_{\Delta'}[X^2] \setminus \{0\} \\ x_1 & , \text{otherwise} \end{cases}$$

The function g_2 does exactly the same as g_1 but assumes the pair (x_{n-1}, x_n) to be coded into z in wrong order, namely as (x_n, x_{n-1}), plus it cares for the diagonal. Now consider

$$h(x_1, ..., x_n) = g_2(g_1(x_1, ..., x_{n-2}, p_\Delta(x_{n-1}, x_n)), x_2, ..., x_{n-2}, p_{\Delta'}(x_n, x_{n-1})).$$

All functions which occur in h are almost unary with at most $n-1$ variables. We claim that $h = f$. Indeed, if $x_{n-1} < x_n$, then $p_\Delta(x_{n-1}, x_n) \neq 0$ and g_1 yields f. But $p_{\Delta'}(x_n, x_{n-1}) = 0$ and so g_2 returns $g_1 = f$. If on the other hand $x_n \leq x_{n-1}$, then $p_\Delta(x_{n-1}, x_n) = 0$ and $g_1 = x_1$, whereas $p_{\Delta'}(x_n, x_{n-1}) \neq 0$, which implies $h = f(g_1, x_2, ..., x_n) = f(x_1, ..., x_n)$. ☐

The following lemma will be crucial for our investigation of clones containing T_1.

Corollary 6. *Let $\mathscr{C}$ be a clone containing T_1. Then $\mathscr{C}$ is downward closed, that is, if $f \in \mathscr{C}$, then also $g \in \mathscr{C}$ for all $g \leq f$.*

Proof. If $f \in \mathscr{C}^{(n)}$ and $g \in \mathscr{O}^{(n)}$ with $g \leq f$ are given, define $h_g(x_1, ..., x_{n+1}) = \min(g(x_1, ..., x_n), x_{n+1})$. Then $h_g \leq x_{n+1}$ and consequently, $h_g \in \langle T_1 \rangle \subseteq \mathscr{C}$. Now $g = h_g(x_1, ..., x_n, f(x_1, ..., x_n)) \in \mathscr{C}$. ☐

2.2. Wildness of functions. We have seen in the last section that the interval $[\mathscr{U}, \mathscr{O}]$ is about growth of functions as all clones in that interval are downward closed. But mind we are not talking about how rapidly functions are growing in the sense of polynomial growth, exponential growth and so forth since we are considering clones modulo $\mathscr{O}^{(1)}$ (and so we can make functions as steep as we like); the growth of a function will be determined by which of its variables are responsible for the function to obtain many values. The following definition is due to M. Goldstern and S. Shelah [5]. Recall that $N = \{1, ..., n\}$.

Definition 7. Let $f \in \mathscr{O}^{(n)}$. We call a set $\emptyset \neq A \subseteq N$ *f-strong* iff for all $a \in X^A$ the set $\{f(a \cup x) : x \in X^{-A}\}$ is small. A is *f-weak* iff it is not f-strong. In order to use the defined notions more freely, we define the empty set to be f-strong iff f has small range.

Thus, a set of indices of variables of f is strong iff f is bounded whenever those variables are. For example, a function is almost unary iff it has a one-element strong set. Here, we shall rather think in terms of the complements of weak sets.

Definition 8. Let $f \in \mathscr{O}^{(n)}$ and let $A \subsetneq N$ and $a \in X^{-A}$. We say A is (f,a)-*wild* iff the set $\{f(a \cup x) : x \in X^A\}$ is large. The set A is called f-*wild* iff there exists $a \in X^{-A}$ such that A is (f,a)-wild. We say that A is f-*insane* iff A is (f,a)-wild for all $a \in X^{-A}$. The set N itself we call f-wild and f-insane iff f is unbounded.

Observe that if $A \subseteq B \subseteq N$ and A is f-wild, then B is f-wild as well. Obviously, $A \subseteq N$ is f-wild iff $-A$ is f-weak. It is useful to state the following trivial criterion for a function to be almost unary.

Lemma 9. *Let $n \geq 2$ and $f \in \mathscr{O}^{(n)}$. f is almost unary iff there exists a subset of N with $n-1$ elements which is not f-wild.*

Proof. If f is almost unary, then there is a one-element f-strong subset of N and the complement of that set is not f-wild. If on the other hand there exists $k \in N$ such that $N \setminus \{k\}$ is not f-wild, then $\{k\}$ is f-strong and so f is almost unary. □

We will require the following fact from [5].

Fact 10. *If $f \in Pol(T_1)^{(n)}$ and $A_1, A_2 \subseteq N$ are f-wild, then $A_1 \cap A_2 \neq \emptyset$.*

We observe that the converse of this statement holds as well.

Lemma 11. *Let $f \in \mathscr{O}^{(n)}$ be any n-ary function. If all pairs of f-wild subsets of N have a nonempty intersection, then $f \in Pol(T_1)$.*

Proof. Let $g_1, ..., g_n \in T_1$ be given and set $A_1 = \{k \in N : \exists \gamma \in \mathscr{O}^{(1)}\, (g_k(x_1, x_2) \leq \gamma(x_1))\}$ and $A_2 = -A_1$. Since $A_1 \cap A_2 = \emptyset$ either A_1 or A_2 cannot be f-wild. Thus $f(g_1, ..., g_n)$ is bounded by a unary function of x_2 in the first case and by a unary function of x_1 in the second case. □

The equivalence yields a first description of $Pol(T_1)$ with an interesting consequence.

Theorem 12. *A function $f \in \mathscr{O}^{(n)}$ is an element of $Pol(T_1)$ iff all pairs of f-wild subsets of N have a nonempty intersection.*

2.3. Descriptive set theory.

We show now that for countable X, this description implies that $Pol(T_1)$ is a Borel set with respect to the natural topology on $\mathscr{O}$. The reader not interested in the topic can skip this part and proceed directly to the next section.

We first explain the very basics of descriptive set theory; for more details consult [6]. Let $\mathscr{T} = (T, \Upsilon)$ be a *Polish space*, that is, a complete, metrizable, separable topological space. The *Borel sets* of $\mathscr{T}$ are the smallest σ-algebra on T which contains the open sets. These sets can be ordered according to their complexity: One starts by defining $\Sigma^0_1 = \Upsilon \subseteq \mathscr{P}(T)$ to consist exactly of the open sets and Π^0_1 of the closed sets. Then one continues inductively for all $1 < \alpha < \omega_1$ by setting Π^0_α to contain precisely the complements of Σ^0_α sets, and Σ^0_α to consist of all countable unions of sets which are elements of $\bigcup_{1\leq\delta<\alpha}\Pi^0_\delta$. The sequences $(\Sigma^0_\alpha)_{1\leq\alpha<\omega_1}$ and $(\Pi^0_\alpha)_{1\leq\alpha<\omega_1}$ are increasing and the union over either of the two sequences yields the Borel sets.

