Monoids above the permutations

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Support through JSPS

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Fact. X infinite $\to |Cl(X)| = 2^{2^\kappa}$.

Clones above $\mathscr{O}^{(1)}$

Thm. (Gavrilov) $\kappa = \aleph_0$. There exist exactly 2 precomplete clones $\mathrm{Pol}(T_1), \mathrm{Pol}(T_2)$ above $\mathscr{O}^{(1)}$.

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Thm. κ regular. The precomplete clones containing $\mathscr{S}$ but not $\mathscr{O}^{(1)}$ are the polymorphism clones of the following monoids:

1. $\mathscr{A} = \{f \in \mathscr{O}^{(1)} :$ $f^{-1}[\{y\}]$ is small for almost all $y \in X\}$
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4. $\mathscr{F} = \{f \in \mathscr{O}^{(1)} :$ f is almost surjective or constant$\}$
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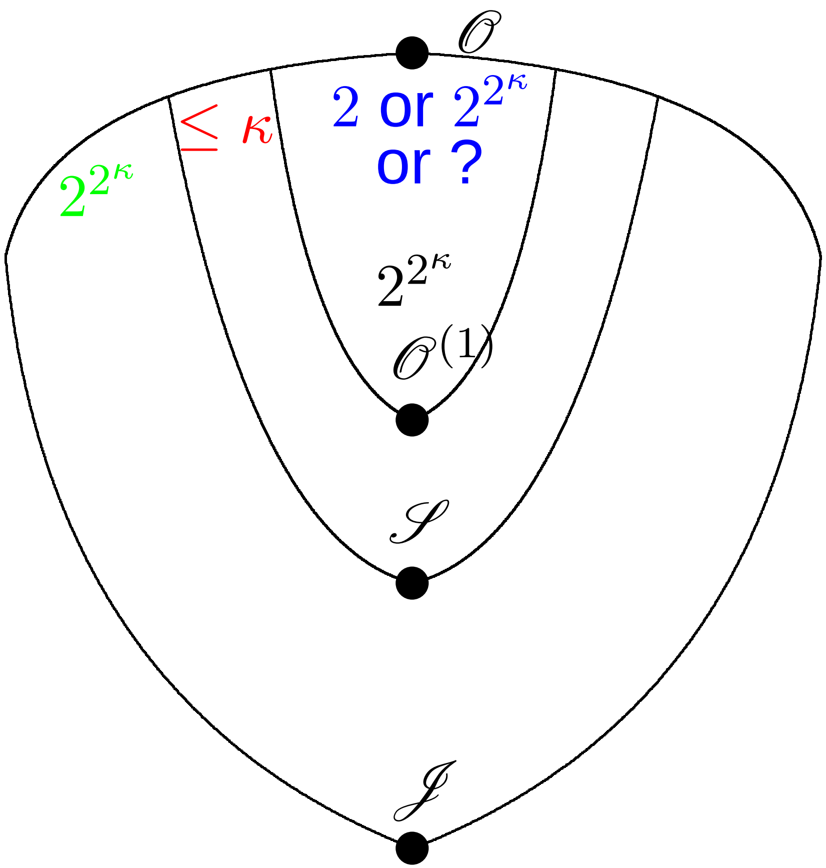
Cor. If $\kappa = \aleph_\alpha$, the number of such clones is $\max(\aleph_0, |\alpha|)$.

Maximal clones

Question. Can we determine $[\mathscr{S}, \mathscr{O}] \setminus [\mathscr{O}^{(1)}, \mathscr{O}]$?



