Some Observations on Minimal Clones

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Abstract

A minimal clone is an atom of the lattice of clones. We propose a new method to study minimal clones by considering the base set E_k with k elements as a finite field and by expressing each function as a polynomial over E_k.

For $k = 3$ we present the list of all binary minimal polynomials over GF(3) derived from Csákány's result. Then, we discuss some properties of binary minimal linear polynomials and of binary minimal monomials.

Keywords: Clone; minimal clone; Galois field

1 Introduction

Let $k(> 2)$ be a fixed integer and $E_k = \{0, 1, \ldots, k-1\}$. Denote by $\mathcal{O}_k^{(n)}$ the set of all functions from $(E_k)^n$ into E_k, and set

$$\mathcal{O}_k = \bigcup_{n=1}^{\infty} \mathcal{O}_k^{(n)}.$$

For any $f \in O_k^{(n)}$ and any $g_1, \ldots, g_n \in O_k^{(m)}$ the (functional) *composition* $f[g_1, \ldots, g_n]$ of f and $g_1, \ldots, g_n$ is a function in $O_k^{(m)}$ defined by

$$\begin{aligned} & f[g_1, \ldots, g_n](x_1, \ldots, x_m) \\ = \; & f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)) \end{aligned}$$

for all $x_1, \ldots, x_m \in E_k$.

Denote by $\mathcal{J}_k$ the set of all *projections* e_i^n over E_k, where e_i^n for every $1 \leq i \leq n$ is defined by $e_i^n(x_1, \ldots, x_i, \ldots, x_n) = x_i$ for all $x_1, \ldots, x_n \in E_k$.

A subset C of $\mathcal{O}_k$ is a *clone* on E_k if (i) C is closed under (functional) composition and (ii) C contains $\mathcal{J}_k$. Denote by $\mathcal{L}_k$ the set of all clones on E_k. The set $\mathcal{L}_k$ is a lattice with respect to inclusion and called the *lattice of clones* on E_k.

An atom of the lattice $\mathcal{L}_k$ is called a *minimal clone*. In other words, a minimal clone is a minimal element in the partially ordered set $\mathcal{L}_k \setminus \{\mathcal{J}_k\}$. It is clear that a minimal clone C is generated by a single function f in $\mathcal{O}_k$, i.e., $C = \langle f \rangle$. A function in $\mathcal{O}_k$ which generates a minimal clone is called a *minimal function*.

For $k = 2$, the structure of $\mathcal{L}_2$ is completely known by E. Post [Po 41] and, consequently, all minimal clones are known. There are 7 minimal clones in $\mathcal{L}_2$.

For $k = 3$, B. Csákány [Cs 83] discovered all minimal clones and listed generators, i.e., minimal functions, for all of them. There are 84 minimal clones in $\mathcal{L}_3$.

For $k = 4$, B. Szczepara [Szc 95] found all minimal clones which are generated by binary functions. However, the determination of all minimal clones in $\mathcal{L}_4$ is not yet settled.

Other than the above mentioned works, minimal clones and minimal functions have been discussed by several authors: J. Dudek, J. Ježek, K. A. Kearnes, P. P. Pálfy, R. W. Quackenbush, I. G. Rosenberg, Á. Szendrei, T. Waldhauser, etc. Many of their results contain interesting aspects of and deep insight to minimal clones. However, the problem of determining all minimal clones seems to be extremely difficult. Even the determination of minimal clones for any single k greater than 3 is far beyond our present capability.

In this paper, we propose a new method to attack this problem of determining minimal functions. If k is a power of a prime number, i.e., $k = p^e$ for some prime p and $e \geq 1$, we can introduce the algebraic structure of a field into the set E_k and E_k may be viewed as a finite field (Galois field) GF(p^e). An n-ary function on E_k is then expressed as a polynomial in n variables over GF(p^e). Polynomials corresponding to minimal functions may have some common characteris-

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tics which distinguish them from polynomials that are not minimal functions. Although we have not yet fulfilled, the aim of our work, which has just been started, is to discover such common characteristics embraced by minimal functions. Our hope is to find common characteristics with respect to the form of polynomials shared by all, or possibly some, of minimal functions.

