Polynomials as Generators of Minimal Clones

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Abstract

A minimal clone is an atom of the lattice of clones. A minimal function is a function which generates a minimal clone. We consider the base set with k elements, for a prime k, as a finite field and treat functions as polynomials.

Starting from binary minimal functions over GF(3), we generalize some of them and obtain binary minimal functions, as polynomials, over GF(k) for any prime $k \geq 3$.

Keywords: Clone; minimal clone; Galois field

1 Introduction

To begin with, consider two Boolean functions f and g expressed as polynomials over GF(2) :

$$\begin{aligned} f(x,y) &= xy+1 \\ g(x,y) &= xy+x+y \end{aligned}$$

The question is : Which function is stronger with respect to the 'productive power' by (functional) composition ?

Answer is clear: f is stronger and g is weaker. In fact, $f(x,y)$ is NAND(x,y) which is so strong as to produce all Boolean functions whereas $g(x,y)$ is OR(x,y) which generates a minimal clone (whose definition appears below).

Next, consider three polynomials over GF(3):

$$\begin{aligned} u(x,y) &= x^2y^2+xy^2+x^2y+2xy+x+y \\ v(x,y) &= x^2y^2+xy^2+x^2y+xy+x+y \\ w(x,y) &= x^2y^2+xy^2+x^2y+2xy+x+y+1 \end{aligned}$$

The question is : Which function is the weakest ?

In this case, the answer may not be so obvious. All these functions look more or less similar, but actually u is the weakest among these three functions. In fact, (i) $u(x,y)$ generates a minimal clone, (ii) $w(x,y)$ is Webb function which is known to generate all functions and (iii) $v(x,y)$ stays somewhere in-between.

The purpose of our study is to find some nice characterization of polynomials whose productive power is, in a sense, weakest, i.e., which generate minimal clones. As indicated by the above example, this is quite a challenging task.

For an integer $k\,(\geq 2)$ let $E_k = \{0,1,\ldots,k-1\}$. Let $\mathcal{O}_k^{(n)}$ denote the set of all n-variable functions on E_k, i.e., mappings from $(E_k)^n$ into E_k, and set

$$\mathcal{O}_k = \bigcup_{n=1}^{\infty} \mathcal{O}_k^{(n)}.$$

For any $f \in O_k^{(m)}$ and $g_1,\ldots,g_m \in O_k^{(n)}$ the (*functional*) *composition* $f[g_1,\ldots,g_m]$ of f with $g_1,\ldots,g_m$ is a function in $O_k^{(n)}$ defined by

$$\begin{aligned} & f[g_1,\ldots,g_m](x_1,\ldots,x_n) \\ = \;& f(g_1(x_1,\ldots,x_n),\ldots,g_m(x_1,\ldots,x_n)) \end{aligned}$$

for all $(x_1,\ldots,x_n) \in (E_k)^n$.

Let $\mathcal{J}_k$ be the set of all *projections* e_i^n, $1 \leq i \leq n$, over E_k where the i-th projection e_i^n of arity n is defined by $e_i^n(x_1,\ldots,x_i,\ldots,x_n) = x_i$ for all $(x_1,\ldots,x_n) \in (E_k)^n$.

A subset C of $\mathcal{O}_k$ is a *clone* on E_k if (i) C is closed under (functional) composition and (ii) C contains $\mathcal{J}_k$. Denote by $\mathcal{L}_k$ the set of all clones on E_k. The set $\mathcal{L}_k$ is an algebraic lattice with respect to inclusion and called the *lattice of clones* on E_k. It is obvious that the greatest element is $\mathcal{O}_k$ and the least element is $\mathcal{J}_k$ in the lattice $\mathcal{L}_k$.

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For a subset F of $\mathcal{O}_k$ denote by $\langle F\rangle$ the clone generated by F. Thus, $\langle F\rangle$ is the intersection of all clones containing F. In particular, when F is a singleton, i.e., $F = \{f\}$ for some $f \in \mathcal{O}_k$, we simply write $\langle f\rangle$ in stead of $\langle F\rangle$.

