Proceedings of the First Ordener Lectures

October 4 & 5, 2012

185 Rue Ordener, Paris

Organizers: Manuel Bodirsky & Michael Pinsker

The Ordener Program

Thursday October 4

Coffee: Ethiopian Moka

Morning session (10:00-12:45):
Barnaby Martin - Coffee - Hubie Chen - Coffee - Florent Madelaine

Lunch break: Turkish

Afternoon session (14:15-16:30):
Todor Tsankov - Coffee - András Pongrácz

Friday October 5

Coffee: Costa Rica Arabica

Morning session (10:00-12:45):
Johan Thapper - Coffee - David Bradley-Williams

Lunch break: Italian

Afternoon session (14:15-16:30):
Manuel Bodirsky - Coffee - Michael Pinsker

The Ordener Problems

Barnaby Martin

Problem 1. Let $\mathcal{B}$ be a finite structure over a finite signature σ whose domain B is of cardinality $|B|$. For $1 \leq j \leq |B|$, the formula $\exists^{\geq j} x\ \phi(x)$ with *counting quantifier* should be interpreted on $\mathcal{B}$ as stating that there exist at least j distinct elements $b \in B$ such that $\mathcal{B} \models \phi(b)$. Counting quantifiers generalise existential ($\exists := \exists^{\geq 1}$), universal ($\forall := \exists^{\geq |B|}$) and (weak) majority ($\exists^{\geq |B|/2}$) quantifiers.

For $\emptyset \neq X \subseteq \{1, \ldots, |B|\}$, the X-CSP($\mathcal{B}$) takes as input a sentence of the form $\Phi := Q_1 x_1 Q_2 x_2 \ldots Q_m x_m\ \phi(x_1, x_2, \ldots, x_m)$, where ϕ is a conjunction of positive atoms of σ and each Q_i is of the form $\exists^{\geq j}$ for some $j \in X$.

Let $\mathcal{K}_n$ be the complete irreflexive graph on n vertices. A full complexity classification – a trichotomy – is known for the problem X-CSP($\mathcal{K}_n$), $X \subseteq \{1, \ldots, n\}$, except for cases of the form $\{j\}$-CSP($\mathcal{K}_{2j}$). When, $j := 1$ we have $\{1\}$-CSP($\mathcal{K}_2$)=CSP($\mathcal{K}_2$) which is in L. For higher j, the question is both challenging and open.

Hubie Chen

Problem 2. Is the periodic power idempotent, that is, does it hold that, for a relational structure B, that B^{per} is isomorphic to $(B^{\mathsf{per}})^{\mathsf{per}}$?

Problem 3. It was shown by Chen and Müller (LICS '12) that surjective periomorphisms characterize conjunctive-positive definability on ω-categorical structures. Prove or disprove the following: surjective *polymorphisms* characterize conjunctive-positive definability on ω-categorical structures.

Problem 4. Let G be an oligomorphic permutation group, and consider relational structures B where B is a relational structure having all elements of G as automorphisms, and where $[B]_{pH}$ has a constant polymorphism. (Here, $[B]_{pH}$ denotes the set of all relations that are conjunctive-positive/positive Horn definable over B.) Develop techniques for classifying the complexity of QCSP(B).

Note: Chen and Müller (LICS '12) show how to do this classification using periomorphisms in the case where G is the set of all permutations. An interesting open case is where G is the automorphism group of the ordered rationals $(\mathbb{Q}, <)$.

Problem 5. How many Ordener Lecturers are needed to unscrew a lightbulb?

Florent Madelaine

Recall the notion of *exponential* of the structures B and C, denoted by C^B, whose defining property states that for all structures A,

$$A \times B \to C \text{ if and only if } A \to C^B.$$

This abstract definition coincides with the following concrete definition in the case of finite structures and arbitrary structures. The universe of the (concrete) *exponential* C^B consists of all maps from the universe of B to the universe of C. For every symbol R_i and every tuples $f_1, f_2, \ldots, f_{r_i}$ of such maps, we set $R_i\big(f_1, f_2, \ldots, f_{r_i}\big)$ to hold in C^B if, and only if, for every tuple $b_1, b_2, \ldots, b_{r_i}$ of elements of B such that $R_i(b_1, b_2, \ldots, b_{r_i})$ holds in B, it follows that $R_i\big(f_1(b_1), f_2(b_2), \ldots, f_{r_i}(b_{r_i})\big)$ holds in C.

Problem 6. Given two ω-categorical structures over a similar relational structure $\mathcal{C}$ and $\mathcal{B}$, is it the case that there is always some ω-categorical structure homomorphically equivalent to $\mathcal{C}^{\mathcal{B}}$?

One could also take the opposite stance and attempt to prove a negative result.

Problem 7. Same question for the class of countable (or larger) structures. One would expect the core of the exponential to not be in the class.

Our effort in this direction have been fruitless. One would like the exponential of two countable structure to encode something like the reals. However for any reasonable encoding, one ends up having a homomorphism from $\mathcal{C}$ to $\mathcal{B}$ and thus the core of the exponential is trivial.

Todor Tsankov

Problem 8. Is the universal minimal flow of oligomorphic permutation groups always metrizable? How about the universal minimal flow of any Roelcke precompact Polish group?

Problem 9. Is it true that any metrizable minimal flow of an oligomorphic group has a G_δ orbit? (If the universal minimal flow is metrizable, this is equivalent to the universal minimal flow having a G_δ orbit.)

András Pongrácz

Problem 10. Does every homogeneous structure over a finite relational language have an ω -categorical Ramsey expansion? (Is it homogeneous in a finite relational language?)

Problem 11. Assume that $\mathcal{F}$ is a finitely bounded homogeneous Ramsey structure. Is the order forgetful order expansion $\mathcal{F}^* = (\mathcal{F}, \prec)$ finitely bounded?

Problem 12. Characterize the structures $\mathcal{F}$ such that $\mathcal{F}$ is a homogeneous Ramsey structure in a finite relational language, and $|Aut(\mathcal{F}) : Aut(\mathcal{F}^*)|$ is finite, where $\mathcal{F}^* = (\mathcal{F}, \prec)$ is the order forgetful order expansion of $\mathcal{F}$. (Also interesting for $|Aut(\mathcal{F}) : Aut(\mathcal{F}^*)| = 2$.)

Problem 13. Characterize the structures $\mathcal{F}$ such that $\mathcal{F}$ is a homogeneous Ramsey structure in a finite relational language, and $Aut(\mathcal{F})$ is 3-transitive.

Problem 14. Let $\mathcal{F}$ be homogeneous in a finite relational language, and let $\mathcal{F}'$ be an expansion of $\mathcal{F}$ such that $|Aut(\mathcal{F}) : Aut(\mathcal{F}')|$ is finite. Is $\mathcal{F}'$ homogeneous in a finite relational language? (Also interesting for $\mathcal{F}'$ ordered Ramsey.)

Problem 15. Let $\mathcal{F}$ be homogeneous in a finite relational language, and let $\mathcal{F}' = (\mathcal{F}; R_1, \ldots, R_n)$ be a homogeneous ordered Ramsey expansion of $\mathcal{F}$ such that the universal minimal flow $M(Aut(\mathcal{F}))$ is $Aut(\mathcal{F}) \curvearrowright \overline{Aut(\mathcal{F}) \cdot (R_1, \ldots, R_n)}$ (the logic action). Assume that $Aut(\mathcal{F})$ is uniquely ergodic. Is it true that the unique Borel measure is supported by the generic orbit of $M(Aut(\mathcal{F}))$?

Johan Thapper

Problem 16. Classify all ω-categorical structures with totally symmetric polymorphisms of all arities. Are there countably many?

Problem 17. A structure Γ is *f-samplable*, for some $f : \mathbb{N} \to \mathbb{N}$ if, for each n, there is a structure B of size $f(n)$ with $A \to B$ iff $A \to \Gamma$, for all structures A of size n. Find an f-samplable Γ such that,

(1) B cannot always be chosen homomorphic to Γ;
(2) B can always be chosen homomorphic to Γ but not always as an induced substructure of Γ.

David Bradley-Williams

Problem 18. Given an ω-categorical first order theory T with countably infinite model $M \models T$, when is M homogenisable in a finite relational language?

An answer is desired which is dependant only on properties of the theory, or properties of the topological group $\mathrm{Aut}(M)$.

Problem 19. In particular is there a topological group counter example? Are there a pair of groups G and H which are isomorphic as topological groups but that G is the automorphism group of M which is homogeneous in a finite relational language, but $H = \mathrm{Aut}(N)$ where N is not homogenisable in a finite relational language?

Problem 20. For an ω-categorical structure M in a finite relational language $L = \{M; R_1, R_2, ..., R_k\}$, is there an L-definable function $f : M^n \to M$ such that whenever A is a finite subset of M, then the algebraic closure of A is exactly the closure generated by function f;

$$\mathrm{acl}(A) = \langle A \rangle_f.$$

Manuel Bodirsky

Problem 21 (Transition to the core)**.** Let Γ be the reduct of a finitely bounded homogeneous structures. Is the model companion of Γ also the reduct of a finitely bounded homogeneous structure? Same question for the core companion.

Problem 22 (Transition to the core)**.** Let Γ be the reduct of a structure with finite relational signature and the Ramsey property. Is the model companion of Γ also the reduct of a homogeneous structure with finite relational signature and the Ramsey property? Same question for core companion.