In the sequel, we shall concentrate on polynomials of binary idempotent minimal functions. Here a function $f \in \mathcal{O}_k$ is *idempotent* if it satisfies $f(x, \ldots, x) = x$ for all $x \in E_k$.

As an intial step of this work, we present the list of polynomials over GF(3) in 2 variables corresponding to generators of all minimal clones generated by binary functions, which is derived from Csákány's list [Cs 83] and, then, determine all binary minimal linear polynomials as well as all binary minimal monomials.

2 Type Theorem for Minimal Clones

The following theorem due to I. G. Rosenberg [Ro 86] is sometimes called the *type theorem* for minimal clones.

Theorem 2.1 *A minimal function f on E_k whose arity is minimum among arities of functions in $\langle f \rangle \setminus \mathcal{J}_k$ is of one of the following five types.*

(i) *f is a unary function.*

(ii) *f is a binary idempotent function.*

(iii) *f is a ternary majority function. I.e., f satisfies*

$$f(x, x, y) = f(x, y, x) = f(y, x, x) = x$$

for all $x, y \in E_k$.

(iv) *f is a ternary function defined by*

$$f(x, y, z) = x + y + z$$

for an elementary abelian 2-group with addition $+$.

(v) *f is a $k\,(\geq 3)$-ary semiprojection. I.e., there exists $1 \leq i \leq n$ such that*

$$f(x_1, \ldots, x_n) = x_i$$

whenever $|\{x_1, \ldots, x_n\}| < n$ for any $x_1, \ldots, x_n \in E_k$.

In this paper we shall only consider minimal functions of type (ii), i.e., binary idempotent minimal functions.

3 Polynomials on Finite Fields

A field $\mathbf{F}$ consisting of a finite number of elements is called a *finite field* or *Galois field.* It is well-known that the cardinality of a finite field must be a power p^e of some prime number p. When $\mathbf{F}$ is a finite field with $q = p^e$ elements, it is often denoted by GF(q). It is a basic fact that the finite field GF(q) consists of elements x satisfying $x^q = x$.

In particular, assume that $k = p$ is a prime and E_k is a prime field. Then the addition and the multiplication of GF(p) are exactly the addition modulo p and the multiplication modulo p, respectively, and computation can be performed easily.

In the sequel, we shall consider binary functions on E_k. For a prime k, it is known that a function $f \in \mathcal{O}_k^{(2)}$ can be expressed as a polynomial

$$f(x, y) = \sum_{0 \leq i, j < k} a_{ij}\, x^i\, y^j$$

for some $a_{ij} \in E_k$ $(0 \leq i,\, j < k)$.

Note that each $f \in \mathcal{O}_k^{(2)}$ has a unique expression as a polynomial.

4 Csákány's Theorem for $k = 3$

In this section we consider binary idempotent minimal functions on a three element set, i.e., minimal functions of type (ii) of Theorem 2.1 for $k = 3$.

Due to B. Csákány [Cs 83], generators of all minimal clones for $k = 3$ are known. For all minimal clones generated by functions of type (ii), we have expressed generators, i.e., binary minimal functions, as polynomials considering the base set E_3 as the finite field GF(3). In Appendix 1, they are presented in the order of the number of terms in a polynomial. The names of the functions such as b_{11}, b_0, b_{68} etc. come from the Csákány's naming. From Appendix 1 we could observe some properties and draw some conjectures on minimal functions which will be stated in the following section.

