SUBLATTICES OF THE LATTICE OF LOCAL CLONES

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Abstract. We investigate the complexity of the lattice of local clones over a countably infinite base set. In particular, we prove that this lattice contains all algebraic lattices with at most countably many compact elements as complete sublattices, but that the class of lattices embeddable into the local clone lattice is strictly larger than that.

1. Local clones

Fix a countably infinite base set X, and denote for all $n \geq 1$ the set $X^{X^n} = \{f : X^n \to X\}$ of n-ary operations on X by $\mathscr{O}^{(n)}$. Then the union $\mathscr{O} := \bigcup_{n\geq 1} \mathscr{O}^{(n)}$ is the set of all finitary operations on X. A *clone* $\mathscr{C}$ is a subset of $\mathscr{O}$ satisfying the following two properties:

- $\mathscr{C}$ contains all projections, i.e. for all $1 \leq k \leq n$ the operation $\pi^n_k \in \mathscr{O}^{(n)}$ defined by $\pi^n_k(x_1, \ldots, x_n) = x_k$, and
- $\mathscr{C}$ is closed under composition, i.e. whenever $f \in \mathscr{C}$ is n-ary and $g_1, \ldots, g_n \in \mathscr{C}$ are m-ary, then the operation $f(g_1, \ldots, g_n) \in \mathscr{O}^{(m)}$ defined by
$$(x_1, \ldots, x_m) \mapsto f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$$
also is an element of $\mathscr{C}$.

Since arbitrary intersections of clones are again clones, the set of all clones on X, equipped with the order of inclusion, forms a complete lattice $\mathrm{Cl}(X)$. In this paper, we are not interested in all clones of $\mathrm{Cl}(X)$, but only in clones which satisfy an additional topological closure property: Equip X with the discrete topology, and $\mathscr{O}^{(n)} = X^{X^n}$ with the corresponding product topology (Tychonoff topology), for every $n \geq 1$. A clone $\mathscr{C}$ is called *locally closed* or just *local* iff each of its n-ary fragments $\mathscr{C} \cap \mathscr{O}^{(n)}$ is a closed subset of $\mathscr{O}^{(n)}$. Equivalently, a clone $\mathscr{C}$ is local iff it satisfies the following interpolation property:

> For all $n \geq 1$ and all $g \in \mathscr{O}^{(n)}$, if for all finite $A \subseteq X^n$ there exists an n-ary $f \in \mathscr{C}$ which agrees with g on A, then $g \in \mathscr{C}$.

2000 *Mathematics Subject Classification.* Primary 08A40; secondary 08A05.

Key words and phrases. clone; local closure; complete lattice; embedding.

The author is grateful for support through project P17812 as well as through the Erwin Schrödinger Fellowship of the Austrian Science Fund.

Again, taking the set of all local clones on X, and ordering them according to set-theoretical inclusion, one obtains a complete lattice, which we denote by $\mathrm{Cl}_{\mathrm{loc}}(X)$: This is because intersections of clones are clones, and because arbitrary intersections of closed sets are closed. We are interested in the structure of $\mathrm{Cl}_{\mathrm{loc}}(X)$, in particular in how complicated it is as a lattice.

Before we start our investigations, we give an alternative description of local clones which will be useful. Let $f \in \mathscr{O}^{(n)}$ and let $\rho \subseteq X^m$ be a relation. We say that f *preserves* ρ iff $f(r_1, \ldots, r_n) \in \rho$ whenever $r_1, \ldots, r_n \in \rho$, where $f(r_1, \ldots, r_n)$ is calculated componentwise. For a set of relations $\mathscr{R}$, we write $\mathrm{Pol}(\mathscr{R})$ for the set of those operations in $\mathscr{O}$ which preserve all $\rho \in \mathscr{R}$. The operations in $\mathrm{Pol}(\mathscr{R})$ are called *polymorphisms* of $\mathscr{R}$, hence the symbol Pol. The following is due to [Rom77], see also the textbook [Sze86].