Problem 23 (Inverse of Canonization Lemma)**.** Let Γ be ω-categorical, homogeneous, and ordered. Suppose also that for all finite tuples $\bar{a}$ of elements of Γ, and for all $f\colon \Gamma \to \Gamma$ there exists $g\colon \Gamma \to \Gamma$ such that

$$g \in \overline{Aut(\Gamma) f\, Aut(\Gamma, \bar{a})} \tag{1}$$

and g is canonical as a map from $(\Gamma, \bar{a})$ to Γ. Is it then true that Γ is Ramsey?

Problem 24 (Tractability)**.** Let Γ be the reduct of a finitely bounded homogeneous structure. Suppose that Γ has for every n an n-ary polymorphism f and an automorphism α such that for all $\pi \in S_n$

$$\forall \bar{x}.\ f(x_1, \ldots, x_n) = \alpha(f(x_{\pi 1}, \ldots, x_{\pi n}))\ .$$

Is CSP(Γ) in P?

Problem 25 (Finite Model Theory)**.** Is it true that for every problem L in NP there exists an ω-categorical Γ such that L and CSP(Γ) are Ptime equivalent?

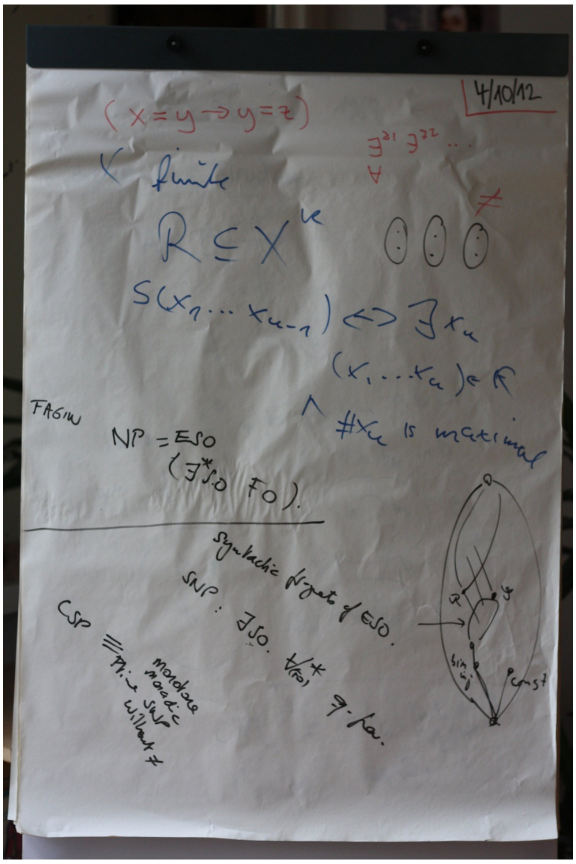
Michael Pinsker

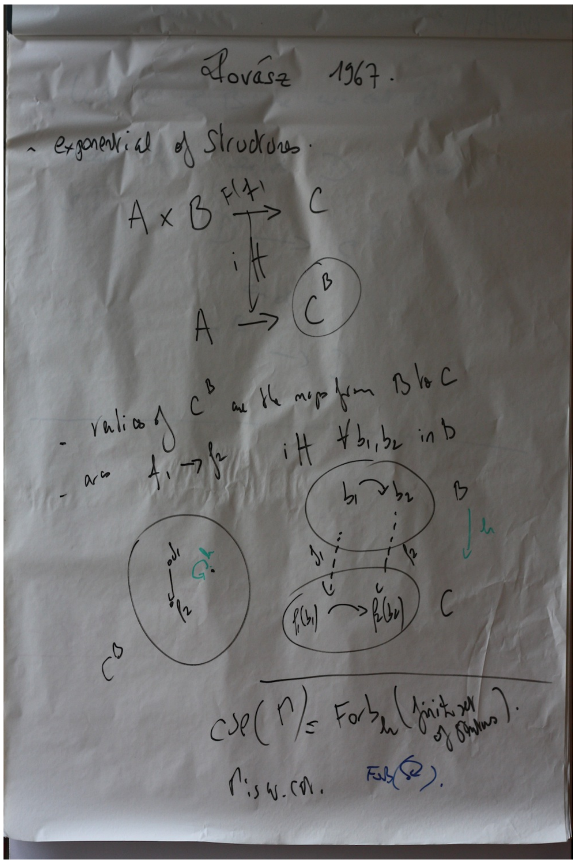
We all know oligomorphic groups. There is a notion of *weakly oligomorphic* for transformation monoids, which is weaker than the monoid simply containing an oligomorphic permutation group, and which is still strong enough to make it possible to prove theorems about structures which have a weakly oligomorphic endomorphism monoid.

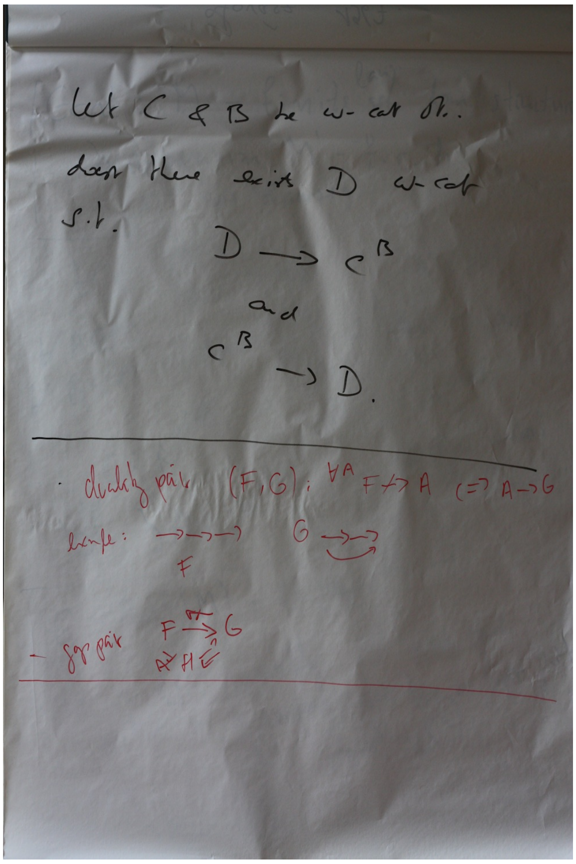
A clone is called *oligomorphic* iff it contains an oligomorphic permutation group. We could also call it *weakly oligomorphic* iff its unary part is a weakly oligomorphic monoid. However, we would like to have a notion which depends on more than just the unary functions of the clone.

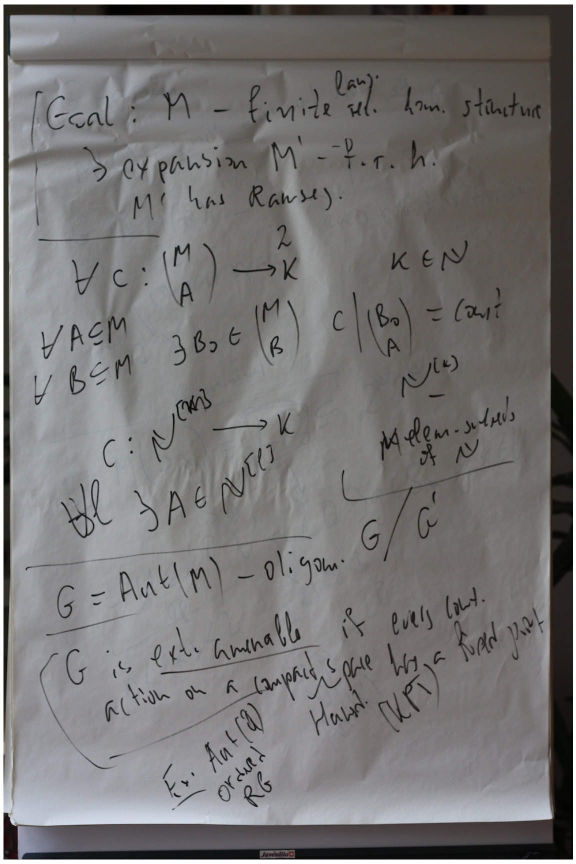
Problem 26. Find the right analog of *oligomorphic* for clones – weaker than the clone just containing a weakly oligomorphic transformation monoid, but strong enough to prove theorems, e.g., Topological Birkhoff.