The permutations of the base set E_3, that is, the renaming of the elements of E_3, can be expressed by the following mappings:

$$\begin{aligned}
\varphi_0 = \mathrm{id} &\;:\; x \longmapsto x \\
\varphi_1 = (012) &\;:\; x \longmapsto x+1 \\
\varphi_2 = (021) &\;:\; x \longmapsto x+2 \\
\varphi_3 = (12) &\;:\; x \longmapsto 2x \\
\varphi_4 = (01) &\;:\; x \longmapsto 2x+1 \\
\varphi_5 = (02) &\;:\; x \longmapsto 2x+2
\end{aligned}$$

Two polynomials are *equivalent* if one is obtained from the other by some permutation on E_3. More precisely, $f(x,y)$ is equivalent to $g(x,y)$ if $g(x,y) = \varphi^{-1}f(\varphi(x),\varphi(y))$ for some permutation φ on E_3. Appendix 2 presents the list of representatives from each equivalence classes of minimal functions of type (ii).

5 Binary Idempotent Minimal Functions

Throughout this section, we assume that k is a prime and E_k is the finite field $\mathrm{GF}(k)$. We shall consider only binary idempotent minimal functions on E_k.

5.1 Irreducible Polynomials

It is obvious that an irreducible polynomial cannot be obtained from reducible polynomials through (functional) composition. Therefore, if some reducible polynomial g is produced by repeated application of composition from an irreducible polynomial f then it immediately implies that f is not a minimal function. Hence we have:

Lemma 5.1 *Let f be an irreducible polynomial over E_k and g be a reducible polynomial over E_k. If $g \in \langle f \rangle$ then f is not a minimal function.*

Example. Let $k = 3$. Consider $f(x,y) = x + x^2 + 2y^2$ as a polynomial over E_3. Clearly, f is an irreducible polynomial over E_3, which is idempotent as well. Define $g(x,y) = f(f(x,y),y)$. Then we have $g(x,y) = xy^2 + x^2y^2 + 2y^2$ which is reducible, since $g(x,y) = (x + x^2 + 2)y^2$. Hence it is concluded by Lemma 5.1 that f is not a minimal function.

5.2 Linear Polynomials

A *linear* polynomial $f(x_1,\dots,x_n)$ on E_k is a polynomial of the form

$$f(x_1,\dots,x_n) \;=\; a_0 + a_1x_1 + \cdots + a_nx_n$$

for some $a_0, a_1, \dots, a_n \in E_k$.

For $k = 3$, we see from the list in Appendix 1 that the only binary linear polynomial on E_3 which is minimal is $2x + 2y$.

In general, we have the following.

Lemma 5.2 *Let $f(x,y) = ax + by + c$ be a linear polynomial on E_k for $a, b, c \in E_k$. If f is idempotent then $a + b \equiv 1 \pmod{k}$ and $c = 0$.*

The proof follows from $f(0,0) = 0$ and $f(1,1) = 1$.

Note that Lemma 5.2 is a special case of a more general result given below.

Proposition 5.3 *Let $f(x,y)$ be a polynomial over E_k expressed as*

$$f(x,\,y) = \sum_{0 \le i,\,j < k} a_{ij}\, x^{\,i}\, y^{\,j}$$

for some $a_{ij} \in E_k$ $(0 \le i,\, j < k)$. For every $0 \le \ell < k$ define α_ℓ as follows: $\alpha_0 = a_{00}$ and α_ℓ for $0 < \ell < k$ is the sum of a_{ij} for which $i + j \equiv \ell \pmod{k-1}$, i.e., $\alpha_\ell = \sum_{i+j\equiv\ell \pmod{k-1}} a_{ij}$. If $f(x,y)$ is idempotent then

$$\begin{aligned}
\ell = 1 &\implies \alpha_\ell \equiv 1 \pmod{k} \\
\ell \ne 1 &\implies \alpha_\ell \equiv 0 \pmod{k}
\end{aligned}$$

This can be verified by (i) the assumption $f(x,x) = x$, (ii) the basic property $x^k = x$ and (iii) the uniqueness of polynomial-form expression of a function for a prime k.

Experiment: For small primes such as $k = 3, 5, 7$ and 11, one can check the following by easy computetion. The following binary linear polynomials are minimal and no other binary linear polynomials are minimal (up to interchange of variables).

$k = 3$: $2x + 2y$

$k = 5$: $2x + 4y$, $3x + 3y$
(Both generate the same minimal clone.)