An atom of the lattice $\mathcal{L}_k$ is called a *minimal clone.* In other words, a minimal clone is a minimal element in the partially ordered set $\mathcal{L}_k \setminus \{\mathcal{J}_k\}$. It is clear that a minimal clone C is generated by a single function f in $\mathcal{O}_k$, i.e., $C = \langle f\rangle$. A function in $\mathcal{O}_k$ which generates a minimal clone is called a *minimal function.*

For $k = 2$ there are 7 minimal clones in $\mathcal{L}_2$, which is easily obtained from full knowledge of $\mathcal{L}_2$ due to E. Post [Po 41]. For $k = 3$ there are 84 minimal clones in $\mathcal{L}_3$. This is due to B. Csákány [Cs 83] who determined all minimal clones and their generators over E_3. Our complete knowledge on minimal clones with respect to the size of the base set E_k is, at present, only up to this point. Even for $k = 4$, in spite of the work of B. Szczepara [Szc 95] who found all minimal clones which are generated by binary functions, the problem of determining all minimal clones in $\mathcal{L}_4$ is still open.

Minimal clones have been studied by many authors, e.g., [Cs 83], [Du 90], [JQ 95], [KS 99], [LP 96], [Ro 86], [Wa 00], etc. Many of them have revealed deep and interesting aspects of minimal clones. However, untill now, the problem of determining all minimal clones stands firm, like an unbreakable fortress, against the attack of those eminent researchers.

This paper is a continuation of our work [MP 06], where we proposed to tackle minimal clones by considering minimal functions as polynomials. We assume that k is a power of a prime and the base set E_k is a finite field. In this paper we only consider minimal clones generated by binary idempotent functions. Starting from Csákány's list of minimal functioins on E_3, expressing them as polynomials over GF(3), we generalize some of them and obtain polynomials generating minimal clones over GF(k) for a prime $k \geq 3$. In the course of discussion, we show some conditions for a function to be minimal.

2 Preliminaries

2.1 Finite Field

A *finite field* (or *Galois field*) is a field $\mathbf{F}$ consisting of a finite number of elements. The number of elements in $\mathbf{F}$ must be a power p^e $(e \geq 1)$ of some prime p. When $\mathbf{F}$ contains $q = p^e$ elements, it is denoted by GF(q). It is fundamental that GF(q) consists of elements x satisfying $x^q = x$.

In particular, when k is a prime and E_k is a prime field, the addition and the multiplication of GF(k) are exactly the addition modulo k and the multiplication modulo k, respectively.

Note that for a prime k every binary function $f \in \mathcal{O}_k^{(2)}$ on E_k is uniquely expressed as a polynomial

$$f(x,\, y) = \sum_{0\,\leq\, i,\, j\,<\, k} a_{\,ij}\, x^{\,i}\, y^{\,j}$$

for some $a_{ij} \in E_k$ $(0 \leq i,\, j < k)$.

2.2 Type Theorem for Minimal Functions

I. G. Rosenberg [Ro 86] classified minimal functions into five types. This is known as the *type theorem* for minimal clones. It states that every minimal function f on E_k whose arity is minimum among arities of functions in $\langle f\rangle \setminus \mathcal{J}_k$ falls in one of the following five categories (types): (i) unary function, (ii) binary idempotent function, (iii) ternary majority function, (iv) ternary function $x + y + z$ for an elementary abelian 2-group and (v) $k\,(\geq 3)$-ary semiprojection.

In the sequel, we concentrate on polynomials of binary idempotent minimal functions, minimal functions of *type* (ii) in the above classification. To recall, a function $f \in \mathcal{O}_k$ is *idempotent* if it satisfies $f(x,\ldots,x) = x$ for all $x \in E_k$. Among five types given above, the second type is, without doubt, the richest in a sense that the number of minimal functions belonging to (ii) is greater than that of functions belonging to any other type. For example, there are 84 minimal clones for $k = 3$ and 48 of them, more than 57%, are minimal clones generated by functions of type (ii).

3 Conditions for Minimality

For $f, g \in \mathcal{O}_k$, we shall write $f \to g$ if $g \in \langle f \rangle$. Note that the binary relation $\to$ on $\mathcal{O}_k$ is a quasi-order, i.e., $\to$ is reflexive and transitive.