Equip our base set $X=\omega$ with the discrete topology. Then the product space $\mathscr{N}=\omega^\omega=\mathscr{O}^{(1)}$ is the so-called *Baire space*. It is obvious that $\mathscr{O}^{(n)}=\omega^{\omega^n}$ is homeomorphic to $\mathscr{N}$. Examples of open sets in $\mathscr{O}^{(n)}$ are the $A^y_x=\{f\in\mathscr{O}^{(n)}: f(x)=y\}$, where $x\in X^n$ and $y\in X$; in fact, these sets form a subbasis of the topology of $\mathscr{O}^{(n)}$. $\mathscr{O}=\bigcup_{n=1}^\infty\mathscr{O}^{(n)}$ is the sum space of ω copies of $\mathscr{N}$: The open sets in $\mathscr{O}$ are those whose intersection with each $\mathscr{O}^{(n)}$ is open in $\mathscr{O}^{(n)}$. With this topology, $\mathscr{O}$ is a Polish space, and in fact again homeomorphic to $\mathscr{N}$.

Since clones are subsets of $\mathscr{O}$, they can divided into *Borel clones* and clones which are no Borel sets. In our case, we find that $Pol(T_1)$ is a very simple Borel set.

Theorem 13. *Let X be countably infinite. Then $Pol(T_1)$ is a Borel set in $\mathscr{O}$.*

Proof. We have to show that $Pol(T_1)^{(n)}$ is Borel in $\mathscr{O}^{(n)}$ for each $n\geq 1$. By the preceding theorem,

$$Pol(T_1)^{(n)}=\{f\in\mathscr{O}^{(n)}:\forall A,B\subseteq N(A,B\,f\text{-wild}\to A\cap B\neq\emptyset)\}$$

There are no (only finite) quantifiers in this definition except for those which might occur in the predicate of wildness (observe that $\exists$-quantifiers correspond to unions and $\forall$-quantifiers to intersections). Now

$$A\subseteq N\,f\text{-wild}\leftrightarrow\exists a\in X^{-A}\forall k\in X\exists b\in X^A(f(a\cup b)>k)$$

For fixed $A\subseteq N$, $a\in X^{-A}$, $k\in X$, and $b\in X^A$, the set of all functions in $\mathscr{O}^{(n)}$ for which $(f(a\cup b)>k)$ is open. Thus, the set of all $f\in\mathscr{O}^{(n)}$ for which A is f-wild is of the form $\bigcup\bigcap\bigcup$ *open*, and hence Σ^0_3 by counting of unions and negations. Observe that all unions which occur in the definition are countable.

Since the predicate of wildness is negated in the definition of $Pol(T_1)^{(n)}$, we conclude that $Pol(T_1)^{(n)}$ is Π^0_3. □

It is readily verified that $\mathscr{U}$ (and hence, T_1) is a Borel set as well. This is interesting in connection with the following:

Above the Borel sets of a Polish space, one can continue the hierarchy of complexity. The next level, Σ^1_1, comprises the so-called *analytic sets*, which are the continuous images of Borel sets; the *co-analytic sets* (Π^1_1) are the complements of analytic sets. It is easy to see that the clone generated by a Borel set of functions in $\mathscr{O}$ is an analytic set. Since $\mathscr{O}^{(1)}$ and all countable sets are Borel, every set which is countably generated over $\mathscr{O}^{(1)}$ is analytic. M. Goldstern showed in [3] that T_2 and $Pol(T_2)$ are relatively complicated:

Theorem 14. *Let X be countably infinite. Then T_2 and $Pol(T_2)$ are co-analytic but not analytic in $\mathscr{O}$. Hence, neither of the two clones $\langle T_2\rangle$ and $Pol(T_2)$ is countably generated over $\mathscr{O}^{(1)}$.*

2.4. What wildness means. We wish to compare the wildness of functions. Write S_N for the set of all permutations on N.

Definition 15. For $f, g \in \mathscr{O}^{(n)}$ we say that f is *as wild as* g and write $f \sim_W g$ iff there exists a permutation $\pi \in S_N$ such that A is f-wild if and only if $\pi[A]$ is g-wild for all $A \subseteq N$. Moreover, g is *at least as wild as* f ($f \leq_W g$) iff there is a permutation $\pi \in S_N$ such that for all f-wild subsets $A \subseteq N$ the image $\pi[A]$ of A under π is g-wild.

Lemma 16. *$\sim_W$ is an equivalence relation and $\leq_W$ a quasiorder extending $\leq$ on the set of n-ary functions $\mathscr{O}^{(n)}$.*

Proof. We leave the verification of this to the reader. □

Lemma 17. *Let $f, g \in \mathscr{O}^{(n)}$. Then $f \sim_W g$ iff $f \leq_W g$ and $g \leq_W f$.*

Proof. It is clear that $f \leq_W g$ (and $g \leq_W f$) if $f \sim_W g$. Now assume $f \leq_W g$ and $g \leq_W f$. Then there are $\pi_1, \pi_2 \in S_N$ which take f-wild and g-wild subsets of N to g-wild and f-wild sets, respectively.

Set $\pi = \pi_2 \circ \pi_1$. Then A is f-wild iff $\pi[A]$ is f-wild for any subset A of N: If A is f-wild, then $\pi_1[A]$ is g-wild, then $\pi_2[\pi_1[A]] = \pi[A]$ is f-wild. If on the other hand $\pi[A]$ is f-wild, then take $k \geq 1$ such that $\pi^k = id_N$ and observe that $\pi^{k-1}[\pi[A]] = \pi^k[A] = A$ is f-wild.

Now we see that A is f-wild iff $\pi_1[A]$ is g-wild for all $A \subseteq N$: If $\pi_1[A]$ is g-wild, then so is $\pi_2 \circ \pi_1[A] = \pi[A]$ and so is A by the preceding observation. Hence, the permutation π_1 shows that $f \sim_W g$. □

Corollary 18. *Let $n \geq 1$. Then $\leq_W / \sim_W$ is a partial order on the $\sim_W$-equivalence classes of $\mathscr{O}^{(n)}$.*

Notation 19. Let $f \in \mathscr{O}^{(n)}$. By $\langle f\rangle_{T_1}$ we mean $\langle\{f\} \cup T_1\rangle$ from now on. $\langle f\rangle_{T_1}$ is the smallest clone containing f as well as all almost unary functions.

We are aiming for the following theorem which tells us why we invented wildness.