Proposition 1. $\mathrm{Pol}(\mathscr{R})$ *is a local clone for all sets of relations* $\mathscr{R}$*. Moreover, every local clone is of this form.*

Similarly, for an operation $f \in \mathscr{O}^{(n)}$ and a relation $\rho \subseteq X^m$, we say that ρ is *invariant* under f iff f preserves ρ. Given a set of operations $\mathscr{F} \subseteq \mathscr{O}$, we write $\mathrm{Inv}(\mathscr{F})$ for the set of all relations which are invariant under all $f \in \mathscr{F}$. Since arbitrary intersections of local clones are local clones again, the mapping on the power set of $\mathscr{O}$ which assigns to every set of operations $\mathscr{F} \subseteq \mathscr{O}$ the smallest local clone $\langle \mathscr{F} \rangle_{loc}$ containing $\mathscr{F}$ is a hull operator, the closed elements of which are exactly the local clones. Using the operators Pol and Inv which connect operations and relations, one obtains the following well-known alternative for describing this operator (confer [Rom77] or [Sze86]).

Proposition 2. *Let* $\mathscr{F} \subseteq \mathscr{O}$*. Then* $\langle \mathscr{F} \rangle_{loc} = \mathrm{Pol}\,\mathrm{Inv}(\mathscr{F})$*.*

As already mentioned, it is the aim of this paper to investigate the structure of the local clone lattice. So far, this lattice has been studied only sporadically, e.g. in [RS82], [RS84]. There, the emphasis was put on finding local completeness criteria for sets of operations $\mathscr{F} \subseteq \mathscr{O}$, i.e. on how to decide whether or not $\langle \mathscr{F} \rangle_{loc} = \mathscr{O}$. Only very recently has the importance of the local clone lattice to questions from model theory and theoretical computer science been revealed:

Let $\Gamma = (X, \mathscr{R})$ be a countably infinite structure; that is, X is a countably infinite base set and $\mathscr{R}$ is a set of finitary relations on X. Consider the expansion Γ' of Γ by all relations which are first-order definable from Γ. More precisely, Γ' has X as its base set and its relations $\mathscr{R}'$ consist of all finitary relations which can be defined from relations in $\mathscr{R}$ using first-order formulas. A *reduct* of Γ' is a structure $\Delta = (X, \mathscr{D})$, where $\mathscr{D} \subseteq \mathscr{R}'$. We also call Δ a reduct of Γ, which essentially amounts to saying that we expect our structure Γ to be closed under first-order definitions. Clearly, the set of reducts of Γ is in one-to-one correspondence with the power set of $\mathscr{R}'$, and therefore not of much interest as a partial order. However, it might be more reasonable to consider such reducts up to, say, *first-order interdefinability.*

That is, we may consider two reducts $\Delta_1 = (X, \mathscr{D}_1)$ and $\Delta_2 = (X, \mathscr{D}_2)$ the same iff their first-order expansions coincide, or equivalently iff all relations in $\mathscr{D}_1$ are first-order definable in Δ_2 and vice-versa.

In 1976, P. J. Cameron [Cam76] showed that there are exactly five reducts of $(\mathbb{Q}, <)$ up to first-order interdefinability. Recently, M. Junker and M. Ziegler gave a new proof of this fact, and established that $(\mathbb{Q}, <, a)$, the expansion of $(\mathbb{Q}, <)$ by a constant a, has 114 reducts [JZ05]. S. Thomas proved that the first-order theory of the random graph also has exactly five reducts, up to first-order interdefinability [Tho91].

These examples have in common that the structures under consideration are *ω-categorical*, i.e., their first-order theories determine their countable models up to isomorphism. This is no coincidence: For, given an ω-categorical structure Γ, its reducts up to first-order interdefinability are in one-to-one correspondence with the locally closed permutation groups which contain the automorphism group of Γ, providing a tool for describing such reducts (confer [Cam90]).

A natural variant of these concepts is to consider reducts up to *primitive positive interdefinability*. That is, we consider two reducts Δ_1, Δ_2 of Γ the same iff their expansions by all relations which are definable from each of the structures by primitive positive formulas coincide. (A first-order formula is called *primitive positive* iff it is of the form $\exists \overline{x}(\phi_1 \wedge \cdots \wedge \phi_l)$ for atomic formulas $\phi_1, \ldots, \phi_l$.) It turns out that for ω-categorical structures Γ, the *local clones* containing all automorphisms of Γ are in one-to-one correspondence with those reducts of the first-order expansion of Γ which are closed under primitive positive definitions. This recent connection, which relies on a theorem from [BN06], has already been utilized in [BCP], where the reducts of $(\mathbb{N}, =)$ have been classified by this method (a surprisingly complicated task, as it turned out!).