A basic fact is the following:

Lemma 3.1 *Let $f \in \mathcal{O}_k^{(2)}$ be an essentially binary function. If f satisfies the following two conditions then f is minimal.*

(1) *f is idempotent.*

(2) *For any $g \in \mathcal{O}_k^{(m)}$, $m \geq 2$, satisfying $f \to g$, if g is not a projection then $g \to f$.*

Now, let $m \geq 3$ and $g \in \mathcal{O}_k^{(m)}$. We shall say that g is a *quasi-projection* if g becomes a projection whenever two arguments of g, say, the i-th argument and the j-th argument for $1 \leq i < j \leq m$, are identified, i.e., if $g(x_1, \ldots, x_i, \ldots, x_i, \ldots, x_m)$ is always a projection.

Lemma 3.2 *Let $f \in \mathcal{O}_k^{(2)}$ be a binary idempotent function. Then f is minimal if and only if the following two conditions hold.*

(1) *For any $g \in \mathcal{O}_k^{(2)} \setminus \mathcal{J}_k$, if $f \to g$ then $g \to f$.*

(2) *For any $m \geq 3$ and any $g \in \mathcal{O}_k^{(m)} \setminus \mathcal{J}_k$, if $f \to g$ then g is not a quasi-projection.*

Proof. ($\Rightarrow$) Let f be minimal. Then (1) follows immediately. To show the second condition (2), suppose that there exists a function g in $\mathcal{O}_k^{(m)} \setminus \mathcal{J}_k$, $m \geq 3$, which satisfies $f \to g$ and is a quasi-projection. If we prove $g \not\to f$ then we are done, because $g \not\to f$ together with $f \to g$ implies $\langle g \rangle \subset \langle f \rangle$ which is against the minimality of f.

Now assume on the contrary that $g \to f$. Then f is composed as

$$f(x_1, x_2) = g[\alpha_1, \alpha_2, \ldots, \alpha_m](x_1, x_2)$$

where α_i is an expression constructed by (possibly) repeated compositions from g and projections, i.e., $\alpha_i \in \langle g \rangle$, for $i = 1, 2, \ldots, m$. We claim that, since there are only two variables x_1, x_2 whereas g is an $m\,(\geq 3)$ variable function, α_i must be a projection for each $i = 1, 2, \ldots, m$. (More precisely, this is proved by induction on the depth of composition.) Then at least two of $\alpha_1, \alpha_2, \ldots, \alpha_m$ coincide and $g[\alpha_1, \alpha_2, \ldots, \alpha_m]$ is also a projection. Hence f is a projection which contradicts the assumption that f is minimal.

($\Leftarrow$) Suppose f satisfies the conditions (1) and (2), but is *not* minimal. Then there must exists $g \in \mathcal{O}_k^{(m)} \setminus \mathcal{J}_k$ such that

$$f \to g \quad \text{and} \quad g \not\to f. \qquad (\star)$$

Because of (1), it must be that $m \geq 3$. Let $m_0 \geq 3$ be the least integer for which there exists $g \in \mathcal{O}_k^{(m_0)} \setminus \mathcal{J}_k$ with the property ($\star$).

For any i, j, $1 \leq i < j \leq m_0$, let g_{ij} denote the function $g(\ldots, x_i, \ldots, x_i \ldots)$ where the i-th place and the j-th place have the same variable. Then we have $g_{ij} \in \mathcal{O}_k^{(m_0-1)}$ and $f \to g_{ij}$. By the assumption on m_0, it follows that either (i) $g_{ij} \in \mathcal{J}_k$ or (ii) $g_{ij} \to f$. However, if (ii) holds, then $g \to f$ because $g \to g_{ij}$ and $\to$ is transitive, which is against the assumption on m_0. Therefore (i) must hold, i.e., $g_{ij} \in \mathcal{J}_k$, which completes the proof. □

For $f \in \mathcal{O}_k^{(2)}$ let $\Gamma_f^{(x,y)}$ be the following set of expressions:

$$\{\, f(f(x,y),x), \quad f(f(x,y),y), \quad f(x,f(x,y)), \quad f(y,f(x,y)), \quad f(f(x,y),f(y,x)) \,\}$$

Then $\Gamma_f = \Gamma_f^{(x,y)} \cup \Gamma_f^{(y,x)}$ shall be called the *basic set of compositions* for f.

Lemma 3.3 *Let $f \in \mathcal{O}_k^{(2)}$ be a binary idempotent function which is not a projection. Suppose that, for any $\gamma \in \Gamma_f$, one of the following holds:*

$$\gamma(x,y) \approx f(x,y) \quad \text{or} \quad \gamma(x,y) \approx f(y,x)$$

Then f is minimal.