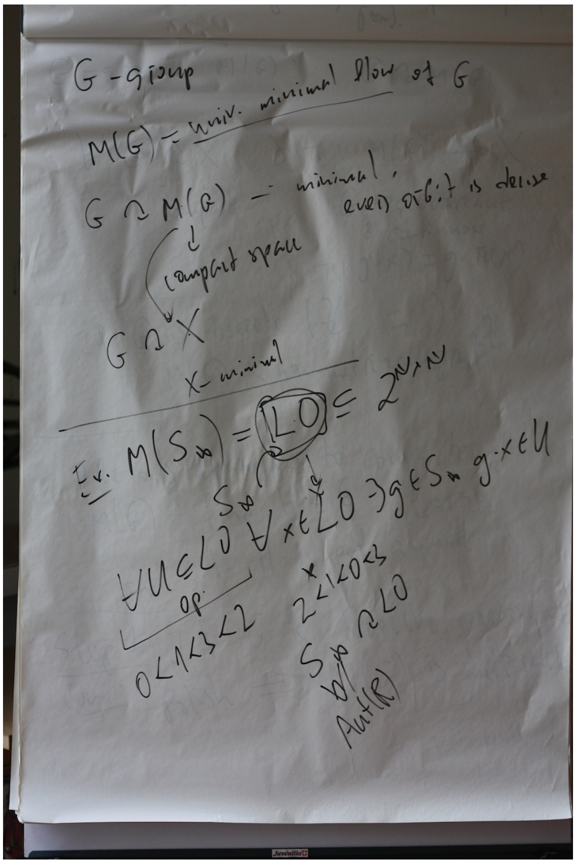
The Ordener Lectures

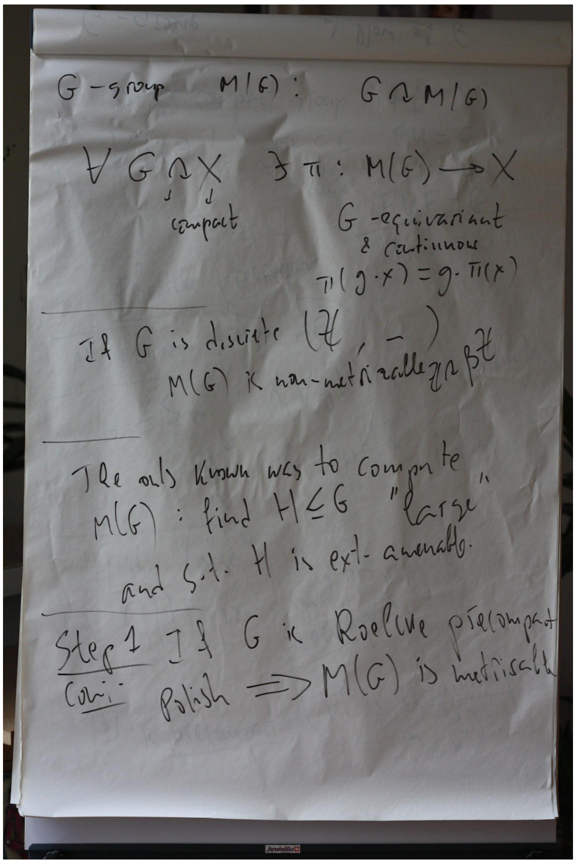

4/10/12
finite
NP = ESO
Syntactic fragment of ESO
SNP
CSP

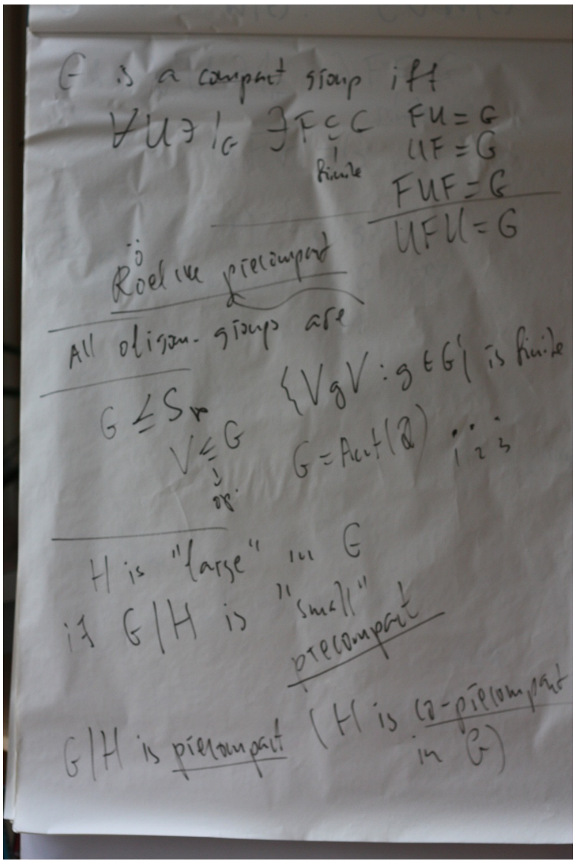

Lovász 1967.
exponential of structures.

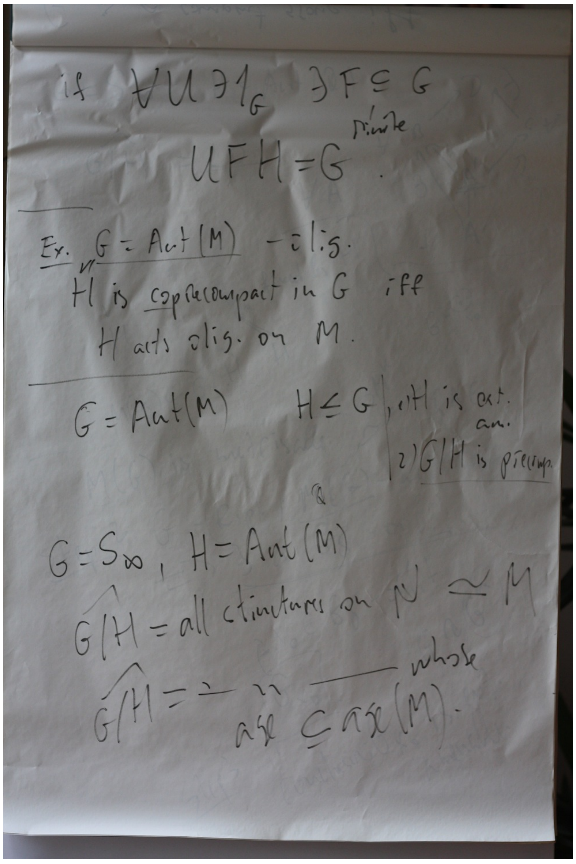

Let C & B be ω-cat str.
then there exists D ω-cat
s.t.
and
duality pair
gap pair

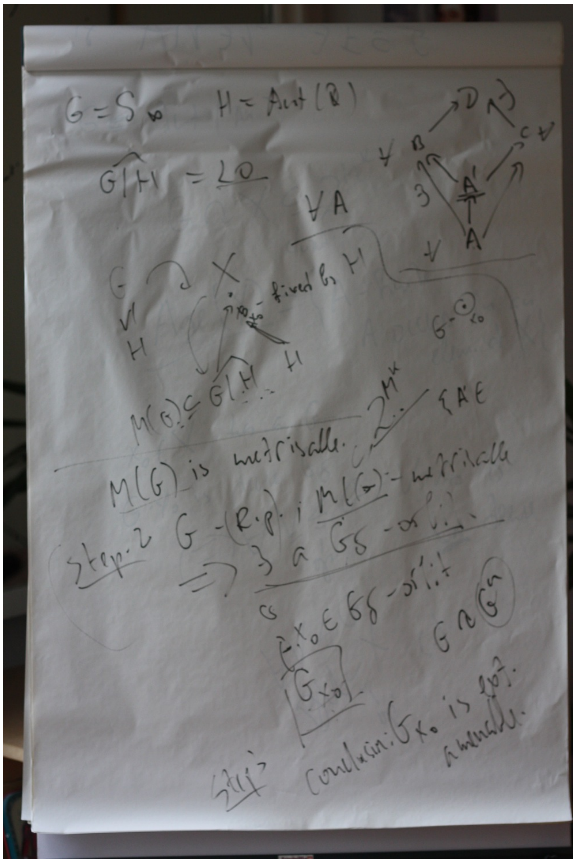

Goal: M - finite lang. rel. hom. structure
∃ expansion M′
M′ has Ramsey).
G = Aut(M) - oligom.
G is ext. amenable if every cont. action on a compact space has a fixed point
Ex: Aut(Q)
ordered RG

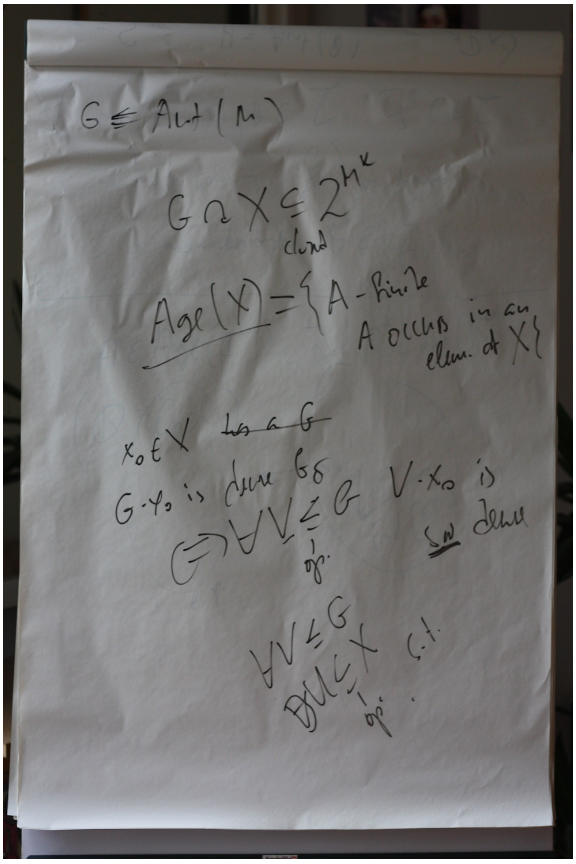

G - group
M(G) = univ. minimal flow of G
G ↷ M(G) — minimal, every orbit is dense
compact space
G ↷ X
X - minimal
Ex. $M(S_\infty) = LO \subseteq 2^{\mathbb{N}\times\mathbb{N}}$
S_∞
$\forall U \subseteq LO\ \forall x \in LO\ \exists g \in S_\infty\ g\cdot x \in U$
op.
$0<1<3<2$
$2<1<0<3$
$S_\infty \curvearrowright LO$
Aut(Q)