We mention in passing that distinguishing relational structures up to primitive positive interdefinability, and therefore understanding the structure of $\mathrm{Cl}_{\mathrm{loc}}(X)$, has recently gained significant importance in theoretical computer science, more precisely for what is known as the Constraint Satisfaction Problem; see [BKJ05] or [Bod04].

2. The structure of the local clone lattice

For our investigations of $\mathrm{Cl}_{\mathrm{loc}}(X)$ we will need the concept of an *algebraic lattice*.

An element a of a complete lattice $\mathfrak{L}$ is called *compact* iff it has the property that whenever $A \subseteq \mathfrak{L}$ and $a \leq \bigvee A$, then there exists a finite $A' \subseteq A$ with $a \leq \bigvee A'$. $\mathfrak{L}$ is called *algebraic* iff every element is the supremum of compact elements. By their very definition, algebraic lattices are determined by their compact elements. More precisely, the compact elements form a join-semilattice, and every algebraic lattice is isomorphic to the lattice of all join-semilattice ideals of the join-semilattice of compact elements, see e.g.

the textbook [CD73]. Whereas the lattice $\mathrm{Cl}(X)$ of all (non-local) clones over X is algebraic, it has been discovered recently in the survey paper [GP] that the local clone lattice $\mathrm{Cl}_{\mathrm{loc}}(X)$ is far from being so; since that paper is yet to appear, we include a sketch of the short proof here.

Proposition 3. *The only compact element in the lattice* $\mathrm{Cl}_{\mathrm{loc}}(X)$ *is the clone of projections.*

Proof. Fix a linear order $\leq$ on X without last element. Denote the arity of every $f \in \mathscr{O}$ by n_f. For each $a \in X$ let

$$\begin{aligned} \mathscr{C}_a &:= \{f \in \mathscr{O} : \forall x \in X^{n_f}\,(f(x) \leq a)\} \quad \text{and} \\ \mathscr{D}_a &:= \{f \in \mathscr{O} : \forall x \in X^{n_f}\,(\max(x) \geq a \;\Rightarrow\; f(x) \geq \max(x))\,\}. \end{aligned}$$

Then

(1) $\langle \mathscr{C}_a \rangle_{loc} = \mathscr{C}_a \cup \{\, \pi^n_k : 1 \leq k \leq n < \omega \,\}$.
(2) $\langle \mathscr{D}_a \rangle_{loc}$ is the set of all operations which are essentially in $\mathscr{D}_a$ (i.e., except for dummy variables).
(3) If $a \leq a'$, then $\mathscr{C}_a \subseteq \mathscr{C}_{a'}$ and $\mathscr{D}_a \subseteq \mathscr{D}_{a'}$, hence every finite union of clones $\langle \mathscr{C}_a \rangle_{loc}$ (or $\langle \mathscr{D}_a \rangle_{loc}$, respectively) is again a clone of this form.
(4) The local closure of $\bigcup_a \langle \mathscr{D}_a \rangle_{loc}$, as well as the local closure of $\bigcup_a \langle \mathscr{C}_a \rangle_{loc}$, is the clone of all operations $\mathscr{O}$.
(5) If $f \in \mathscr{O}$ has unbounded range, then $f \notin \bigcup_a \langle \mathscr{C}_a \rangle_{loc}$ (unless f is a projection).
(6) If $f \in \mathscr{O}$ has bounded range, then $f \notin \bigcup_a \langle \mathscr{D}_a \rangle_{loc}$.
(7) No local clone $\mathscr{C}$ (other than the clone of projections) is compact in $\mathrm{Cl}_{loc}(X)$: If $\mathscr{C}$ contains a nontrivial unbounded operation, this is witnessed by the family $(\langle \mathscr{C}_a \rangle_{loc} : a \in X)$, and if $\mathscr{C}$ contains a bounded operation this is witnessed by the family $(\langle \mathscr{D}_a \rangle_{loc} : a \in X)$.