Here, by $h_1(x,y) \approx h_2(x,y)$ for $h_1, h_2 \in \mathcal{O}_k^{(2)}$ we mean $h_1(x,y) = h_2(x,y)$ for all $(x,y) \in E_k^2$.

Proof. (Sketch) Let $g \in \mathcal{O}_k^{(m)}$ be constructed from f and the projections by repeated composition. If the depth of construction is greater than 1, by suitable identification of variables, if necessary, each of the innermost parts of the construction is altered to some form γ in Γ_f. Then $\gamma(x,y)$ may be replaced by $f(x,y)$ or $f(y,x)$, reducing the depth of the construction. Eventually we reach f, showing that $g \to f$. □

4 From Csákány's List

As already mentioned, generators of all minimal clones for $k = 3$ are known by B. Csákány [Cs 83]. For all minimal clones generated by functions of type (ii), i.e., binary minimal functions, we present generators as polynomials over the field GF(3) in Appendix. For the reader's sake, the names of the functions such as b_{11}, b_0, b_{68} are preserved from [Cs 83].

5 Binary Minimal Functions

Starting from Csákány's results for $k = 3$, we attempt to generalize and obtain binary idempotent minimal functions for arbitrary prime $k \geq 3$.

Throughout this section, we assume that $k\,(\geq 3)$ is a prime and E_k is the finite field GF(k). We consider only binary idempotent minimal functions.

5.1 Linear Polynomials and Monomials

In [MP 06], we generalized Csákány's results on linear polynomials and monomials for $k = 3$ to any prime $k \geq 3$. (Also, refer to [Szc 95].) Before we go further, we review those results.

A binary *linear* polynomial on E_k is a polynomial of the form $a_0 + a_1 x + a_2 y$ for some $a_0, a_1, a_2 \in E_k$. From Appendix, we see that $2x + 2y$ is the only binary linear polynomial on E_3 which is minimal. This generalizes to the following:

Theorem 5.1 *For a prime $k\,(\geq 3)$, let $f(x, y)$ be a binary linear polynomial on E_k. Then f is minimal if and only if $f(x, y) = ax + (k + 1 - a)y$ for some $1 < a < k$.*

Example. For $k = 5$, the linear polynomial $f(x, y) = 2x + 4y$ is expressed by the Cayley table as follows:

$f(x, y) = 2x + 4y$

$x\backslash y$	0	1	2	3	4
0	0	4	3	2	1
1	2	1	0	4	3
2	4	3	2	1	0
3	1	0	4	3	2
4	3	2	1	0	4

Note: Lemma 3.2 can be used to prove Theorem 5.1.

Secondly, a binary *monomial* on E_k is a polynomial with two variables consisting of a single term, i.e., $a\,x^{i_1} y^{i_2}$ for some $a, i_1, i_2 \in E_k$. In Appendix, we see that there is only one binary monomial on E_3, $x\,y^2$, which is minimal. To generalize, we proved:

Theorem 5.2 *For a prime $k\,(\geq 3)$ and $1 \leq s \leq t < k$, let $f(x, y) = x^s y^t$ be a binary monomial on E_k. Then f is minimal if and only if $s = 1$ and $t = k - 1$.*

Hence we see that $f(x, y) = x\,y^{k-1}$ is the unique monomial on E_k which is minimal (up to the interchange of variables).

Example. For $k = 5$, $f(x, y) = x\,y^4$ is a function which is expressed by the Cayley table as follows:

$f(x, y) = x\,y^4$

$x\backslash y$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	2	2	2	2
3	0	3	3	3	3
4	0	4	4	4	4

5.2 More Generalizations of Csákány's Results

We achieve the following procedure:

Step 1: Take arbitrary $f(x, y) \in \mathcal{O}_3^{(2)}$ from Appendix.
Step 2: Search for a polynomial $g(x, y) \in \mathcal{O}_k^{(2)}$ defined on E_k for $k \geq 3$ whose counterpart for $k = 3$ is $f(x, y)$.
Step 3: Examine if g is minimal.