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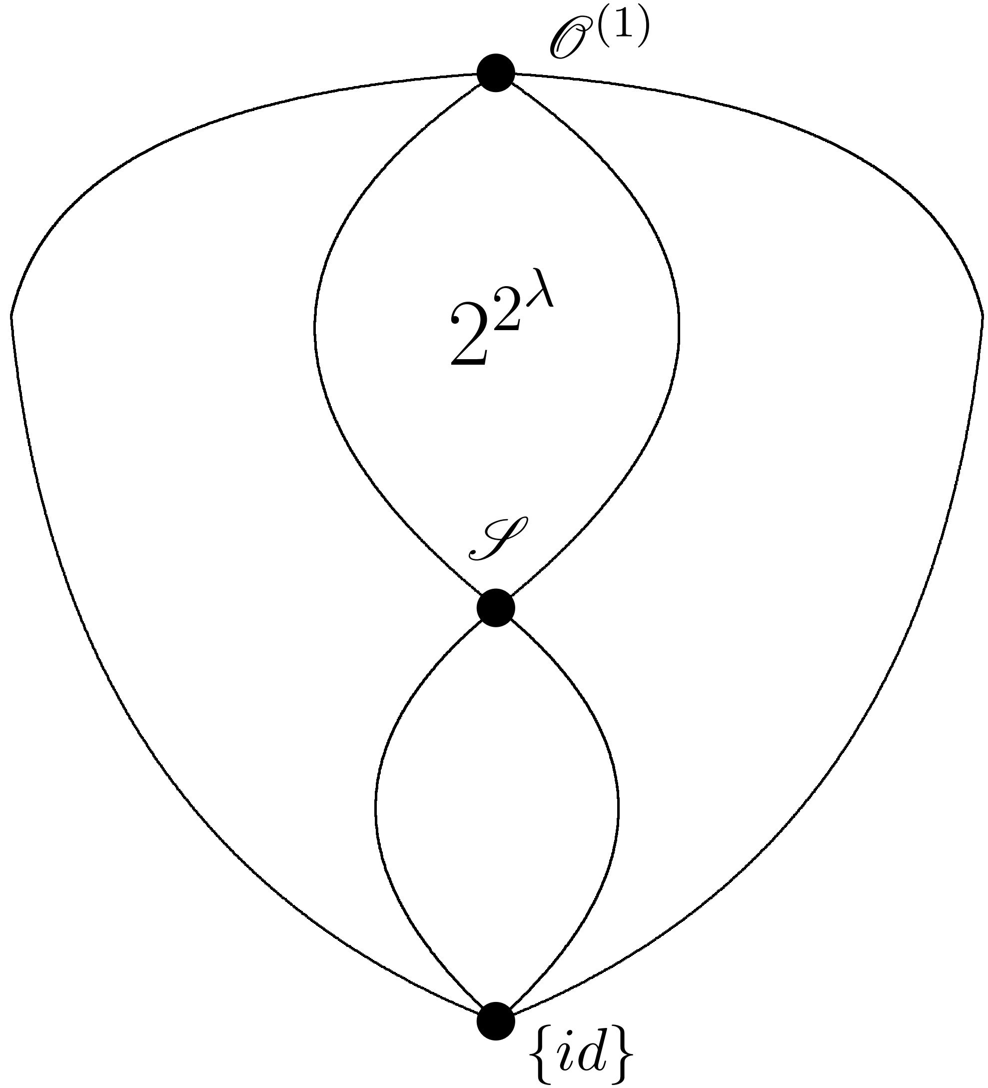
Cor. $|[\mathscr{S}, \mathscr{O}^{(1)}]| = 2^{2^\lambda}$.

Many monoids containing $\mathscr{S}$



Precomplete monoids

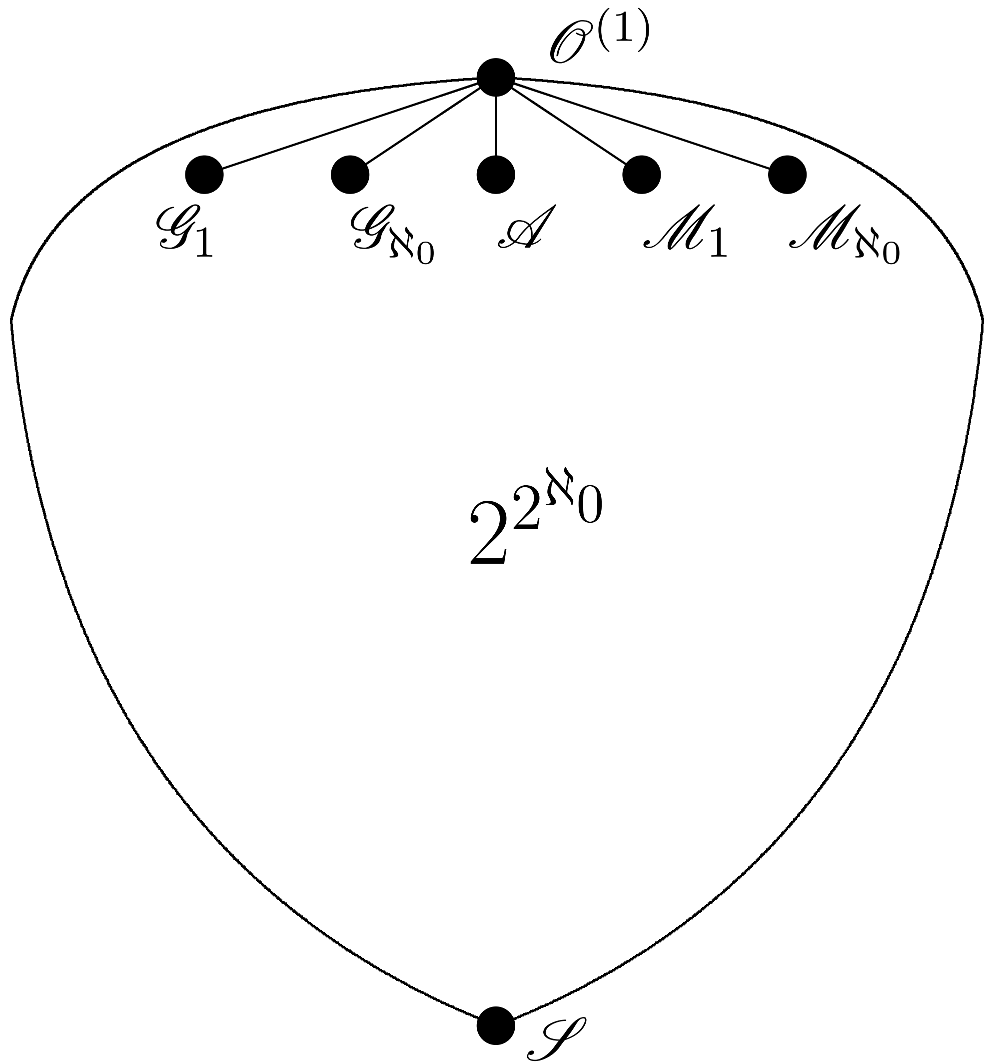
Thm. (Gavrilov) $|X| = \aleph_0$. The precomplete submonoids of $\mathscr{O}^{(1)}$ containing $\mathscr{S}$ are $\mathscr{A}$, $\mathscr{G}_{\aleph_0}$, $\mathscr{G}_1$, $\mathscr{M}$ and $\mathscr{N}$, where