We leave the easily verifiable details to the reader. □

How complicated is $\mathrm{Cl}_{\mathrm{loc}}(X)$, in particular, which lattices does it contain as sublattices? The latter question has been posed as "Problem V" in the survey paper [GP]. The following is a first easy observation which tells us that there is practically no hope that $\mathrm{Cl}_{\mathrm{loc}}(X)$ can ever be fully described, since it is believed that already the clone lattice over a three-element set is too complex to be fully understood.

Proposition 4. *Let* $\mathrm{Cl}(A)$ *be the lattice of all clones over a finite set* A*. Then* $\mathrm{Cl}(A)$ *is an isomorphic copy of an interval of* $\mathrm{Cl}_{\mathrm{loc}}(X)$*.*

Proof. Assume without loss of generality that $A \subseteq X$. Assign to every operation $f(x_1, \ldots, x_n)$ on A a set of n-ary operations $\mathscr{S}_f \subseteq \mathscr{O}^{(n)}$ on X as follows: An operation $g \in \mathscr{O}^{(n)}$ is an element of $\mathscr{S}_f$ iff g agrees with f on A^n. Let σ map every clone $\mathscr{C}$ on the base set A to the set $\bigcup\{\mathscr{S}_f : f \in \mathscr{C}\}$. Then the following hold:

(1) For every clone $\mathscr{C}$ on A, $\sigma(\mathscr{C})$ is a local clone on X.
(2) σ maps the clone of all operations on A to $\mathrm{Pol}(\{A\})$.

(3) All local clones (in fact: all clones) which contain $\sigma(\{f : f \text{ is a projection on } A\})$ (i.e., which contain the local clone on X which, via σ, corresponds to the clone of projections on A) and which are contained in $\mathrm{Pol}(\{A\})$ are of the form $\sigma(\mathscr{C})$ for some clone $\mathscr{C}$ on A.
(4) σ is one-one and order preserving.

(1) and (2) are easy verifications and left to the reader. To see (3), let $\mathscr{D}$ be any clone in the mentioned interval, and denote by $\mathscr{C}$ the set of all restrictions of operations in $\mathscr{D}$ to appropriate powers of A. Since $\mathscr{D} \subseteq \mathrm{Pol}(\{A\})$, all such restrictions are operations on A, and since $\mathscr{D}$ is closed under composition and contains all projections, so does $\mathscr{C}$. Thus, $\mathscr{C}$ is a clone on A. We claim $\mathscr{D} = \sigma(\mathscr{C})$. By the definitions of $\mathscr{C}$ and σ, we have that $\sigma(\mathscr{C})$ clearly contains $\mathscr{D}$. To see the less obvious inclusion, let $f \in \sigma(\mathscr{C})$ be arbitrary, say of arity m. The restriction of f to A^m is an element of $\mathscr{C}$, hence there exists an m-ary $f' \in \mathscr{D}$ which has the same restriction to A^m as f. Define $s(x_1, \ldots, x_m, y) \in \mathscr{O}^{(m+1)}$ by

$$s(x_1, \ldots, x_m, y) = \begin{cases} y & , \ \mathrm{if}(x_1, \ldots, x_m) \in A^m \\ f(x_1, \ldots, x_m) & , \ \text{otherwise.} \end{cases}$$

Since s behaves on A^{m+1} like the projection onto the last coordinate, and since $\mathscr{D}$ contains $\sigma(\{f : f \text{ is a projection on } A\})$, we infer $s \in \mathscr{D}$. But $f(x_1, \ldots, x_m) = s(x_1, \ldots, x_m, f'(x_1, \ldots, x_m))$, proving $f \in \mathscr{D}$.
(4) is an immediate consequence of (1) and the definitions. $\square$

It is known that all countable products of finite lattices embed into the clone lattice over a four-element set [Bul94], so by the preceding proposition they also embed into $\mathrm{Cl_{loc}}(X)$. However, there are quite simple countable lattices which do not embed into the clone lattice over any finite set: The lattice M_ω consisting of a countably infinite antichain plus a smallest and a greatest element is an example [Bul93]. We shall see now that the class of lattices embeddable into $\mathrm{Cl_{loc}}(X)$ properly contains the class of lattices embeddable into the clone lattice over a finite set. In fact, the structure of $\mathrm{Cl_{loc}}(X)$ is at least as complicated as the structure of any algebraic lattice with $\aleph_0$ compact elements.