$k = 7$: $2x + 6y$, $3x + 5y$, $4x + 4y$
(All generate the same minimal clone.)

$k = 11$: $ax + (12 - a)y$ for all $1 < a < 11$
(All generate the same minimal clone.)

Example. For $k = 5$, the binary linear polynomial $f(x, y) = 2x + 4y$ can be expressed by the Cayley table as follows:

$f(x, y) = 2x + 4y$

$x \backslash y$	0	1	2	3	4
0	0	4	3	2	1
1	2	1	0	4	3
2	4	3	2	1	0
3	1	0	4	3	2
4	3	2	1	0	4

From the above observation, we are lead to conjecture more generally that a similar situation holds for any prime k. In fact, this turns out to be true from the result of Á. Szendrei [Sze 05].

Theorem 5.4 *Let k be a prime. Let $f(x, y)$ be a binary linear polynomial on E_k. Then f is minimal if and only if*

$$f(x, y) = ax + (k + 1 - a)y$$

for some $1 < a < k$.

5.3 Monomials

Now we consider the monomials on E_k which are minimal. A *monomial* $f(x_1, \ldots, x_n)$ on E_k is a polynomial consisting of a single term, i.e., it is a polynomial of the form

$$f(x_1, \ldots, x_n) = a\, x_1^{i_1} \cdots x_n^{i_n}$$

for some $a, i_1, \ldots, i_n \in E_k$.

For the case of $k = 3$, from Appendix 1 we see that the only binary monomial on E_3 which is minimal is the monomial $x\, y^2$ (up to interchange of variables).

For a prime k, we shall consider a binary monomial

$$f(x, y) = x^s\, y^t$$

for $1 \le s \le t < k$.

Lemma 5.5 *For any $c, s, t \in E_k$, let $f(x, y)$ be a monomial $f(x, y) = c\, x^s y^t$. If f is idempotent then* (i) $c = 1$ *and* (ii) $s + t = k$.

Proof. (i) Obvious. (Substitute $x = y = 1$ to $f(x, y)$.)
(ii) This follows from the basic fact on a finite field that $x^k = x$ for every $x \in E_k$. □

For the minimality of a binary monomial, we have a following example.

Proposition 5.6 *If $f(x, y) = x\, y^{k-1}$ then f is minimal.*

Example. For $k = 5$, $f(x, y) = x\, y^4$ is a function which can be expressed by the Cayley table as follows:

$f(x, y) = x\, y^4$

$x \backslash y$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	2	2	2	2
3	0	3	3	3	3
4	0	4	4	4	4

Proof of Proposition 5.6. First, notice that 2-variable functions obtained from f by repeated application of functional composition are either $f(x, y)$ or $f(y, x)$. This can be verified in a straghtforward way: E.g., $f(f(x, y), y) = f(x, y)$, $f(f(y, x), y) = f(y, x)$, $f(x, f(x, y)) = f(x, y)$, etc.

Secondly, one might get $n(> 2)$-variable function $g(x_1, \ldots, x_n)$ by repeated application of functional composition from f. Let $h(x, y)$ be a 2-variable function obtained from g, e.g., $h(x, y) = g(x, y, \ldots, y)$. Then, since f is a monomial, h can never be essentially unary, but essentially binary.

These facts show that from any function $g \in \langle f \rangle \setminus \mathcal{J}_k$ one can recover f, i.e., $f \in \langle g \rangle$. This concludes that $\langle f \rangle$ is a minimal clone. □

Experiment. It has been verified that xy^{k-1} is the only monomial which is minimal (up to interchange of variables) for small primes such as $k = 3, 5, 7, 11$.

This experiment suggests us to go a step further and conjecture the following: For any prime k, xy^{k-1} is the *only* monomial which is minimal (up to interchange of variables). In the rest of the paper, we show that this in fact holds true (Theorem 5.8).

Lemma 5.7 *For any* $1 < s < k$ *we have*

$$xy^{k-1} \in \langle x^s\, y^{k-s} \rangle.$$

Proof. There are two cases to be considered.