$$\mathscr{M} = \{f \in \mathscr{O}^{(1)} : f \text{ surjective or not injective}\}$$

and

$$\mathscr{N} = \{f \in \mathscr{O}^{(1)} : f \text{ almost surjective or not almost injective}\}.$$

Maximal monoids above $\mathscr{S}$ ($\kappa = \aleph_0$)



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Thm. κ regular: The precomplete submonoids of $\mathscr{O}^{(1)}$ which contain $\mathscr{S}$ are exactly $\mathscr{A}$ and $\mathscr{G}_\lambda$ and $\mathscr{M}_\lambda$ for $\lambda = 1$ and $\aleph_0 \leq \lambda \leq \kappa$, λ a cardinal, where

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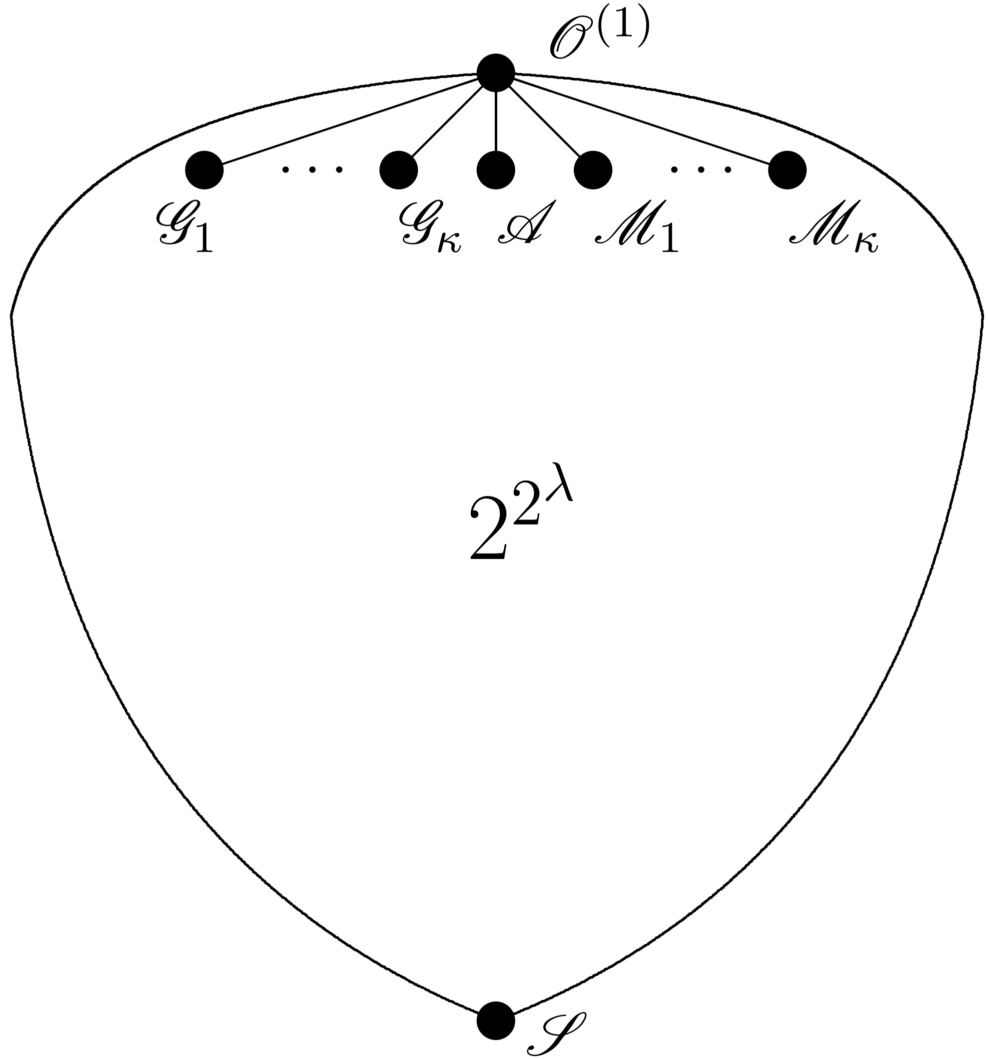
κ singular: Replace $\mathscr{A}$ by $\mathscr{A}'$, where