Theorem 20. *Let $f, g \in \mathscr{O}^{(n)}$. If $f \leq_W g$, then $f \in \langle g\rangle_{T_1}$. In words, if g is at least as wild as f, then it generates f modulo T_1.*

Corollary 21. *Let $f, g \in \mathscr{O}^{(n)}$. If $f \sim_W g$, then $\langle f\rangle_{T_1} = \langle g\rangle_{T_1}$.*

We split the proof of Theorem 20 into a sequence of lemmas. In the next lemma we see that it does not matter which $a \in X^{-A}$ makes a set $A \subseteq N$ wild.

Lemma 22. *Let $g \in \mathscr{O}^{(n)}$. Then there exists $g' \in \langle g \rangle_{T_1}^{(n)}$ such that for all $A \subseteq N$ the following holds: If A is g-wild, then A is $(g', 0^{-A})$-wild.*

Proof. Fix for all g-wild $A \subseteq N$ a tuple $a_A \in X^{-A}$ such that $\{g(x \cup a_A) : x \in X^A\}$ is large. For an n-tuple $(x_1, ..., x_n)$ write $P = P(x_1, ..., x_n) = \{l \in N : x_l \neq 0\}$ for the set of indices of positive components in the tuple. Define for $1 \leq i \leq n$ functions

$$\gamma_i(x_1, ..., x_n) = \begin{cases} x_i & , x_i \neq 0 \vee P(x_1, ..., x_n) \text{ not } g\text{-wild} \\ (a_P)_i & , \text{otherwise} \end{cases}$$

In words, if the set P of indices of positive components in $(x_1, ..., x_n)$ is a wild set, then the γ_i leave those positive components alone and send the zero components to the respective values making P wild. Otherwise, they act just like projections. It is obvious that γ_i is almost unary, $1 \leq i \leq n$. Set $g' = g(\gamma_1, ..., \gamma_n) \in \langle g \rangle_{T_1}$. To prove that g' has the desired property, let $A \subseteq N$ be g-wild. Choose any minimal g-wild $A' \subseteq A$. Then by the definition of wildness the set $\{g(x \cup a_{A'}) : x \in X^{A'}\}$ is large. Take a large $B \subseteq X^{A'}$ such that the sequence $(g(x \cup a_{A'}) : x \in B)$ is one-one. Select further a large $C \subseteq B$ such that each component in the sequence of tuples $(x : x \in C)$ is either constant or injective and such that 0 does not occur in any of the injective components (it is a simple combinatorial fact that this is possible). If one of the components were constant, then A' would not be minimal g-wild; hence, all components are injective. Now we have

$$\begin{aligned} |X| &= |\{g(x \cup a_{A'}) : x \in C\}| \\ &= |\{g'(x \cup 0^{-A'}) : x \in C\}| \quad \leq |\{g'(x \cup 0^{-A}) : x \in X^A\}| \end{aligned}$$

and so A is $(g', 0^{-A})$-wild. □

We prove that we can assume functions to be monotone.

Lemma 23. *Let $g \in \mathscr{O}^{(n)}$. Then there exists $g'' \in \langle g \rangle_{T_1}^{(n)}$ such that $g \leq g''$ and g'' is monotone with respect to the pointwise order $\leq$.*

Proof. We will define a mapping γ from X^n to X^n such that $\gamma_i = \pi_i^n \circ \gamma$ is almost unary for $1 \leq i \leq n$ and such that $g'' = g \circ \gamma$ has the desired property. We fix for every g-wild $A \subseteq N$ a sequence $(\alpha_\xi^A)_{\xi \in X}$ of elements of X^n so that all components of α_ξ^A which lie not in A are constant and so that $(g(\alpha_\xi^A))_{\xi \in X}$ is monotone and unbounded.

Let $x \in X^n$. The *order type* of x is the unique n-tuple $(j_1, ..., j_n)$ of indices in N such that $\{j_1, ..., j_n\} = \{1, ..., n\}$ and such that $x_{j_1} \leq ... \leq x_{j_n}$ and such that $j_k < j_{k+1}$ whenever $x_{j_k} = x_{j_{k+1}}$. Let $1 \leq k \leq n$ be the largest element with the property that the set $\{j_k, ..., j_n\}$ is g-wild. We call the set $\{j_k, ..., j_n\}$ the *pushing set* $Push(x)$ and $\{j_1, ..., j_{k-1}\}$ the *holding set* of x with respect to g.

We define by transfinite recursion

$$\gamma : \begin{array}{ccc} X^n & \to & X^n \\ x & \mapsto & \alpha_{\lambda(x)}^{Push(x)} \end{array}$$

where

$$\lambda(x) = \min\{\xi : g(\alpha_\xi^{Push(x)}) \geq \sup(\{g''(y) : y < x\} \cup \{g(x)\})\}.$$

This looks worse than it is: We simply map x to the first element of the sequence $(\alpha_\xi^{Push(x)})_{\xi \in X}$ such that all values of g'' already defined as well as $g(x)$ are topped. By definition, $g'' = g \circ \gamma$ is monotone and $g \leq g''$. It only remains to prove that all γ_i, $1 \leq i \leq n$, are almost unary to see that $g'' \in \langle g \rangle_{T_1}$.

Suppose not, and say that γ_k is not almost unary for some $1 \leq k \leq n$. Then there exists a value $c \in X$ and a sequence of n-tuples $(\beta_\xi)_{\xi \in X}$ with constant value c in the k-th component such that $(\gamma_k(\beta_\xi))_{\xi \in X}$ is unbounded. Since there exist only finitely many order types of n-tuples, we can assume that all β_ξ have the same order type $(j_1, ..., j_n)$; say without loss of generality $(j_1, ..., j_n) = (1, ..., n)$. Then all β_ξ have the same pushing set $Push(\beta)$ of indices. If k was an element of the holding set of the tuples β_ξ, then $(\gamma_k(\beta_\xi) : \xi \in X)$ would be constant so that k must be in $Push(\beta)$. Clearly, $(\lambda(\beta_\xi))_{\xi \in X}$ has to be unbounded as otherwise $(\gamma_k(\beta_\xi))_{\xi \in X}$ would be bounded. Since by definition the value of λ increases only when it is necessary to keep $g \leq g''$, the set $\{g(y) : \exists \xi \in X (y \leq \beta_\xi)\}$ is unbounded. But because of the order type of the β_ξ, whenever $i \leq k$, then we have $(\beta_\xi)_i^n \leq c$ for all $\xi \in X$ so that the components of the β_ξ with index in the set $\{1, ..., k\}$ are bounded. Thus, $\{k + 1, ..., n\}$ is g-wild, contradicting the fact that k is in the pushing set $Push(\beta)$. $\square$

In a next step we shall see that modulo T_1, wildness is insanity.