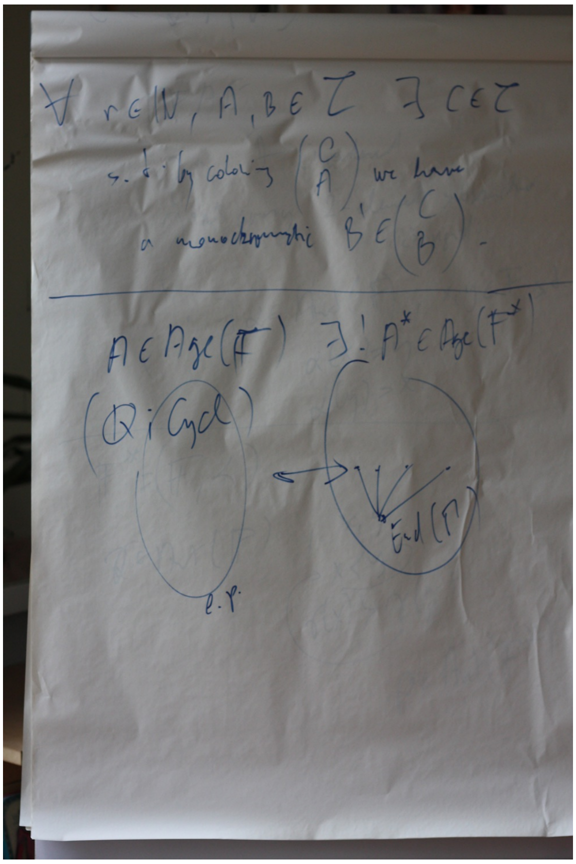

G - group M(G): G ↷ M(G)
$\forall\, G \curvearrowright X\ \exists \pi : M(G) \to X$
compact
G - equivariant & continuous
$\pi(g\cdot x) = g\cdot \pi(x)$
If G is discrete ($\mathbb{Z}$, –)
M(G) is non-metrizable $\supsetneq \beta\mathbb{Z}$
The only known way to compute M(G): find $H \leq G$ "large" and s.t. H is ext. amenable.
Step 1 If G is Roelcke precompact
Cor: Polish ⟹ M(G) is metrisable

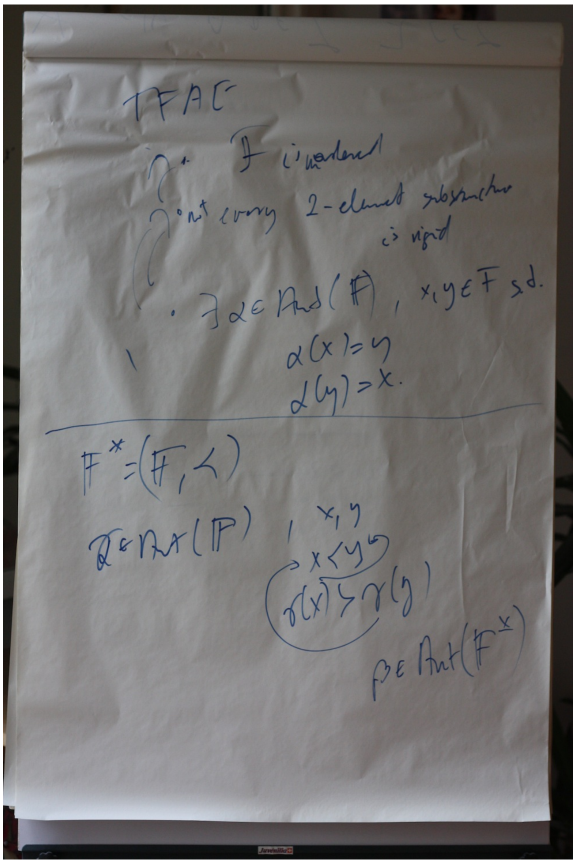

G is a compact group iff
$\forall U \ni 1_G\ \exists F \subseteq G$ finite $FU = G$, $UF = G$, $FUF = G$, $UFU = G$
Roelcke precompact
All oligom. groups are
$G \leq S_\infty$ $V \leq G$ op. $\{VgV : g \in G\}$ is finite
$G = Aut(\mathbb{Q})$
H is "large" in G if G/H is "small" precompact
G/H is precompact (H is co-precompact in G)

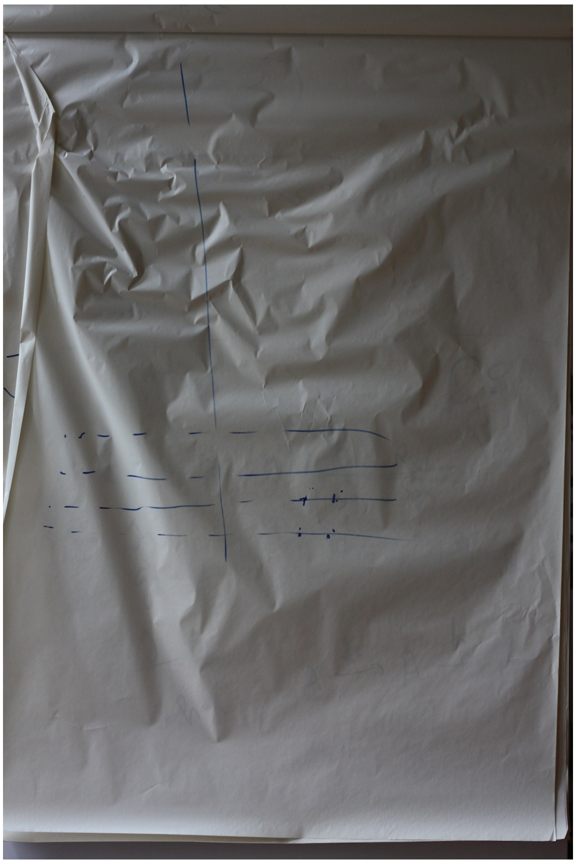

if $\forall U \ni 1_G\ \exists F \subseteq G$ finite $UFH = G$.
Ex. G = Aut(M) - olig.
H is co-precompact in G iff H acts olig. on M.
$G = Aut(M)$ $H \leq G$ 1) H is ext. am. 2) G/H is precomp.
$G = S_\infty$, $H = Aut(M)$
$\widehat{G/H}$ = all structures on $\mathbb{N} \simeq M$
$\widehat{G/H}$ = " " whose age ⊆ age(M).

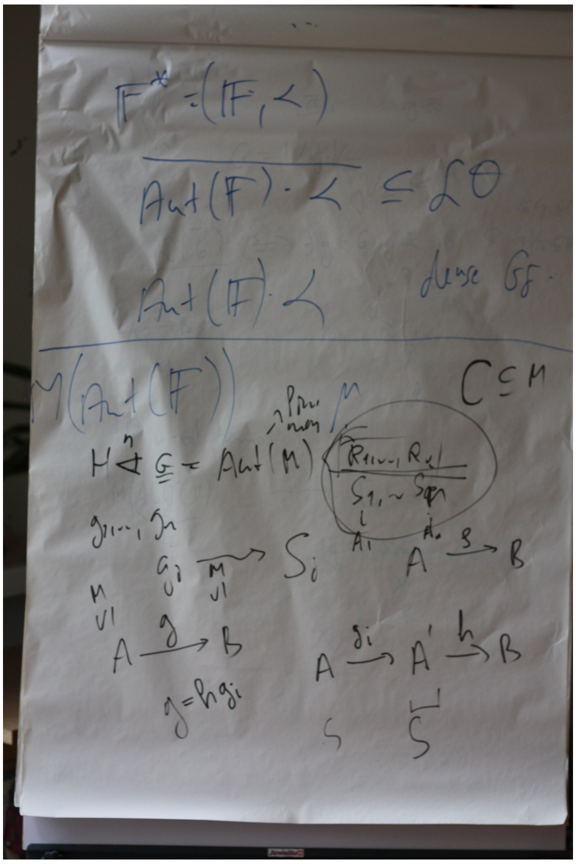

G = S∞ H = Aut(Q)
G/H = LO
∀A
M(G) is metrisable.
Step 2
Conclusion: G·x₀ is not amenable.

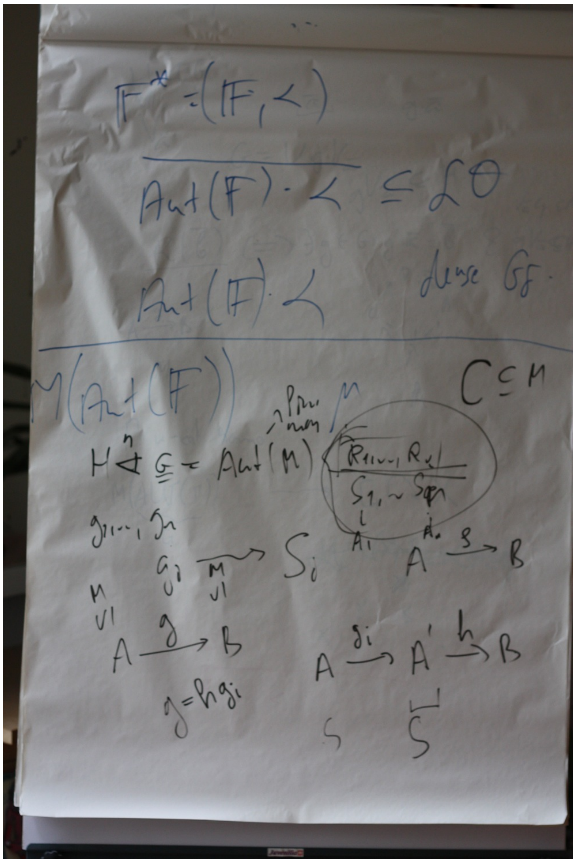

G ≤ Aut(M)
G ↷ X ⊆ 2^{M^k}
closed
Age(X) = {A - finite
A occurs in an
elem. of X}
x₀ ∈ X
G·x₀ is dense Gδ
V·x₀ is
SW dense
∀V ≤ G
op.
∃U ≤ X
op.