Theorem 5. *Every algebraic lattice with a countable number of compact elements is a complete sublattice of* $\mathrm{Cl_{loc}}(X)$.

To prove Theorem 5, we cite the following deep theorem from [Tům89].

Theorem 6. *Every algebraic lattice with a countable number of compact elements is isomorphic to an interval in the subgroup lattice of a countable group.*

Proof of Theorem 5. Let $\mathfrak{L}$ be the algebraic lattice to be embedded into $\mathrm{Cl_{loc}}(X)$. Let $\mathfrak{X} = (X, +, -, 0)$ be the group provided by Theorem 6. For every $a \in X$, define a unary operation $f_a \in \mathscr{O}^{(1)}$ by $f_a(x) = a + x$. Clearly, we have $f_a(f_b(x)) = a+b+x = f_{a+b}(x)$ for all $a, b \in X$. Using this, it is easy to verify that for all $S \subseteq X$, the (not necessarily local) clone $\mathscr{C}_S$ generated by $\mathscr{F}_S := \{f_a : a \in S\}$ essentially (that is, up to fictitious variables and projections) consists of all operations f_a for which a is in the subsemigroup of $(X, +)$ generated by S. Let $[\mathfrak{G}_1, \mathfrak{G}_2]$ be the interval in the subgroup lattice of $\mathfrak{X}$ that $\mathfrak{L}$ is isomorphic to. Define a mapping $\sigma : [\mathfrak{G}_1, \mathfrak{G}_2] \to \mathrm{Cl}_{\mathrm{loc}}(X)$ sending every group $\mathfrak{H} = (H, +, -, 0)$ in the interval to $\mathscr{C}_H$. It follows readily from our observation above that the operations in $\mathscr{C}_H$ are up to fictitious variables the f_a, where $a \in H$, and the projections; in particular, the unary operations in $\mathscr{C}_H$ equal $\mathscr{F}_H$ (plus the identity operation, which is an element of $\mathscr{F}_H$ anyway since it equals f_0). Therefore, σ is injective and order-preserving.

We still have to check that all $\mathscr{C}_H$ are locally closed. To see this, let $f \in \langle \mathscr{C}_H \rangle_{loc}$; then f depends on only one variable, since all operations in $\mathscr{C}_H$ depend on only one variable and dependence on several variables is witnessed on finite sets. Assume therefore without loss of generality $f \in \mathscr{O}^{(1)}$. We claim $f \in \mathscr{F}_H$. To see this, observe that f agrees with some $f_a \in \mathscr{F}_H$ on the finite set $\{0\} \subseteq X$. Suppose that there is $b \in X$ such that $f(b) \neq f_a(b) = a + b$. Then $f \in \langle \mathscr{F}_H \rangle_{loc}$ implies that there exists $f_c \in \mathscr{F}_H$ such that f and f_c agree on $\{0, b\}$. But then $c = f_c(0) = f(0) = f_a(0) = a$, and thus $f(b) = f_c(b) = c + b = a + b = f_a(b) \neq f(b)$, an obvious contradiction. Hence, $f = f_a \in \mathscr{F}_H$ and we are done.

With the explicit description of the $\mathscr{C}_H$ and given that they are indeed local clones, a straightforward check shows that σ preserves arbitrary meets and joins. ☐

Since in particular, $\mathrm{Cl}_{\mathrm{loc}}(X)$ contains M_ω as a sublattice, and since according to [Bul93], M_ω is not a sublattice of the clone lattice over any finite set, we have the following corollary to Theorem 5.

Corollary 7. $\mathrm{Cl}_{\mathrm{loc}}(X)$ *does not embed into the clone lattice over any finite set.*

Observe also that Theorem 5 is a strengthening of Proposition 4 in so far as the clone lattice over a finite set is an example of an algebraic lattice with countably many compact elements. However, in that proposition we obtain an embedding as an interval, not just as a complete sublattice.