Case 1: $\gcd(s, k-1) = 1$

Fermat's theorem asserts that

$$s^{\varphi(k-1)} \equiv 1 \pmod{k-1}$$

where φ is the Euler's function, which implies

$$x^{s^{\varphi(k-1)}}\, y^{k-s^{\varphi(k-1)}} = xy^{k-1}.$$

It is easy to see that $x^{s^{\varphi(k-1)}}\, y^{k-s^{\varphi(k-1)}}$ can be obtained from $x^s\, y^{k-s}$ by repeated application of functional composition. So the assertion of the lemma follows.

Now put $t = k - s$ for $1 < s < k$.

The case $\gcd(t, k-1) = 1$ is handled similarly.

Case 2: $\gcd(s, k-1) \neq 1$ and $\gcd(t, k-1) \neq 1$

In this case we prove the following claim.

Claim. For some $c > 1$, $s^c + t^c - (st)^c \equiv 1 \pmod{k-1}$

Proof of Claim. Since k is a prime, $\gcd(s, t) = 1$. Let $k - 1 = \alpha \cdot \beta \cdot \gamma$ such that $\gcd(s, \beta\gamma) = 1$ and $\gcd(t, \alpha\gamma) = 1$. Then, again, by Fermat's theorem we have

$$s^{\varphi(\beta\gamma)} \equiv 1 \pmod{\beta\gamma} \quad \text{and} \quad t^{\varphi(\alpha\gamma)} \equiv 1 \pmod{\alpha\gamma}.$$

Let $c = \varphi(\alpha\gamma) \cdot \varphi(\beta\gamma)$ then $c > 1$ and c satisfies

$$s^c \equiv 1 \pmod{\beta\gamma} \quad \text{and} \quad t^c \equiv 1 \pmod{\alpha\gamma}.$$

This means that there exists $m, n \in \mathbf{N}$ such that

$$s^c = 1 + m(\beta\gamma) \quad \text{and} \quad t^c = 1 + n(\alpha\gamma),$$

from which it follows that

$$(s^c - 1)(t^c - 1) = (\alpha\beta\gamma)(mn\gamma).$$

Hence we have

$$s^c + t^c - (st)^c \equiv 1 \pmod{k-1}$$

for some $c > 1$. This completes the proof of Claim.

As in Case 1, it is not difficult to see that $x^u y^{k-u}$ for $u = s^c + t^c - (st)^c$ can be obtained from $x^s\, y^{k-s}$ by repeated application of functional composition. Therefore the assertion of the lemma holds. □

On the other hand, it is readily verified that $x^s\, y^{k-s}$ for $1 < s < k$ cannot be obtained from xy^{k-1}. Hence we have the following.

Theorem 5.8 *Let* k *be a prime and* $1 < s < k$. *Then* $x^s\, y^{k-s}$ *is not minimal. Hence* xy^{k-1} *is a unique monomial which is minimal* (*up to the interchange of variables*).

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Appendix 1

Generators of all minimal clones of type (ii) over E_3 (Originally from B. Csákány [Cs 83])

$$b_{11} = xy^2$$

$$b_{624} = 2x + 2y$$
$$b_{68} = 2x + 2xy^2$$
$$b_0 = 2x^2y + 2xy^2$$

$$b_{449} = x + y + 2x^2y$$
$$b_{368} = x + y^2 + 2x^2y^2$$
$$b_{692} = x + 2y^2 + x^2y^2$$
$$b_{33} = x + 2x^2y + xy^2$$
$$b_{41} = x^2 + xy^2 + 2x^2y^2$$
$$b_{71} = 2x^2 + xy^2 + x^2y^2$$

$$b_{26} = 2x + x^2 + 2xy + 2x^2y$$
$$b_{37} = 2x + 2x^2 + xy + 2x^2y$$
$$b_{17} = 2x + x^2 + 2xy^2 + 2x^2y^2$$
$$b_{38} = 2x + 2x^2 + 2xy^2 + x^2y^2$$
$$b_{10} = xy + 2x^2y + 2xy^2 + 2x^2y^2$$
$$b_{20} = 2xy + 2x^2y + 2xy^2 + x^2y^2$$