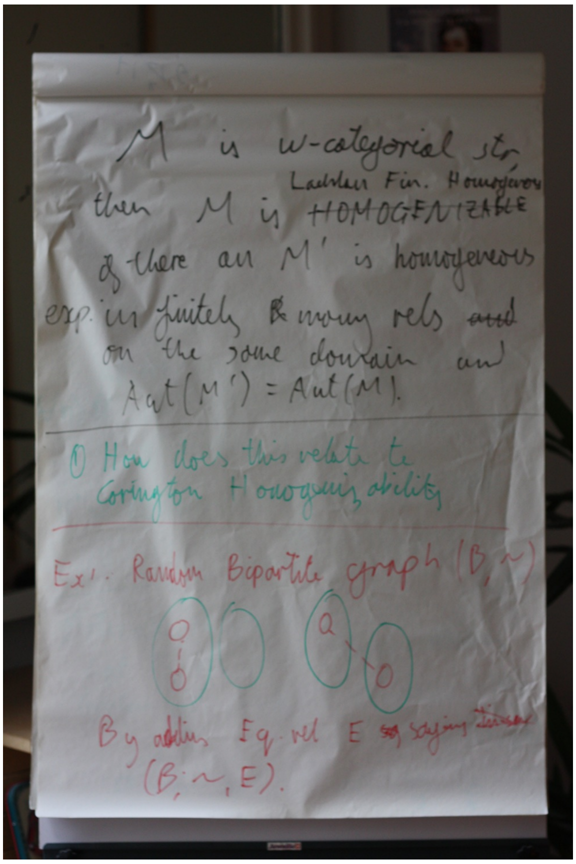

∀ r ∈ ℕ, A, B ∈ 𝒞 ∃ C ∈ 𝒞
s.t. by coloring (C A) we have
a monochromatic B′ ∈ (C B).
A ∈ Age(𝔽) ∃! A* ∈ Age(𝔽*)
(ℚ; Cycl)
End(ℙ)
e.p.

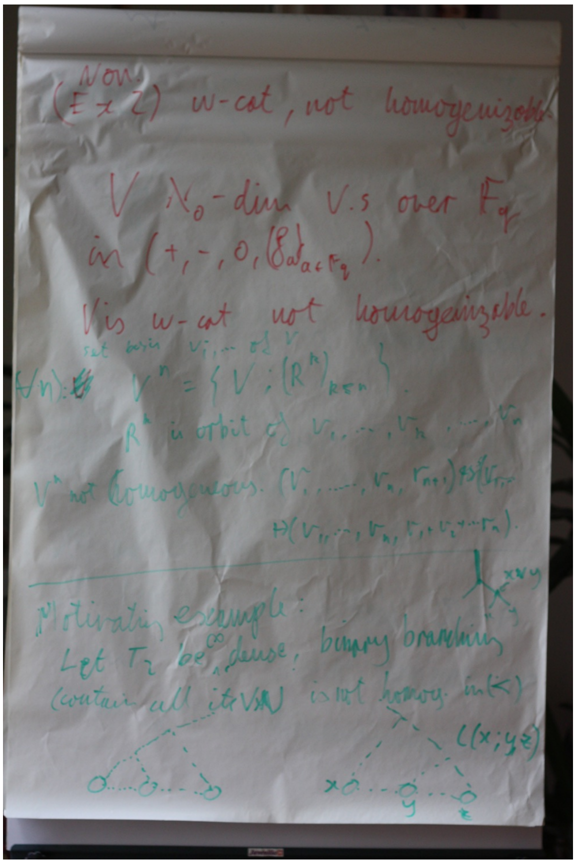

TFAE
𝔽 is ordered
every 2-element substructure
is rigid
∃ α ∈ Aut(𝔽), x, y ∈ F s.t.
α(x) = y
α(y) = x.
𝔽* = (𝔽, <)
γ ∈ Aut(ℙ)
x, y
x < y
γ(x) > γ(y)
β ∈ Aut(𝔽*)

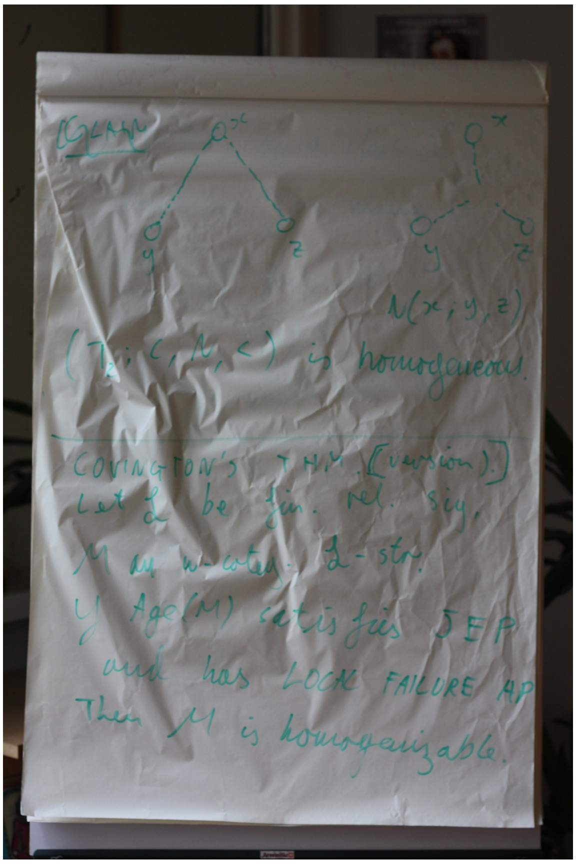


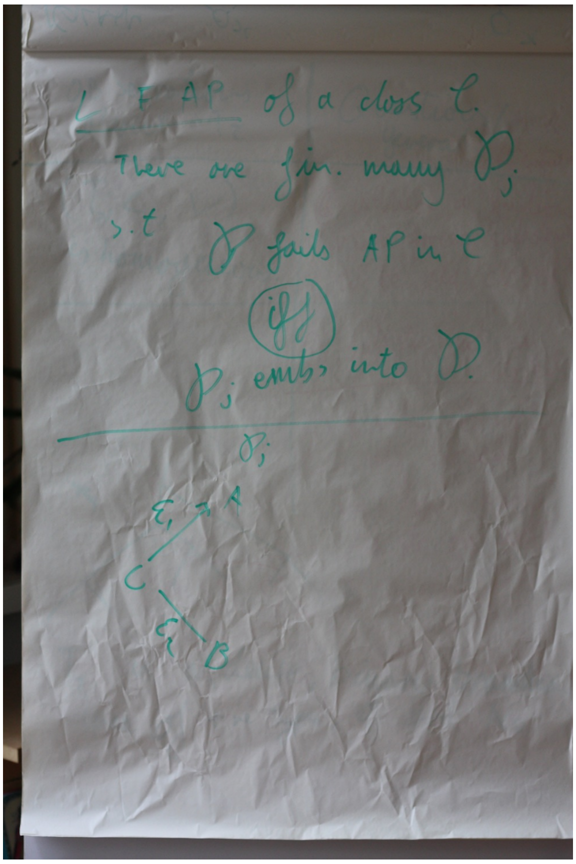

$\mathbb{F}^* = (\mathbb{F}, <)$
$\mathrm{Aut}(\mathbb{F})\cdot < \subseteq \mathcal{LO}$
$\mathrm{Aut}(\mathbb{F})\cdot <$ dense G_δ
$M(\mathrm{Aut}(\mathbb{F}))$
$C \subseteq M$
$H \trianglelefteq G = \mathrm{Aut}(M)$
$g_i \ldots, g_n$
$A \xrightarrow{g} B$
$g = h g_i$
$A \xrightarrow{g_i} A' \xrightarrow{h} B$