What about other lattices, i.e. lattices which are more complicated or larger than algebraic lattices with countably many compact elements? The following proposition puts a restriction on which lattices can be sublattices of $\mathrm{Cl}_{\mathrm{loc}}(X)$.

Proposition 8. $\mathrm{Cl}_{\mathrm{loc}}(X)$ *embeds as a suborder into the power set of ω. In particular, it does not contain any uncountable ascending or descending chains.*

Corollary 9. *The size of* $\mathrm{Cl}_{\mathrm{loc}}(X)$ *is* $2^{\aleph_0}$.

Proof of Corollary 9. The fact that all algebraic lattices with at most $\aleph_0$ compact elements embed into $\mathrm{Cl}_{\mathrm{loc}}(X)$ shows that it must contain at least

$2^{\aleph_0}$ elements (since for example the power set of ω with inclusion is such an algebraic lattice). The upper bound is a consequence of Proposition 8. $\square$

In order to see the truth of Proposition 8, the following definition will be convenient.

A *partial clone of finite operations* on X is a set of partial operations of finite domain on X which contains all restrictions of the projections to finite domains and which is closed under composition. The set of partial clones of finite operations on X forms a complete algebraic lattice, the compact elements of which are precisely the finitely generated partial clones.

Proposition 10. *The mapping σ from $\mathrm{Cl}_{\mathrm{loc}}(X)$ into the lattice of partial clones of finite operations on X which sends every $\mathscr{C} \in \mathrm{Cl}_{\mathrm{loc}}(X)$ to the partial clone of all restrictions of its operations to finite domains is one-to-one and preserves arbitrary joins.*

Proof. It is obvious that $\sigma(\mathscr{C})$ is a partial clone of finite operations, for all local (in fact: also non-local) clones $\mathscr{C}$.
Let $\mathscr{C}, \mathscr{D} \in \mathrm{Cl}_{\mathrm{loc}}(X)$ be distinct. Say without loss of generality that there is an n-ary $f \in \mathscr{C} \setminus \mathscr{D}$; then since $\mathscr{D}$ is locally closed, there exists some finite set $A \subseteq X^n$ such that there is no $g \in \mathscr{D}$ which agrees with f on A. The restriction of f to A then witnesses that $\sigma(\mathscr{C}) \neq \sigma(\mathscr{D})$.
We show that $\sigma(\mathscr{C}) \vee \sigma(\mathscr{D}) = \sigma(\mathscr{C} \vee \mathscr{D})$; the proof for arbitrary joins works the same way. It follows directly from the definition of σ that it is order-preserving. Thus, $\sigma(\mathscr{C} \vee \mathscr{D})$ contains both $\sigma(\mathscr{C})$ and $\sigma(\mathscr{D})$ and hence also their join. Now let $f \in \sigma(\mathscr{C}) \vee \sigma(\mathscr{D})$. This means that it is a composition of partial operations in $\sigma(\mathscr{C}) \cup \sigma(\mathscr{D})$. All partial operations used in this composition have extensions to operations in $\mathscr{C}$ or $\mathscr{D}$, and if we compose these extensions in the same way as the partial operations, we obtain an operation in $\mathscr{C} \vee \mathscr{D}$ which agrees with f on the domain of the latter. Whence, $f \in \sigma(\mathscr{C} \vee \mathscr{D})$. $\square$

Note that the preceding proposition immediately implies Proposition 8: The number of partial operations with finite domain on X is countable, and therefore partial clones of finite operations can be considered as subsets of ω.

Until today, no other restriction to embeddings into $\mathrm{Cl}_{\mathrm{loc}}(X)$ except for Proposition 8 is known, and we ask:

Question 11. *Does every lattice which is order embeddable into the power set of ω have a lattice embedding into $\mathrm{Cl}_{\mathrm{loc}}(X)$?*

However, it seems difficult to embed even the simplest lattices which are not covered by Theorem 5 into $\mathrm{Cl}_{\mathrm{loc}}(X)$. For example, we do not know:

Question 12. *Does the lattice $M_{2^{\aleph_0}}$, which consists of an antichain of length $2^{\aleph_0}$ plus a smallest and a largest element, embed into $\mathrm{Cl}_{\mathrm{loc}}(X)$?*

So far, we only know

Proposition 13. *There exists a join-preserving embedding as well as a meet-preserving embedding of $M_{2^{\aleph_0}}$ into* $\mathrm{Cl}_{\mathrm{loc}}(X)$.