Lemma 24. *Let $g \in \mathscr{O}^{(n)}$. Then there exists $g'' \in \langle g \rangle_{T_1}^{(n)}$ such that g'' is monotone and for all $A \subseteq N$ the following holds: If A is g-wild, then A is g''-insane.*

Proof. Let $g' \in \langle g \rangle_{T_1}^{(n)}$ be provided by Lemma 22 and make a monotone g'' out of it with the help of the preceding lemma. We claim that g'' already has both desired properties. To prove this, consider an arbitrary g-wild $A \subseteq N$. By construction of g', A is $(g', 0^{-A})$-wild and so it is also $(g'', 0^{-A})$-wild as $g' \leq g''$. But $0^{-A} \leq a$ for all $a \in X^{-A}$; hence the fact that g'' is monotone implies that A is (g, a)-wild for all $a \in X^{-A}$ which means exactly that A is g''-insane. $\square$

Lemma 25. *Let $f, g \in \mathscr{O}^{(n)}$. If $f \leq_W g$, then there exists $h \in \langle g \rangle_{T_1}^{(n)}$ such that $f \leq h$.*

Proof. Without loss of generality, we assume that the permutation $\pi \in S_N$ taking f-wild subsets of N to g-wild sets is the identity on N. We take $g'' \in \langle g \rangle_{T_1}$ according to the preceding lemma. We wish to define $\gamma \in \mathscr{O}^{(1)}$ with $f \leq \gamma \circ g''$. For $x \in X$ write $U_x = g''^{-1}[\{x\}]$ for the preimage of x under g''. Now set

$$\gamma(x) = \begin{cases} \sup\{f(y) : y \in U_x\} & , U_x \neq \emptyset \\ 0 & , \text{otherwise} \end{cases}$$

We claim that γ is well-defined, that is, the supremum in its definition always exists in X. For suppose there is an $x \in X$ such that the set $\{f(y) : y \in U_x\}$ is unbounded. Choose a large subset $B \subseteq U_x$ making the sequence $(f(y) : y \in B)$ one-one. Take further a large $C \subseteq B$ so that all components in the sequence $(y : y \in C)$ are either one-one or constant. Set $A \subseteq N$ to consist of the indices of the injective components. Obviously, A is f-wild; therefore it is g''-insane. Since g'' is also monotone, the set $\{g''(y) : y \in C\}$ is large, contradicting the fact that g'' is constant on U_x. Thus, γ is well-defined and clearly $f \leq h \in \langle g \rangle_{T_1}$ where $h = \gamma \circ g''$. □

Proof of Theorem 20. The assertion is an immediate consequence of the preceding lemma and the fact that all clones above $\mathscr{U}$ are downward closed. □

Remark 26. Unfortunately, the converse does not hold: If $f, g \in \mathscr{O}^{(n)}$ and $f \in \langle g \rangle_{T_1}$ then it need not be true that $f \leq_W g$. We will see an example at the end of the section.

2.5. med_3 and T_1 generate $Pol(T_1)$.

We are now ready to prove the explicit description of $Pol(T_1)$.

Definition 27. For all $n \geq 1$ and all $1 \leq k \leq n$ we define a function

$$m_k^n(x_1, ..., x_n) = x_{j_k} \quad , \text{if } x_{j_1} \leq ... \leq x_{j_n}.$$

For example, m_n^n is the maximum function $\max_n$ and m_1^n the minimum function $\min_n$ in n variables. Note that $\min_n \in Pol(T_1)$ (it is even almost unary) but $\max_n \notin Pol(T_1)$ (and hence $\langle \max_n \rangle_{T_1} = \mathscr{O}$). If n is an odd number then we call $m^n_{\frac{n+1}{2}}$ the n-th median function and denote this function by med_n.

For fixed odd n it is easily verified (check the wild sets and apply Theorem 12) that med_n it is the largest of the m_k^n which still lies in $Pol(T_1)$: $m_k^n \in Pol(T_1)$ iff $k \leq \frac{n+1}{2}$. It is for this reason that we are interested in the median functions on our quest for a nice generating system of $Pol(T_1)$. As a consequence of the following theorem from [7] it does not matter which of the median functions we consider.

Theorem 28. *Let $k, n \geq 3$ be odd natural numbers. Then $\mathrm{med}_k \in \langle \{\mathrm{med}_n\} \rangle$. In other words, a clone contains either no median function or all median functions.*

The following lemma states that within the restrictions of functions of $Pol(T_1)$ (Fact 10), we can construct functions of arbitrary wildness with the median.

Lemma 29. *Let $n \geq 1$ and let $\mathscr{A} = \{A_1, ..., A_k\} \subseteq \mathscr{P}(N)$ be a set of subsets of N with the property that $A_i \cap A_j \neq \emptyset$ for all $1 \leq i, j \leq k$. Then there exists monotone $t_{\mathscr{A}} \in \langle\{\text{med}_3\}\rangle^{(n)}$ such that all members of $\mathscr{A}$ are $t_{\mathscr{A}}$-insane.*

Proof. We prove this by induction over the size k of $\mathscr{A}$. If $\mathscr{A}$ is empty there is nothing to show. If $k = 1$, we can set $t_{\mathscr{A}} = \pi_i^n$, where i is an arbitrary element of A_1. Then A_1 is obviously $t_{\mathscr{A}}$-insane. If $k = 2$, then define $t_{\mathscr{A}} = \pi_i^n$, where $i \in A_1 \cap A_2$ is arbitrary. Clearly, both A_1 and A_2 are $t_{\mathscr{A}}$-insane. Finally, assume $k \geq 3$. By induction hypothesis, there exist monotone terms $t_{\mathscr{B}}, t_{\mathscr{C}}, t_{\mathscr{D}} \in \langle\{\text{med}_3\}\rangle^{(n)}$ for the sets $\mathscr{B} = \{A_1, ..., A_{k-1}\}$, $\mathscr{C} = \{A_1, ..., A_{k-2}, A_k\}$ and $\mathscr{D} = \{A_{k-1}, A_k\}$ such that all sets in $\mathscr{B}$ (and $\mathscr{C}, \mathscr{D}$ respectively) are $t_{\mathscr{B}}$-insane ($t_{\mathscr{C}}$-insane, $t_{\mathscr{D}}$-insane). Set

$$t_{\mathscr{A}} = \text{med}_3(t_{\mathscr{B}}, t_{\mathscr{C}}, t_{\mathscr{D}}).$$

Then each A_i is insane for two of the three terms in med_3. Thus, if we fix the variables outside A_i to arbitrary values, then at least two of the three subterms in med_3 are still unbounded and so is $t_{\mathscr{A}}$ by the monotonicity of its subterms. Hence, every A_i is $t_{\mathscr{A}}$-insane, $1 \leq i \leq k$. Obviously $t_{\mathscr{A}}$ is monotone. □

Lemma 30. *Let $f \in Pol(T_1)^{(n)}$. Then there exists $t_f \in \langle\{\text{med}_3\}\rangle$ such that $f \leq_W t_f$.*

Proof. Write $\mathscr{A} = \{A_1, ..., A_k\}$ for the set of f-wild subsets of N. By Fact 10, $A_i \cap A_j \neq \emptyset$ for all $1 \leq i, j \leq k$. Apply the preceding lemma to $\mathscr{A}$. □

Theorem 31. $Pol(T_1) = \langle \text{med}_3 \rangle_{T_1}$.