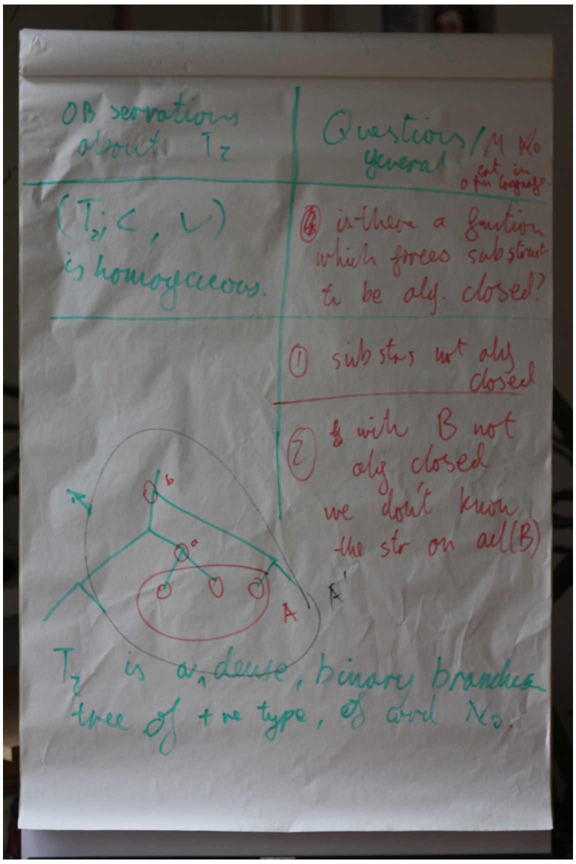

$\mathbb{F}^* = (\mathbb{F}, <)$
$\mathrm{Aut}(\mathbb{F})\cdot < \subseteq \mathcal{LO}$
$\mathrm{Aut}(\mathbb{F})\cdot <$ dense G_δ
$M(\mathrm{Aut}(\mathbb{F}))$
$C \subseteq M$
$H \trianglelefteq G = \mathrm{Aut}(M)$
$g_i \ldots, g_n$
$A \xrightarrow{g} B$
$g = h g_i$
$A \xrightarrow{g_i} A' \xrightarrow{h} B$

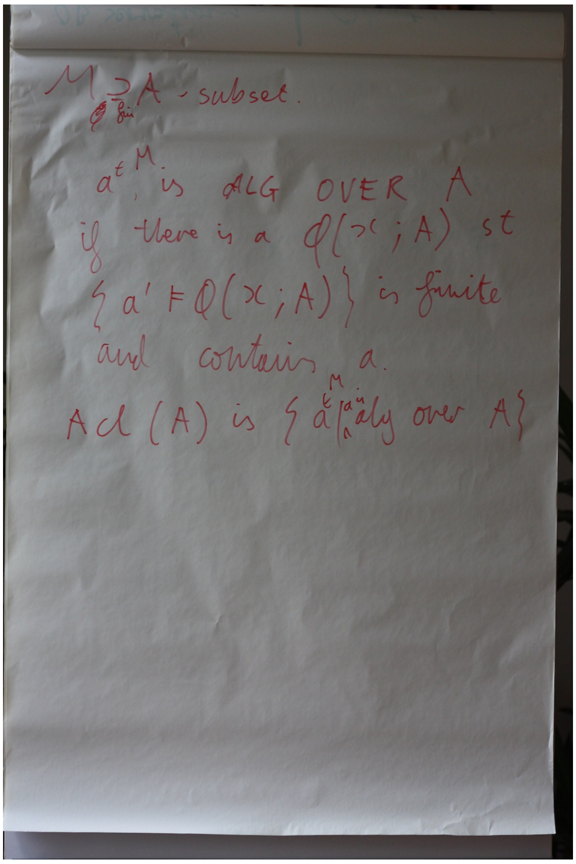

M is ω-categorical str,
Lachlan Fin. Homogeneous
then M is ~~HOMOGENIZABLE~~
if there an M' is homogeneous
exp. in finitely many rels
on the same domain an
$\mathrm{Aut}(M') = \mathrm{Aut}(M)$.
Q How does this relate to
Covington Homogenizability
Ex: Random Bipartite graph $(B, \sim)$
By adding Eq. rel E saying
$(B; \sim, E)$.

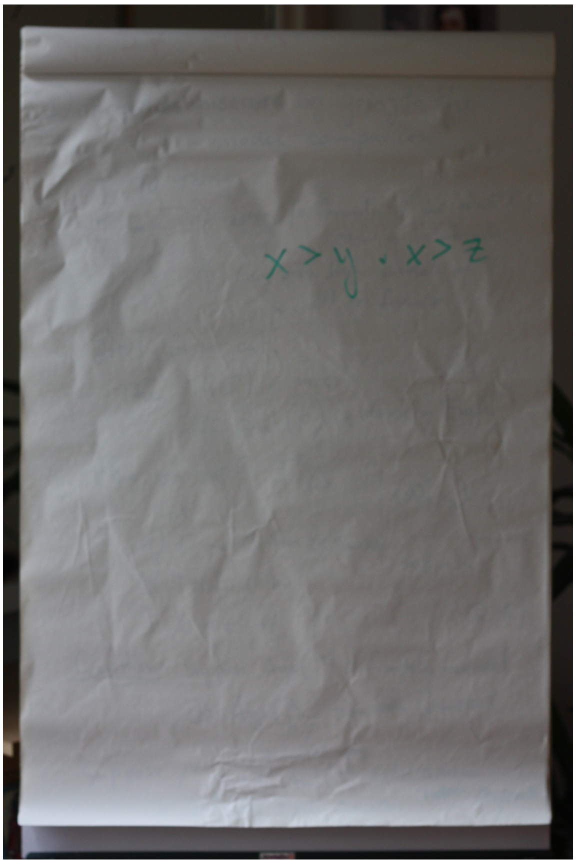

Now
(Ex 2) ω-cat, not homogenizable
V ℵ0-dim v.s over Fq
in (+, −, 0, {fa}a∈Fq).
V is ω-cat not homogenizable.
Motivating example:
Let T2 be dense, binary branching

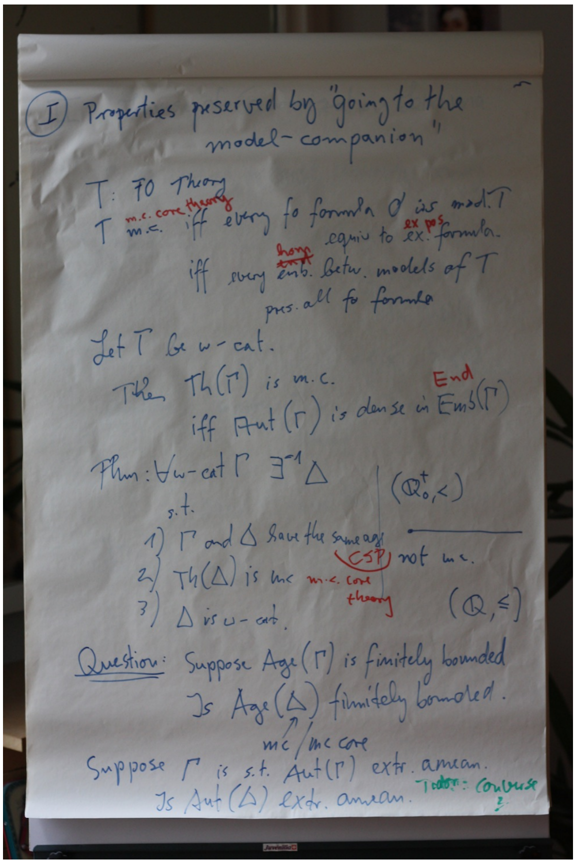

CLAIM
(T2; C, L, <) is homogeneous.
COVINGTON'S THM [version]
Let L be fin. rel. sig,
M an ω-cat L-str.
If Age(M) satisfies JEP
and has LOCAL FAILURE AP
Then M is homogenizable.

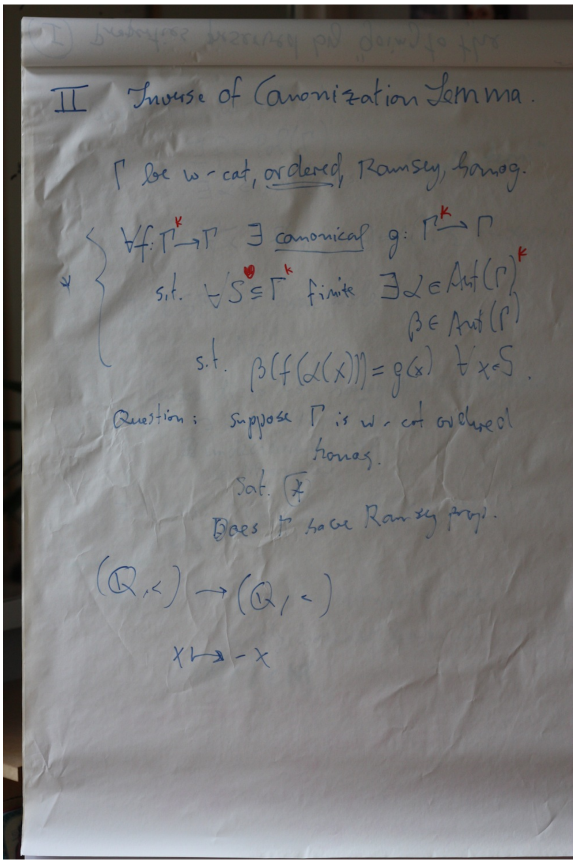

LFAP of a class C.
There are fin. many D,
s.t
D fails AP in C
iff
D, embs into D.