Proof. Denote by 0 and 1 the smallest and the largest element of $M_{2^{\aleph_0}}$, respectively, and enumerate the elements of its antichain by $(a_i)_{i\in 2^{\aleph_0}}$.
We first construct a join-preserving embedding. Enumerate the non-empty proper subsets of X by $(A_i)_{i\in 2^{\aleph_0}}$. Consider the mapping σ which sends 0 to the clone of projections, 1 to $\mathscr{O}$, and every a_i to $\mathrm{Pol}(\{A_i\})$. Now it is well-known (see [RS84]) that for any non-empty proper subset A of X, $\mathrm{Pol}(\{A\})$ is covered by $\mathscr{O}$, i.e. there exist no local (in fact even no global) clones between $\mathrm{Pol}(\{A\})$ and $\mathscr{O}$. Hence, we have that $\sigma(a_i)\vee\sigma(a_j) = \langle \mathrm{Pol}(A_i)\cup \mathrm{Pol}(A_j)\rangle_{loc} = \mathscr{O} = \sigma(1)$ for all $i\neq j$. Since clearly $\sigma(a_i)$ contains $\sigma(0)$ for all $i\in 2^{\aleph_0}$, the mapping σ indeed preserves joins.
To construct a meet embedding, fix any distinct $a,b\in X$ and define for every non-empty subset A of $X\setminus\{a,b\}$ an operation $f_A\in\mathscr{O}^{(1)}$ by

$$f_A(x)=\begin{cases} a, & \text{if } x\in A\\ b, & \text{otherwise.}\end{cases}$$

Enumerate the non-empty subsets of $X\setminus\{a,b\}$ by $(B_i : i\in 2^{\aleph_0})$. Denote the constant unary operation with value b by c_b. Let the embedding σ map 0 to $\langle\{c_b\}\rangle_{loc}$, for all $i\in 2^{\aleph_0}$ map a_i to $\langle\{f_{B_i}\}\rangle_{loc}$, and let it map 1 to $\mathscr{O}$. One readily checks that $\sigma(a_i)=\langle\{f_{B_i}\}\rangle_{loc}$ contains only projections and, up to fictitious variables, the operations f_{B_i} and c_b. Therefore, for $i\neq j$ we have $\sigma(a_i)\wedge\sigma(a_j)=\langle\{c_b\}\rangle_{loc}=\sigma(0)$. Since clearly $\sigma(a_i)\subseteq\sigma(1)=\mathscr{O}$ for all $i\in 2^{\aleph_0}$, we conclude that σ does indeed preserve meets.

□

Simple as the preceding proposition is, it still shows us as a consequence that Theorem 5 is not optimal.

Corollary 14. $\mathrm{Cl}_{\mathrm{loc}}(X)$ *is not embeddable into any algebraic lattice with countably many compact elements.*

Proof. It is well-known and easy to check (confer also [CD73]) that any algebraic lattice $\mathfrak{L}$ with countably many compact elements can be represented as the subalgebra lattice of an algebra over the base set ω. The meet in the subalgebra lattice $\mathfrak{L}$ is just the set-theoretical intersection. Now there is certainly no uncountable family of subsets of ω with the property that any two distinct members of this family have the same intersection D; for the union of such a family would have to be uncountable. Consequently, $\mathfrak{L}$ cannot have $M_{2^{\aleph_0}}$ as a meet-subsemilattice. But $\mathrm{Cl}_{\mathrm{loc}}(X)$ has, hence $\mathfrak{L}$ cannot have $\mathrm{Cl}_{\mathrm{loc}}(X)$ as a sublattice. □

Observe that this corollary is a strengthening of Corollary 7, since the clone lattice over a finite set is an algebraic lattice with countably many compact elements.

We conclude by remarking that the lattice $\mathrm{Cl}(X)$ of all (not necessarily local) clones on X is infinitely more complicated than $\mathrm{Cl}_{\mathrm{loc}}(X)$: It contains all algebraic lattices with at most $2^{\aleph_0}$ compact elements, and in particular all lattices of size continuum, as complete sublattices [Pin07].

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