Proof. It is clear that $Pol(T_1) \supseteq \langle \text{med}_3 \rangle_{T_1}$. On the other hand we have just seen that if $f \in Pol(T_1)$, then there exists $t_f \in \langle\{\text{med}_3\}\rangle$ such that $f \leq_W t_f$, whence $f \in \langle \text{med}_3 \rangle_{T_1}$. □

Corollary 32. *$Pol(T_1)$ is the $\leq$-downward closure of the clone generated by* med_3 *and the unary functions $\mathscr{O}^{(1)}$.*

Proof. Given $f \in Pol(T_1)$, by Lemma 30 there exists $t_f \in \langle\{\text{med}_3\}\rangle$ such that $f \leq_W t_f$. By Lemma 29, t_f is monotone and each t_f-wild set is in fact even t_f-insane. Now one follows the proof of Lemma 25 to obtain $\gamma \in \mathscr{O}^{(1)}$ such that $f \leq \gamma \circ t_f$. □

Corollary 33. *$Pol(T_1) = \langle\{\text{med}_3, p_\Delta\} \cup \mathscr{O}^{(1)}\rangle$. In particular, $Pol(T_1)$ is finitely generated over the unary functions.*

Proof. Remember that $\langle\{p_\Delta\} \cup \mathscr{O}^{(1)}\rangle = \langle T_1 \rangle$ (Fact 4) and apply Theorem 31. □

Now we can give the example promised in Remark 26. Set

$$g(x_1, ..., x_4) = \text{med}_3(x_1, x_2, x_3)$$

and

$$f(x_1, ..., x_4) = \mathrm{med}_5(x_1, x_1, x_2, x_3, x_4).$$

It is obvious that $\langle g\rangle_{T_1} = \langle \mathrm{med}_3\rangle_{T_1} = Pol(T_1)$. Next observe that $\langle f\rangle_{T_1} \subseteq \langle \mathrm{med}_5\rangle_{T_1} = Pol(T_1)$ and that $f(x_1, x_2, x_3, x_3) = \mathrm{med}_3$ which implies $Pol(T_1) = \langle \mathrm{med}_3\rangle_{T_1} \subseteq \langle f\rangle_{T_1}$. Thus, $\langle g\rangle_{T_1} = \langle f\rangle_{T_1}$. Consider on the other hand the 2-element wild sets of the two functions: Exactly $\{1, 2\}, \{1, 3\}$ and $\{2,3\}$ are g-wild, and $\{1, 2\}, \{1, 3\}, \{1, 4\}$ are the wild sets of two elements for f. Now the intersection of first group is empty, whereas the one of the second group is not; so there is no permutation of the set $\{1, 2, 3, 4\}$ which takes the first group to the second or the other way. Hence, neither $f \leq_W g$ nor $g \leq_W f$.

3. The interval $[\mathscr{U}, \mathscr{O}]$

3.1. A chain in the interval. Now we shall show that the open interval $(\langle T_1\rangle, Pol(T_1))$ is not empty by exhibiting a countably infinite descending chain therein with intersection $\mathscr{U}$.

Notation 34. For a natural number $n \geq 2$, we write $\mathscr{M}_n = \langle\{m_2^n\} \cup T_1\rangle$.

Observe that since $m_2^2 = \max_2 \notin Pol(T_1)$, Theorem 1 implies that $\mathscr{M}_2 = \mathscr{O}$. Moreover, $m_2^3 = \mathrm{med}_3$ and hence, $\mathscr{M}_3 = Pol(T_1)$.

Lemma 35. *Let $n \geq 2$. Then $\mathscr{M}_n^{(k)} = \mathscr{U}^{(k)}$ for all $1 \leq k < n$. That is, all functions in $\mathscr{M}_n$ of arity less than n are almost unary.*

Proof. Given n, k we show by induction over terms that if $t \in \mathscr{M}_n^{(k)}$, then t is almost unary. To start the induction we note that the only k-ary functions in the generating set of $\mathscr{M}_n$ are almost unary. Now assume $t = f(t_1, t_2)$, where $f \in T_1$ and $t_1, t_2 \in \mathscr{M}_n^{(k)}$. By induction hypothesis, t_1 and t_2 are almost unary and so is t as the almost unary functions are closed under composition. Finally, say $t = m_2^n(t_1, ..., t_n)$, where the t_i are almost unary k-ary functions, $1 \leq i \leq n$. Since $k < n$, there exist $i, j \in N$ with $i \neq j$, $l \in \{1, ..., k\}$ and $\gamma, \delta \in \mathscr{O}^{(1)}$ such that $t_i \leq \gamma(x_l)$ and $t_j \leq \delta(x_l)$. Then, $t \leq \max(\gamma, \delta)(x_l)$ and so t is almost unary as well. ☐

Corollary 36. *If $n \geq 2$, then $m_2^n \notin \mathscr{M}_{n+1}$. Consequently, $\mathscr{M}_n \nsubseteq \mathscr{M}_{n+1}$.*

Lemma 37. *If $n \geq 2$, then $m_2^{n+1} \in \mathscr{M}_n$. Consequently, $\mathscr{M}_{n+1} \subseteq \mathscr{M}_n$.*

Proof. Set

$$f(x_1, ..., x_{n+1}) = m_2^n(x_1, ..., x_n) \in \mathscr{M}_n.$$

Then every n-element subset of $\{1, ..., n + 1\}$ is f-wild. Hence, $m_2^{n+1} \leq_W f$ and so $m_2^{n+1} \in \langle f\rangle_{T_1} \subseteq \mathscr{M}_n$. ☐

Theorem 38. *The sequence $(\mathscr{M}_n)_{n\geq 2}$ forms a countably infinite descending chain:*

$$\mathscr{O} = \mathscr{M}_2 \supsetneqq \mathscr{M}_3 = Pol(T_1) \supsetneqq \mathscr{M}_4 \supsetneqq ... \supsetneqq \mathscr{M}_n \supsetneqq \mathscr{M}_{n+1} \supsetneqq ...$$

Moreover,

$$\bigcap_{n\geq 2} \mathscr{M}_n = \mathscr{U}.$$

Proof. The first statement follows from Corollary 36 and Lemma 37. The second statement a direct consequence of Lemma 35. □

3.2. Finally, this is the interval. We will now prove that there are no more clones in the interval $[\mathscr{U}, \mathscr{O}]$ than the ones we already exhibited. We first state a technical lemma.