$$\mathscr{A}' = \{f \in \mathscr{O}^{(1)} : \exists \lambda < \kappa \ (\{x \in X : |f^{-1}[x]| > \lambda\} \text{ small })\}.$$

Maximal monoids above $\mathscr{S}$



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κ singular and $\lambda < \kappa$: Same.

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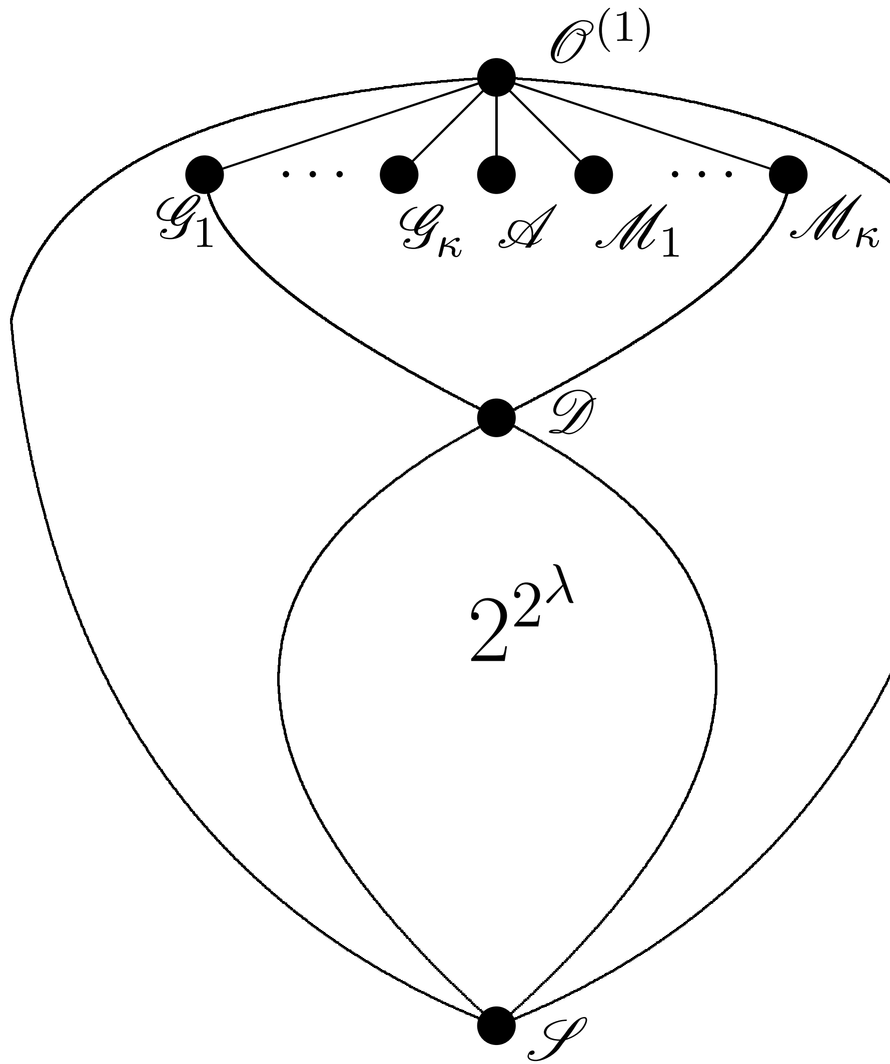

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Cor. Whether or not $|[\mathscr{S}, \mathscr{A}']| = |[\mathscr{S}, \mathscr{M}_1]|$ is independent from ZFC.

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