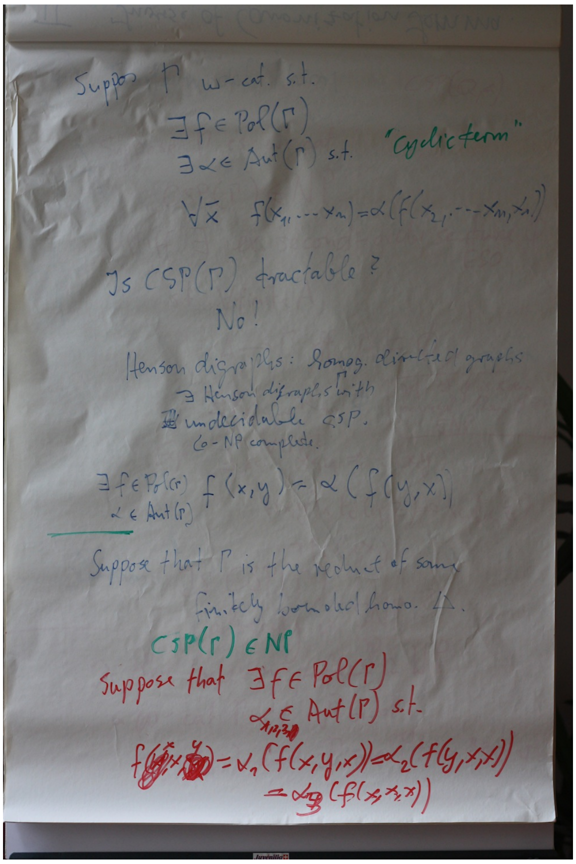

OBservations about T2
Questions / M ℵ0 general
(T2; <, L)
is homogeneous.
is there a function which forces substructures to be alg. closed?
① sub strs not alg. closed
② with B not alg. closed
we don't know the str on acl(B)
T2 is a dense, binary branching tree of + ve type, of card ℵ0.

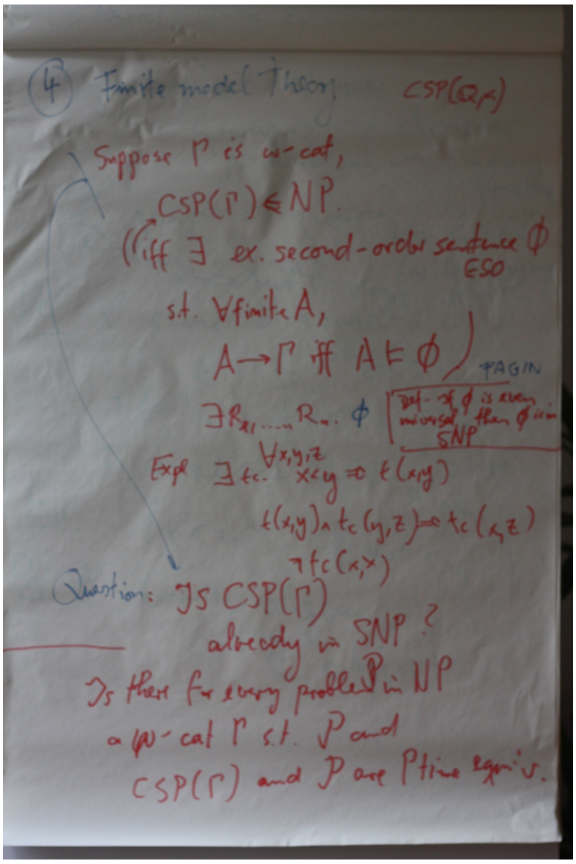

M ⊇ A - subset.
a ∈ M is ALG OVER A
if there is a φ(x;A) st
{a' ⊨ φ(x;A)} is finite
and contains a.
Acl(A) is {a ∈ M alg over A}

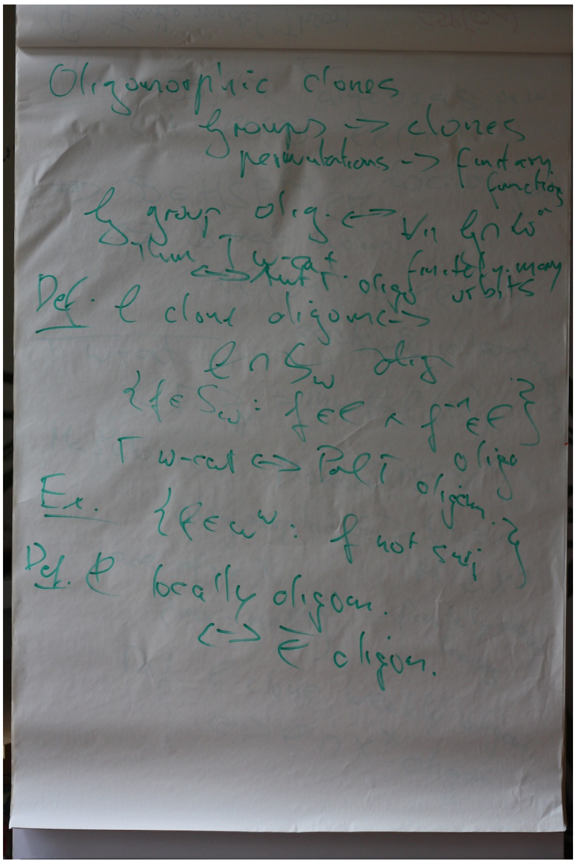

x > y · x > z

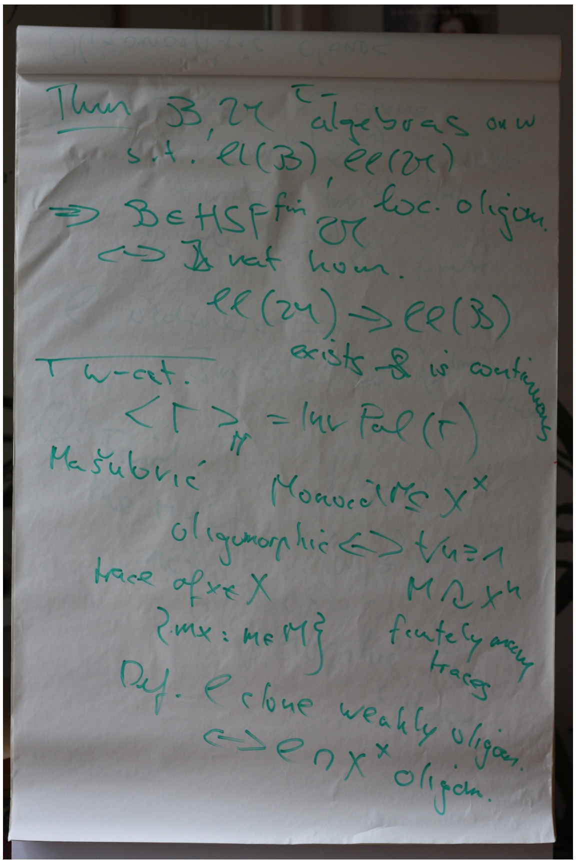

I Properties preserved by "going to the model-companion"
T: FO Theory
T m.c. iff every fo formula φ is mod. T equiv to ex. formula.
m.c. core theory
ex pos
iff every emb. betw. models of T pres. all fo formula
homo
Let Γ be ω-cat.
Then Th(Γ) is m.c. iff Aut(Γ) is dense in Emb(Γ)
End
Thm: ∀ω-cat Γ ∃! Δ
s.t.
1) Γ and Δ have the same age
CSP
2) Th(Δ) is mc
m.c. core theory
3) Δ is ω-cat.
(Q,≤)
not m.c.
Question: Suppose Age(Γ) is finitely bounded
Is Age(Δ) finitely bounded.
mc / mc core
Suppose Γ is s.t. Aut(Γ) extr. amean.
Is Aut(Δ) extr. amean.
Todor: Converse ?

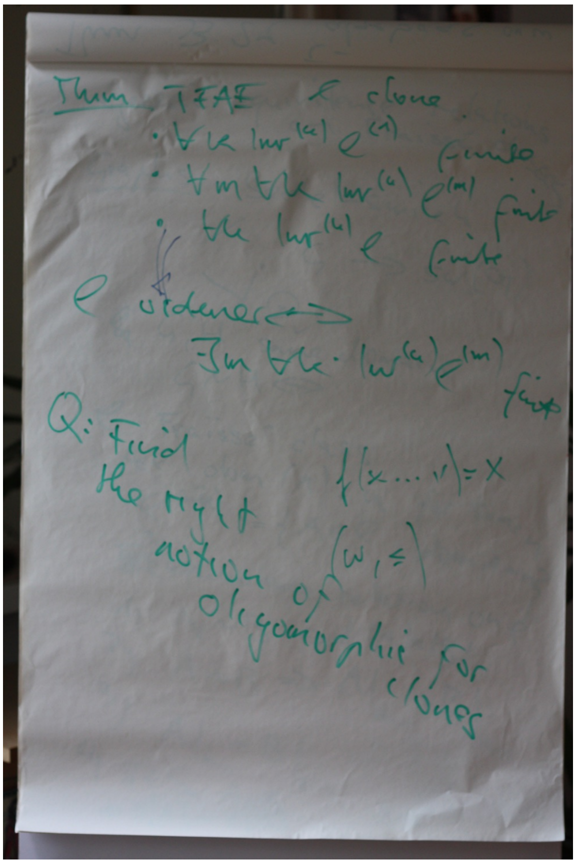

II Inverse of Canonization Lemma.
Γ be ω-cat, ordered, Ramsey, homog.
∀f: Γ^k → Γ ∃ canonical g: Γ^k → Γ
s.t. ∀S ⊆ Γ^k finite ∃α ∈ Aut(Γ)^k
β ∈ Aut(Γ)
s.t. β(f(α(x))) = g(x) ∀x∈S.
Question: suppose Γ is ω-cat ordered homog.
Sat. (*)
Does Γ have Ramsey prop.
(Q,<) → (Q,<)
x ↦ -x