(1) From Appendix, we see that $x + y + 2\,x\,y^2$ is minimal for $k = 3$. A generalization is:

Proposition 5.3 *For a prime $k\,(\geq 3)$ the function $f(x, y) = x + y + (k - 1)\,x\,y^{k-1}$ is minimal.*

Proof. For any γ in the basic set Γ_f of compositions for f, we can compute and see that

$$f(f(x, y), y) \approx f(x, f(x, y)) \approx f(y, f(x, y)) \approx f(x, y)$$

and

$$f(f(x, y), x) \approx f(f(x, y), f(y, x)) \approx f(y, x).$$

Hence, by Lemma 3.3, f is minimal. $\square$

Example. For $k = 5$, the Cayley table of $f(x, y) = x + y + 4\,x\,y^4$ is as follows:

$$\underline{f(x,y) \;=\; x + y + 4\,x\,y^4}$$

$x\backslash y$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	2	3	4
2	2	1	2	3	4
3	3	1	2	3	4
4	4	1	2	3	4

(2) Take a minimal function $x + 2y^2 + x^2\,y^2$ for $k = 3$ from Appendix. It generalizes as follows:

Proposition 5.4 *For a prime* $k\,(\geq\, 3)$ *the function* $f(x, y) \;=\; x + (k-1)y^{k-1} + x^{k-1}\,y^{k-1}$ *is minimal.*

Proof. The proof is similar to the previous proposition. In this case,

$$f(f(x,y),x) \approx f(f(x,y),y) \approx f(x,f(x,y))$$
$$\approx f(f(x,y),f(y,x)) \approx f(x,y)$$

and

$$f(y,f(x,y)) \approx f(y,x).$$ $\square$

Example. For $k = 5$, the Cayley table of $f(x, y) = x + 4\,y^4 + x^4\,y^4$ is as follows:

$$\underline{f(x,y) \;=\; x + 4\,y^4 + x^4\,y^4}$$

$x\backslash y$	0	1	2	3	4
0	0	4	4	4	4
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4

(3) The next target is a minimal function $x\,y^2 + 2\,x^2 + x^2\,y^2$ for $k = 3$. It generalizes as follows:

Proposition 5.5 *For a prime* $k\,(\geq\, 3)$ *the function* $f(x, y) = x\,y^{k-1} + (k-1)x^{k-1} + x^{k-1}\,y^{k-1}$ *is minimal.*

Proof. The proof requires a subtle change in the discussion as we have:

$$f(f(x,y),x) \approx f(f(x,y),y) \approx f(f(x,y),f(y,x))$$
$$\approx f(x,y) \quad \text{and} \quad f(y,f(x,y)) \approx f(y,x),$$

but

$$f(x,f(x,y)) \approx x.$$

However, this can be overcome without difficulty if x_1 and x_3 are identified, instead of x_1 and x_2, when one needs to modify 3-variable function $f(x_1, f(x_2, x_3))$ to 2-variable function. $\square$

Example. For $k = 5$, the Cayley table of $f(x, y) = x + 4\,y^4 + x^4\,y^4$ is as follows:

$$\underline{f(x,y) \;=\; x\,y^4 + 4\,x^4 + x^4\,y^4}$$

$x\backslash y$	0	1	2	3	4
0	0	0	0	0	0
1	4	1	1	1	1
2	4	2	2	2	2
3	4	3	3	3	3
4	4	4	4	4	4

(4) Finally, we show an example which requires somewhat better skill even to find a candidate of generalization. The target to generalize is a minimal function

$$2\,x^2 y + 2\,x\,y^2$$

for $k = 3$. In this case simple replacements of 2 by $k-1$ does not work. Our generalization is the following:

Proposition 5.6 *For a prime* $k\,(\geq\, 3)$ *the function* $f(x, y) \;=\; (k-1)\sum_{i=1}^{k-1} x^{k-i}\,y^i$ *is minimal.*

Proof. First, it is easy to see that $f(x, x) = x$ if we notice $(k-1)^2 = 1$. For $x \neq y$, let $D(= D(x, y)) = \sum_{i=1}^{k-1} x^{k-i}\,y^i$. We have

$$xy^{-1}D = D.$$

Hence $xD = yD$ and $(x - y)D = 0$. Since $x \neq y$, it follows that $D = 0$. Therefore $f(x, y) = x$ if $x = y$ and $f(x, y) = 0$ if $x \neq y$. It is not hard to examine that f is minimal. $\square$