Lemma 39. *Let $f \in \mathscr{O}^{(n)}$ be a monotone function such that all f-wild subsets of N are f-insane. Define for $i, j \in N$ with $i \neq j$ functions*

$$f^{(i,j)}(x_1, ..., x_n) = f(x_1, ..., x_{i-1}, x_j, x_{i+1}, ..., x_n)$$

which replace the i-th by the j-th component and calculate f. Then the following implications hold for all f-wild $A \subseteq N$ and all $i, j \in N$ with $i \neq j$:

(i) *If $i \notin A$, then A is $f^{(i,j)}$-insane.*
(ii) *If $j \in A$, then A is $f^{(i,j)}$-insane.*

Proof. We have to show that if we fix the variables outside A to constant values, then $f^{(i,j)}$ is still unbounded; because f is monotone, we can assume all values are fixed to 0. Fix a sequence $(\alpha_\xi : \xi \in X)$ of elements of X^n such that all components outside A are zero for all tuples of the sequence and such that $(f(\alpha_\xi) : \xi \in X)$ is unbounded. Define a sequence of n-tuples $(\beta_\xi : \xi \in X)$ by

$$(\beta_\xi)_k^n = \begin{cases} 0 & , k \notin A \\ \xi & , \text{otherwise} \end{cases}$$

For each $\xi \in X$ there exist a $\lambda \in X$ such that $\alpha_\xi \leq \beta_\lambda$. Then $f(\alpha_\xi) \leq f(\beta_\lambda)$. In either of the cases (i) or (ii), $f(\beta_\lambda) \leq f^{(i,j)}(\beta_\lambda)$. Thus, $(f^{(i,j)}(\beta_\xi) : \xi \in X)$ is unbounded. □

Lemma 40. *Let $f \in \mathscr{O}^{(n)}$ not almost unary. Then there exists $n_0 \geq 2$ such that $\langle f\rangle_{T_1} = \langle m_2^{n_0}\rangle_{T_1}$.*

Proof. We shall prove this by induction over the arity n of f. If $n = 1$, there are no not almost unary functions so there is nothing to show. Now assume our assertion holds for all $1 \leq k < n$. We distinguish two cases:

First, consider f such that all f-wild subsets of N have size at least $n-1$. Then $f \sim_W m_2^n$ and so $\langle f\rangle_{T_1} = \langle m_2^n\rangle_{T_1}$.

Now assume there exists an f-wild subset of N of size $n-2$, say without loss of generality that $\{2, ..., n-1\}$ is such a set. By Lemma 24 and Theorem 20 there exists a monotone $\hat{f}$ with $\langle f\rangle_{T_1} = \langle \hat{f}\rangle_{T_1}$ and with the property that all f-wild subsets of N are $\hat{f}$-insane. Since we could replace f by $\hat{f}$, we assume that f is monotone and that all f-wild sets are f-insane.

Consider the $f^{(i,j)}$ as defined in the preceding lemma. Formally, these functions are still n-ary, but in fact they depend only on $n-1$ variables. Thus, all of the $f^{(i,j)}$ which are not almost unary satisfy the induction hypothesis. Set

$$n_0 = \min\{k : \exists i, j \in N \, \langle f^{(i,j)}\rangle_{T_1} = \langle m_2^k\rangle_{T_1}\}.$$

The minimum is well-defined: Because $\{2, ..., n-1\}$ is f-insane, $f^{(n,1)}$ is not almost unary so that it generates the same clone as some m_2^n modulo T_1; thus, the set is not empty. Clearly, $m_2^{n_0} \in \langle f\rangle_{T_1}$. We show that $m_2^{n_0}$ is strong enough to generate f. Since $\mathscr{M}_n \subseteq \mathscr{M}_{n_0}$ for all $n \geq n_0$ we have $f^{(i,j)} \in \langle m_2^{n_0}\rangle_{T_1}$ for all $i, j \in N$ with $i \neq j$. Now define

$$t(x_1, ..., x_n) = f^{(n,1)}(x_1, f^{(1,2)}, f^{(1,3)}, ..., f^{(1,n-1)}) \in \langle m_2^{n_0}\rangle_{T_1}.$$

We claim that $f \leq_W t$. Indeed, let $A \subseteq N$ be f-wild and whence f-insane by our assumption.

If $1 \notin A$, then A is $f^{(1,j)}$-insane for all $2 \leq j \leq n-1$ by the preceding lemma. So A is insane for all components in the definition of t except the first one. Hence, because f is monotone, A must be t-insane as otherwise $f^{(n,1)}$ would be almost unary.

If $1 \in A$, then by the preceding lemma A is still $f^{(1,j)}$-insane whenever $j \in A$. Thus, increasing the components with index in A increases the first component in t plus all subterms $f^{(1,j)}$ with $j \in A$; but by the definition of $f^{(n,1)}$, that is the same as increasing the variables $A \cup \{n\} \supseteq A$ in f. Whence, A is t-insane.

This proves $f \leq_W t$ and thus $f \in \langle m_2^{n_0}\rangle_{T_1}$. □

So here it is, the interval and the end of our quest.

Theorem 41. *Let $\mathscr{C} \supsetneqq \mathscr{U}$ be a clone. Then there exists $n \geq 2$ such that $\mathscr{C} = \mathscr{M}_n$.*

Proof. Set

$$n_{\mathscr{C}} = \min\{n \geq 2 : \mathscr{M}_n \subseteq \mathscr{C}\}.$$

Since $\mathscr{C}$ contains a function which is not almost unary, the preceding lemma implies that the set over which we take the minimum is nonempty. Obviously, $\mathscr{M}_{n_{\mathscr{C}}} \subseteq \mathscr{C}$. Now let f be an arbitrary function in $\mathscr{C}$ which is not almost unary. Then by the preceding lemma, there exists n_0 such that $\langle m_2^{n_0}\rangle_{T_1} = \langle f\rangle_{T_1}$. Clearly, $n_0 \geq n_{\mathscr{C}}$ so that $f \in \mathscr{M}_{n_0} \subseteq \mathscr{M}_{n_{\mathscr{C}}}$. □

We state a lemma describing how the k-ary parts of the $\mathscr{M}_n$ for arbitrary k relate to each other.