$$b_{43} = x + xy + 2x^2y + xy^2 + 2x^2y^2$$
$$b_{53} = x + 2xy + 2x^2y + xy^2 + x^2y^2$$
$$b_{35} = x + xy + x^2y + 2xy^2 + 2x^2y^2$$
$$b_{42} = x + 2xy + x^2y + 2xy^2 + x^2y^2$$
$$b_{530} = x + y + y^2 + 2x^2y + 2x^2y^2$$
$$b_{125} = x + y + 2y^2 + 2x^2y + x^2y^2$$
$$b_{116} = x + y + xy + 2y^2 + 2xy^2$$
$$b_{528} = x + y + 2xy + y^2 + 2xy^2$$
$$b_{206} = x + 2y + y^2 + x^2y + 2x^2y^2$$
$$b_{287} = x + 2y + 2y^2 + x^2y + x^2y^2$$
$$b_{215} = x + 2y + 2xy + y^2 + xy^2$$
$$b_{286} = x + 2y + xy + 2y^2 + xy^2$$
$$b_{122} = y + x^2 + 2y^2 + 2x^2y + xy^2$$
$$b_{557} = y + 2x^2 + y^2 + 2x^2y + xy^2$$
$$b_{16} = 2x + x^2 + xy + 2x^2y + x^2y^2$$
$$b_{47} = 2x + 2x^2 + 2xy + 2x^2y + 2x^2y^2$$
$$b_{178} = 2x + 2y + x^2 + xy + y^2$$
$$b_{290} = 2x + 2y + 2x^2 + 2xy + 2y^2$$
$$b_{40} = x^2 + xy + 2x^2y + 2xy^2 + x^2y^2$$
$$b_{80} = 2x^2 + 2xy + 2x^2y + 2xy^2 + 2x^2y^2$$
$$b_{364} = x^2 + xy + y^2 + 2x^2y + 2xy^2$$
$$b_{728} = 2x^2 + 2xy + 2y^2 + 2x^2y + 2xy^2$$

$$b_{448} = x + y + xy + x^2y + xy^2 + 2x^2y^2$$
$$b_{458} = x + y + 2xy + x^2y + xy^2 + x^2y^2$$
$$b_{205} = x + 2y + xy + y^2 + xy^2 + x^2y^2$$
$$b_{296} = x + 2y + 2xy + 2y^2 + xy^2 + 2x^2y^2$$
$$b_{188} = 2x + 2y + x^2 + 2xy + y^2 + 2x^2y^2$$
$$b_{280} = 2x + 2y + 2x^2 + xy + 2y^2 + x^2y^2$$
$$b_8 = 2x + x^2 + xy + x^2y + xy^2 + x^2y^2$$
$$b_{36} = 2x + 2x^2 + 2xy + x^2y + xy^2 + 2x^2y^2$$

$$b_{179} = 2x + 2y + x^2 + y^2 + x^2y + 2xy^2 + x^2y^2$$
$$b_{281} = 2x + 2y + 2x^2 + 2y^2 + x^2y + 2xy^2 + 2x^2y^2$$

Appendix 2

Representatives of generators of minimal clones over E_3 from each equivalence class

$$b_{11} = xy^2$$
$$b_0 = 2x^2y + 2xy^2$$
$$b_{68} = 2x + 2xy^2$$
$$b_{624} = 2x + 2y$$
$$b_{41} = x^2 + xy^2 + 2x^2y^2$$
$$b_{368} = x + y^2 + 2x^2y^2$$
$$b_{33} = x + 2x^2y + xy^2$$
$$b_{449} = x + y + 2x^2y$$
$$b_{10} = xy + 2x^2y + 2xy^2 + 2x^2y^2$$
$$b_{17} = 2x + x^2 + 2xy^2 + 2x^2y^2$$
$$b_{16} = 2x + x^2 + xy + 2x^2y + x^2y^2$$
$$b_{178} = 2x + 2y + x^2 + xy + y^2$$