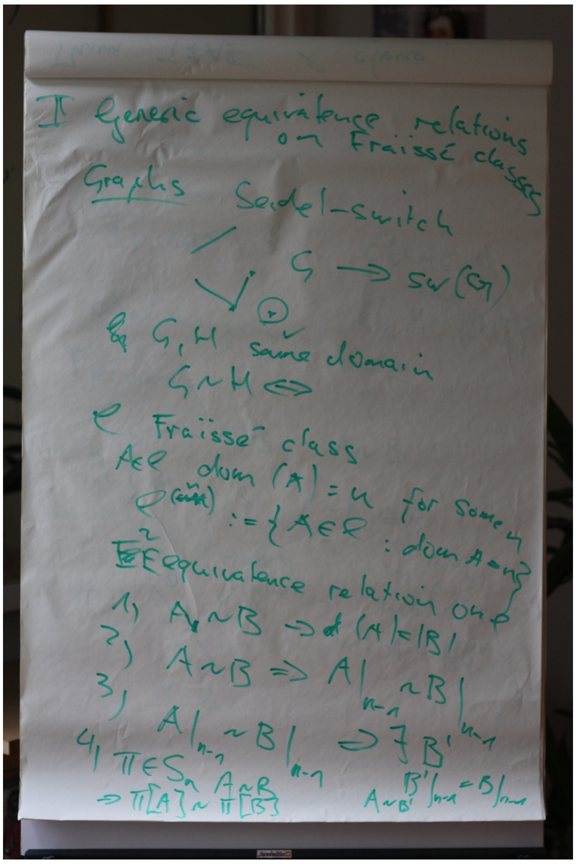

Suppose Γ ω-cat. s.t.
∃f ∈ Pol(Γ)
∃α ∈ Aut(Γ) s.t.
"cyclic term"
∀x̄ f(x1, ..., xn) = α(f(x2, ..., xn, x1))
Is CSP(Γ) tractable?
No!
Henson digraphs: homog. directed graphs
∃ Henson digraphs with undecidable CSP.
co-NP complete.
∃f ∈ Pol(Γ) f(x,y) = α(f(y,x))
α ∈ Aut(Γ)
Suppose that Γ is the reduct of some finitely bounded homo. Δ.
CSP(Γ) ∈ NP
Suppose that ∃f ∈ Pol(Γ)
α1,2,3 ∈ Aut(Γ) s.t.
f(y,x,x) = α1(f(x,y,x)) = α2(f(y,x,x))
= α3(f(x,x,x))

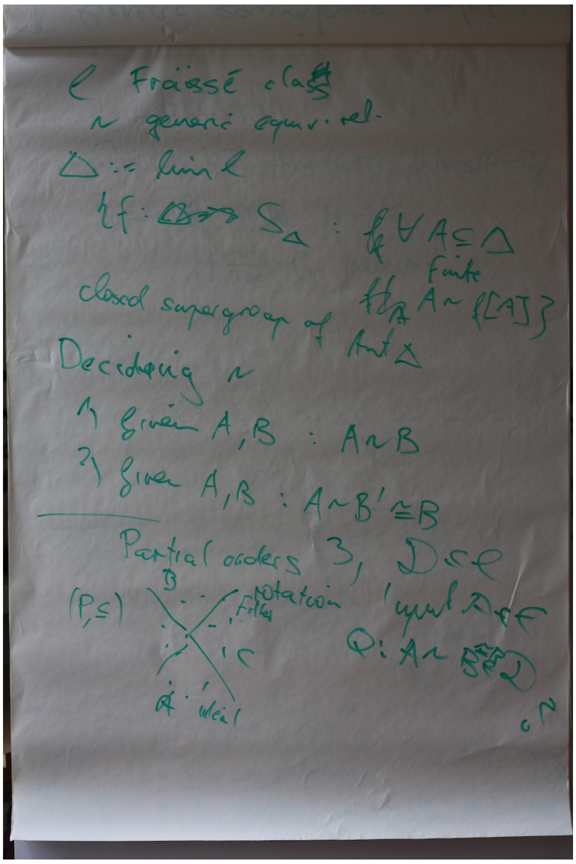

4 Finite model theory
CSP(Q;<)
Suppose Γ is ω-cat,
CSP(Γ) ∈ NP
(iff ∃ ex. second-order sentence Φ
ESO
s.t. ∀ finite A,
A → Γ iff A ⊨ Φ)
FAGIN
∃R1, ..., Rn. ϕ
Def. ∃ϕ is every universal then ϕ is in SNP
Exp ∃tc. ∀x,y,z x<y ⇒ tc(x,y)
tc(x,y) ∧ tc(y,z) ⇒ tc(x,z)
¬tc(x,x)
Question: Is CSP(Γ) already in SNP?
Is there for every problem P in NP
a ω-cat Γ s.t. P and
CSP(Γ) and P are Ptime equiv.

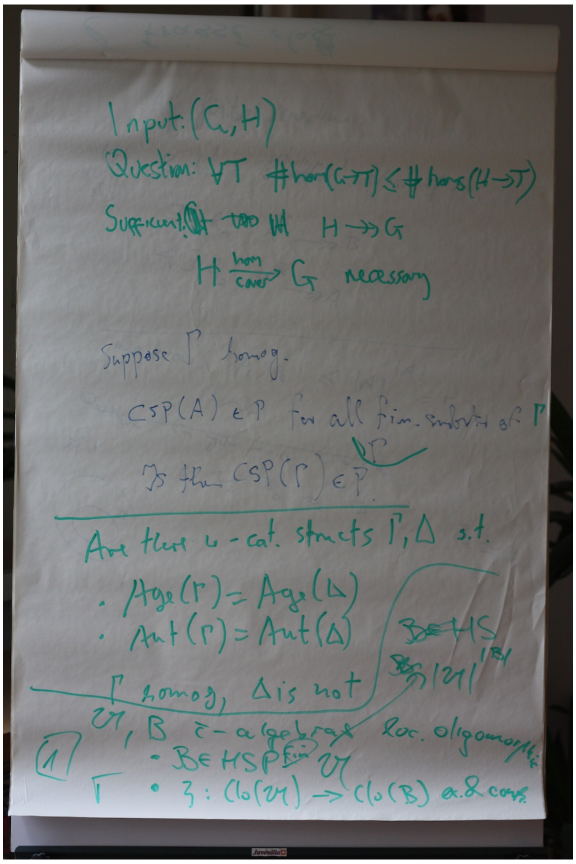

Oligomorphic clones
groups → clones
permutations → finitary functions
G group olig. ⇔ ∀n G ↷ ω^n
then ∃ ω-cat ⇔ Aut Γ. olig finitely many orbits
Def. C clone oligom ⇔
C ∩ S_ω olig
{f ∈ S_ω : f ∈ C ∧ f^{-1} ∈ C}
Γ ω-cat ⇔ Pol Γ oligo
Ex. {f ∈ ω^ω : f not surj} oligom.
Def. C locally oligom.
⇔ C̄ oligom.


Thm B, A τ-algebras on ω
s.t. Cl(B), Cl(A) loc. oligom.
⇒ B ∈ HSP^fin A
⇔ ∃ cat hom.
Cl(A) → Cl(B)
exists & is continuous
Γ ω-cat.
⟨Γ⟩_H = Inv Pol(Γ)
Mašulović Monoid M ⊆ X^X
oligomorphic ⇔ ∀n ≥ 1
trace of x ∈ X M ↷ X^n
{mx : m ∈ M} finitely many traces
Def. C clone weakly oligom.
⇔ C ∩ X^X oligom.


Thm TFAE C clone
· ∀k Inv^(k) C^(1) finite
· ∀m ∀k Inv^(k) C^(m) finite
· ∀k Inv^(k) C finite
C oligomorphic ⟺
∃m ∀k: Inv^(k) C^(m) finite
Q: Find the right notion of oligomorphic for clones
f(x … x) = x
(ω, ≤)


II Generic equivalence relations on Fraïssé classes
Graphs Seidel-switch
G → sw(G)
G, H same domain
G ~ H ⟺
C Fraïssé class
A ∈ C dom(A) = n for some n
C^(n) := {A ∈ C : dom A = n}
~ equivalence relation on C
1) A ~ B ⇒ |A| = |B|
2) A ~ B ⇒ A|n−1 ~ B|n−1
3) A|n−1 ~ B|n−1 ⇒ ∃B′ A ~ B′ B′|n−1 = B|n−1
4) π ∈ S_n A ~ B ⇒ π[A] ~ π[B]


C Fraïssé class
~ generic equiv. rel.
Δ := lim C
{f : Δ → S_Δ : ∀A ⊆ Δ finite A ~ f[A]}
closed supergroup of Aut Δ
Deciding ~
1) given A, B : A ~ B
2) given A, B : A ~ B′ ≅ B
Partial orders
(P, ≤)
rotation
filter
ideal
Q: A ~


Input: (G, H)
Question: ∀T #hom(G→T) ≤ #hom(H→T)
Sufficient: H ↠ G
H →(hom cover) G necessary
Suppose Γ homog.
CSP(A) ∈ P for all fin. substr. of Γ
Is then CSP(Γ) ∈ P
Are there ω-cat. structs Γ, Δ s.t.
· Age(Γ) = Age(Δ)
· Aut(Γ) = Aut(Δ)
Γ homog, Δ is not
B ∈ HS
𝔄, B τ-algebras loc. oligomorphic
· B ∈ HSP^fin 𝔄
· ξ : Clo(𝔄) → Clo(B) cont.