Lemma 42. *Let $m > n \geq 2$ and $k \geq 2$. If $k \geq n$ (that is, if $\mathscr{M}_n^{(k)}$ is nontrivial), then $\mathscr{M}_n^{(k)} \supsetneqq \mathscr{M}_m^{(k)}$.*

Proof. We know that $\mathscr{M}_n^{(k)} \supseteq \mathscr{M}_m^{(k)}$. To see the inequality of the two sets, observe that

$$f(x_1, ..., x_k) = m_2^n(x_1, ..., x_n)$$

is an element of $\mathscr{M}_n^{(k)}$ but definitely not one of $\mathscr{M}_m^{(k)}$. □

Corollary 43. *Let $k \geq 2$. Then*

$$\mathscr{M}_2^{(k)} \underset{\neq}{\supseteq} \mathscr{M}_3^{(k)} \underset{\neq}{\supseteq} ... \underset{\neq}{\supseteq} \mathscr{M}_k^{(k)} \underset{\neq}{\supseteq} \mathscr{M}_{k+1}^{(k)} = \mathscr{U}^{(k)}$$

Consequently, there are k different k-ary parts of clones of the interval $[\mathscr{U}, \mathscr{O}]$ for each k.

In general, if $\mathscr{C}$ is a clone, then

$$Pol(\mathscr{C}^{(1)}) \supseteq Pol(\mathscr{C}^{(2)}) \supseteq ... \supseteq Pol(\mathscr{C}^{(n)}) \supseteq ...$$

Moreover,

$$Pol(\mathscr{C}^{(n)})^{(n)} = \mathscr{C}^{(n)} \quad \text{and} \quad \bigcap_{n \geq 1} Pol(\mathscr{C}^{(n)}) = \mathscr{C}.$$

It is natural to ask whether or not for $\mathscr{C} = \mathscr{U}$ this chain coincides with the chain we discovered.

Theorem 44. *Let $n \geq 1$. Then $\mathscr{M}_{n+1} = Pol(\mathscr{U}^{(n)})$.*

Proof. Clearly, $\mathscr{M}_2 = Pol(\mathscr{U}^{(1)}) = \mathscr{O}$, so assume $n \geq 2$. Consider m_2^{n+1} and let $f_1, ..., f_{n+1}$ be functions in $\mathscr{U}^{(n)}$. Then two of the f_j are bounded by unary functions of the same variable. Thus $m_2^{n+1}(f_1, ..., f_{n+1})$ is bounded by a unary function of this variable. This shows $m_2^{n+1} \in Pol(\mathscr{U}^{(n)})$ and hence $\mathscr{M}_{n+1} \subseteq Pol(\mathscr{U}^{(n)})$. Now consider m_2^n and observe that $m_2^n \notin \mathscr{U}^{(n)} = Pol(\mathscr{U}^{(n)})^{(n)}$; this proves $\mathscr{M}_n \nsubseteq Pol(\mathscr{U}^{(n)})$. Whence, $\mathscr{M}_{n+1} = Pol(\mathscr{U}^{(n)})$. □

3.3. The m_k^n in the chain. As an example, we will show where the clones generated by the m_k^n (as in Definition 27) and T_1 can be found in the chain.

Notation 45. For $1 \leq k \leq n$ we set $\mathscr{M}_n^k = \langle m_k^n \rangle_{T_1}$.

Note that if $k = 1$, then $\mathscr{M}_n^k = \mathscr{U}$, and if $k > \frac{n+1}{2}$, then $\mathscr{M}_n^k = \mathscr{O}$. Observe also that $\mathscr{M}_n = \mathscr{M}_n^2$ for all $n \geq 2$.

Notation 46. For a positive rational number q we write

$$\lfloor q \rfloor = \max\{n \in \mathbb{N} : n \leq q\}$$

and

$$\lceil q \rceil = \min\{n \in \mathbb{N} : q \leq n\}.$$

The remainder of the division $\frac{n}{k}$ we denote by the symbol $R(\frac{n}{k})$.

Lemma 47. *Let $2 \leq k \leq \frac{n+1}{2}$ and let $t \in \mathscr{M}_n^k$ not almost unary. Then all t-wild subsets of N_t have size at least $\frac{n}{k-1} - 1$.*

Proof. Our proof will be by induction over terms. If $t = m^n_k$, then all t-wild subsets of $N_t = N$ have at least $n-k+1$ elements in accordance with our assertion. For the induction step, assume $t = f(t_1, t_2)$, where $f \in T_1$, say $f(x_1, x_2) \leq \gamma(x_1)$ for some $\gamma \in \mathscr{O}^{(1)}$. Then t inherits the asserted property from t_1. Finally we consider the case where $t = m^n_k(t_1, ..., t_n)$. Suppose towards contradiction there exists $A \subseteq N_t$ t-wild with $|A| < \frac{n}{k-1} - 1$. There have to be at least $n-k+1$ terms t_j for which A is t_j-wild so that A can be t-wild. By induction hypothesis, these $n-k+1$ terms are almost unary and bounded by a unary function of a variable with index in A. From the bound on the size of A we conclude that at least

$$\lceil \frac{n-k+1}{|A|} \rceil > \frac{n-k+1}{\frac{n}{k-1}-1} = k-1$$

of the terms t_j are bounded by an unary function of the same variable with index in A. But if k of the t_j have the same one-element strong set, then t is bounded by a unary function of this variable as well in contradiction to the assumption that t is not almost unary. □

Corollary 48. *Let $2 \leq k \leq \frac{n+1}{2}$. Then $\mathscr{M}_{\lceil \frac{n}{k-1} \rceil - 1} \nsubseteq \mathscr{M}^k_n$.*

Proof. With the preceding lemma it is enough to observe that $m_2^{\lceil \frac{n}{k-1} \rceil - 1} \in \mathscr{M}_{\lceil \frac{n}{k-1} \rceil - 1}$ has a wild set of size $\lceil \frac{n}{k-1} \rceil - 2$. □

So we identify now the $\mathscr{M}_j$ which $\mathscr{M}^k_n$ is equal to.