Example. For $k = 5$, the Cayley table of $f(x, y) = 4\,x^4\,y + 4\,x^3y^2 + 4\,x^2y^3 + 4\,x\,y^4$ is as follows:

$$\underline{f(x,y) \;=\; 4\,x^4\,y + 4\,x^3y^2 + 4\,x^2y^3 + 4\,x\,y^4}$$

$x\backslash y$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	0	0
2	0	0	2	0	0
3	0	0	0	3	0
4	0	0	0	0	4

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Appendix

Generators of all minimal clones of type (ii) over GF(3) (Originally from B. Csákány [Cs 83])

$$b_{11} = xy^2$$

$$b_{624} = 2x+2y$$
$$b_{68} = 2x+2xy^2$$
$$b_{0} = 2x^2y+2xy^2$$

$$b_{449} = x+y+2x^2y$$
$$b_{368} = x+y^2+2x^2y^2$$
$$b_{692} = x+2y^2+x^2y^2$$
$$b_{33} = x+2x^2y+xy^2$$
$$b_{41} = x^2+xy^2+2x^2y^2$$
$$b_{71} = 2x^2+xy^2+x^2y^2$$

$$b_{26} = 2x+x^2+2xy+2x^2y$$
$$b_{37} = 2x+2x^2+xy+2x^2y$$
$$b_{17} = 2x+x^2+2xy^2+2x^2y^2$$
$$b_{38} = 2x+2x^2+2xy^2+x^2y^2$$
$$b_{10} = xy+2x^2y+2xy^2+2x^2y^2$$
$$b_{20} = 2xy+2x^2y+2xy^2+x^2y^2$$

$$b_{43} = x+xy+2x^2y+xy^2+2x^2y^2$$
$$b_{53} = x+2xy+2x^2y+xy^2+x^2y^2$$
$$b_{35} = x+xy+x^2y+2xy^2+2x^2y^2$$
$$b_{42} = x+2xy+x^2y+2xy^2+x^2y^2$$
$$b_{530} = x+y+y^2+2x^2y+2x^2y^2$$
$$b_{125} = x+y+2y^2+2x^2y+x^2y^2$$
$$b_{116} = x+y+xy+2y^2+2xy^2$$
$$b_{528} = x+y+2xy+y^2+2xy^2$$
$$b_{206} = x+2y+y^2+x^2y+2x^2y^2$$
$$b_{287} = x+2y+2y^2+x^2y+x^2y^2$$
$$b_{215} = x+2y+2xy+y^2+xy^2$$
$$b_{286} = x+2y+xy+2y^2+xy^2$$
$$b_{122} = y+x^2+2y^2+2x^2y+xy^2$$
$$b_{557} = y+2x^2+y^2+2x^2y+xy^2$$
$$b_{16} = 2x+x^2+xy+2x^2y+x^2y^2$$
$$b_{47} = 2x+2x^2+2xy+2x^2y+2x^2y^2$$
$$b_{178} = 2x+2y+x^2+xy+y^2$$
$$b_{290} = 2x+2y+2x^2+2xy+2y^2$$
$$b_{40} = x^2+xy+2x^2y+2xy^2+x^2y^2$$
$$b_{80} = 2x^2+2xy+2x^2y+2xy^2+2x^2y^2$$
$$b_{364} = x^2+xy+y^2+2x^2y+2xy^2$$
$$b_{728} = 2x^2+2xy+2y^2+2x^2y+2xy^2$$

$$b_{448} = x+y+xy+x^2y+xy^2+2x^2y^2$$
$$b_{458} = x+y+2xy+x^2y+xy^2+x^2y^2$$
$$b_{205} = x+2y+xy+y^2+xy^2+x^2y^2$$
$$b_{296} = x+2y+2xy+2y^2+xy^2+2x^2y^2$$
$$b_{188} = 2x+2y+x^2+2xy+y^2+2x^2y^2$$
$$b_{280} = 2x+2y+2x^2+xy+2y^2+x^2y^2$$
$$b_{8} = 2x+x^2+xy+x^2y+xy^2+x^2y^2$$
$$b_{36} = 2x+2x^2+2xy+x^2y+xy^2+2x^2y^2$$

$$b_{179} = 2x+2y+x^2+y^2+x^2y+2xy^2+x^2y^2$$
$$b_{281} = 2x+2y+2x^2+2y^2+x^2y+2xy^2+2x^2y^2$$