Lemma 49. *Let $2 \leq k \leq n$. Then $\mathscr{M}_{\lceil \frac{n}{k-1} \rceil} \subseteq \mathscr{M}^k_n$.*

Proof. It suffices to show that m^n_k generates $m_2^{\lceil \frac{n}{k-1} \rceil}$. But this is easy:

$$m_2^{\lceil \frac{n}{k-1} \rceil} = m^n_k(x_1, ..., x_1, x_2, ..., x_2, ..., x_{\lceil \frac{n}{k-1} \rceil}, ..., x_{\lceil \frac{n}{k-1} \rceil}),$$

where x_j occurs $k-1$ times if $1 \leq j \leq \lfloor \frac{n}{k-1} \rfloor$ and $R(\frac{n}{k-1}) < k-1$ times if $j = \lfloor \frac{n}{k-1} \rfloor + 1$. For if we evaluate the function for a $\lceil \frac{n}{k-1} \rceil$-tuple with $x_{j_1} \leq ... \leq x_{j_{\lceil \frac{n}{k-1} \rceil}}$, then x_{j_1} occurs at most $k-1$ times in the tuple, but x_{j_1} together with x_{j_2} occur more than k times; thus, the k-th smallest element in the tuple is x_{j_2} and m^n_k returns x_{j_2}. □

Theorem 50. *$\mathscr{M}^k_n = \mathscr{M}_{\lceil \frac{n}{k-1} \rceil}$ for all $2 \leq k \leq n$.*

Proof. By Theorem 41, $\mathscr{M}^k_n$ has to be somewhere in the chain $(\mathscr{M}_n)_{n \geq 2}$. Because of Corollary 48 and Lemma 49 the assertion follows. □

3.4. Further on the chain. We conclude by giving one simple guideline for where to search the clone $\langle f \rangle_{T_1}$ in the chain for arbitrary $f \in \mathscr{O}$.

Lemma 51. *Let $1 \leq k \leq n$ and let $f \in \mathscr{O}^{(n)}$ be a not almost unary function which has a k-element f-wild subset of N. Then $\mathscr{M}_{k+1} \subseteq \langle f \rangle_{T_1}$.*

Proof. We can assume that $\{1, ..., k\}$ and all $A \subseteq N$ with $|A| = n - 1$ are f-insane and that f is monotone. Define

$$g(x_1, ..., x_{k+1}) = f(x_1, ..., x_k, x_{k+1}, ..., x_{k+1}) \in \langle f \rangle_{T_1}.$$

Let $A \subseteq \{1, ..., k+1\}$ with $|A| = k$ be given. If $A = \{1, ..., k\}$ then A is f-wild and so it is g-wild. Otherwise A contains $k+1$ and so it affects $n-1$ components in the definition of g. Therefore A is g-wild by Lemma 9. Hence, $m_2^{k+1} \leq_W g$ and so $\mathscr{M}_{k+1} \subseteq \langle g \rangle_{T_1} \subseteq \langle f \rangle_{T_1}$. □

Remark 52. Certainly it is not true that if the smallest wild set of a function $f \in \mathscr{O}$ has k elements, then $\mathscr{M}_{k+1} = \langle f \rangle_{T_1}$. The m_k^n are an example.

Corollary 53. *Let $f \in Pol(T_1)$ not almost unary and such that there exists a 2-element f-wild subset of N. Then $\langle f \rangle_{T_1} = Pol(T_1)$.*

4. Summary and a nice picture

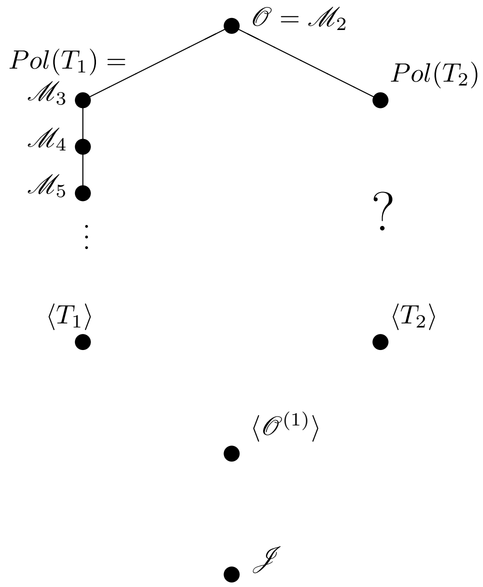
We summarize the main results of this paper: For the interval of clones containing the almost unary functions we have $[\mathscr{U}, \mathscr{O}] = \{\mathscr{M}_2, \mathscr{M}_3, ..., \mathscr{U}\}$, where the $\mathscr{M}_n = \langle \{m_2^n\} \cup \mathscr{U} \rangle = \langle \{m_2^n, p_\Delta\} \cup \mathscr{O}^{(1)} \rangle$ are all finitely generated over $\mathscr{O}^{(1)}$. Alternatively, $\mathscr{M}_n$ can be described as the $\leq$-downward closure of $\langle \{m_2^n\} \cup \mathscr{O}^{(1)} \rangle$. The interval is a chain: $\mathscr{M}_2 = \mathscr{O}^{(1)} \supsetneqq \mathscr{M}_3 = Pol(T_1) \supsetneqq \mathscr{M}_4 \supsetneqq ...$ and $\bigcap_{n \geq 2} \mathscr{M}_n = \mathscr{U}$. Together with the fact that $\mathscr{M}_{n+1} = Pol(\mathscr{U}^{(n)})$ for all $n \geq 1$ this yields that $\mathscr{U}$ an example of a clone $\mathscr{C}$ for which the chain $Pol(\mathscr{C}^{(1)}) \supseteq Pol(\mathscr{C}^{(2)}) \supseteq ... \supseteq \mathscr{C}$ is unrefinable and collapses nowhere. $\mathscr{U}$ is a so-called binary clone, that is, $\langle \mathscr{U}^{(2)} \rangle = \mathscr{U}$.

The $\mathscr{M}_n$ have the property that $\mathscr{M}_n^{(k)} = \mathscr{U}^{(k)}$ whenever $1 \leq k < n$. Furthermore, $\mathscr{M}_n^{(k)} \supsetneqq \mathscr{M}_m^{(k)}$ whenever $m > n \geq 2$ and $k \geq n$. Consequently, for each $k \geq 1$ there exist exactly k different k-ary parts of clones of the interval $[\mathscr{U}, \mathscr{O}]$.

Using wildness, a notion which completely determines a function modulo $\mathscr{U}$, it is possible to calculate for all $2 \leq k \leq n$ that $\mathscr{M}_n^k = \langle \{m_k^n\} \cup \mathscr{U} \rangle = \mathscr{M}_{\lceil \frac{n}{k-1} \rceil}$. In general, if one knows the wild subsets of $\{1, ..., n\}$ of a function $f \in \mathscr{O}^{(n)}$, he can draw certain conclusions about where to find the clone $\langle \{f\} \cup \mathscr{U} \rangle$ in the chain.

On countable X, if we equip $\mathscr{O}$ with the natural topology, then the sets T_1 and $Pol(T_1)$ are Borel sets of low complexity, as opposed to the sets T_2 and $Pol(T_2)$ which have been shown by M. Goldstern to be non-analytic. In fact, with the results of this paper, all clones above the almost unary functions can be shown to be Borel.

If X is countably infinite or weakly compact, we can draw the situation we ran into like this.



$$[\langle T_1 \rangle, \mathscr{O}] = \{\langle T_1 \rangle, ..., \mathscr{M}_3, \mathscr{M}_2\}$$

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