Clones on infinite sets

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Outline

1. The Clone Lattice: Definition and Basic Properties
2. Set Theory $\rightarrow$ Clones
3. Clones $\rightarrow$ Model Theory & Complexity Theory

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Consider an algebra $\mathfrak{A} = (X, \mathcal{F}_{\mathfrak{A}})$ with domain X.

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For algebras $\mathfrak{A}, \mathfrak{B}$ on X, we write $\mathfrak{A} \preceq \mathfrak{B}$ iff $\langle \mathcal{F}_{\mathfrak{A}} \rangle \subseteq \langle \mathcal{F}_{\mathfrak{B}} \rangle$.

We write $\mathfrak{A} \sim \mathfrak{B}$ iff $\langle \mathcal{F}_{\mathfrak{A}} \rangle = \langle \mathcal{F}_{\mathfrak{B}} \rangle$.

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Equivalent definition

For all $n \geq 1$, set $\mathcal{O}^{(n)} := X^{X^n} = \{f : X^n \to X\}$.

Write $\mathcal{O} = \bigcup_n \mathcal{O}^{(n)}$.

A *clone* is a subset $\mathcal{C}$ of $\mathcal{O}$ which

- Contains all projections and
- Is closed under composition.

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- The set of *idempotent* operations (all f that satisfy $f(x, \ldots, x) = x$).
- For a topological space $\mathcal{X} = (X, \mathcal{T})$, the set of all continuous operations which map some product X^n into X.

The lattice operations

Fact

If $\mathcal{C}_i,\ i \in I$ are clones, then so is their intersection.

Thus $\bigwedge_{i\in I} \mathcal{C}_i = \bigcap_{i\in I} \mathcal{C}_i$.

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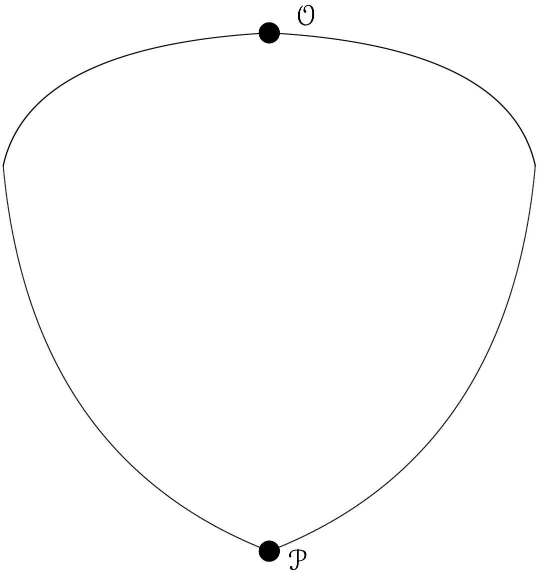
Problem

What does $\mathrm{Cl}(X)$ look like?

Picture of the clone lattice



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Every set of such functions generates a different clone.

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Theorem (2006)

$\mathrm{Cl}(X)$ contains all algebraic lattices with $2^{|X|}$ compact elements as complete sublattices.

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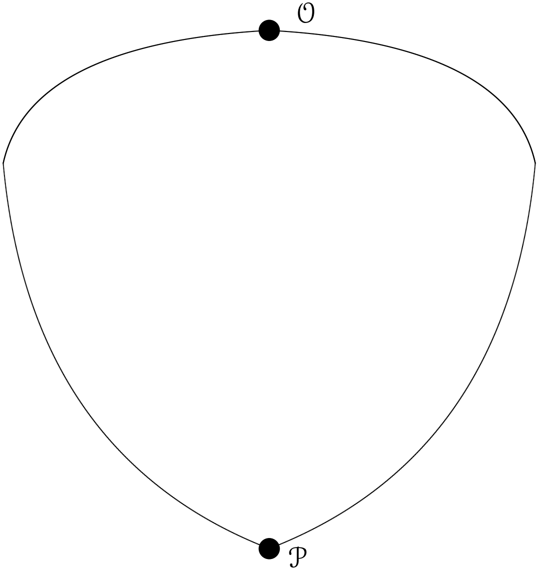

Set Theory $\rightarrow$ Clones

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$\mathcal{C}_U := \{f \in \mathcal{O} : \forall A \notin U \, (f[A^n] \notin U)\}$ is a clone.

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Proof 3

For $f \in \mathcal{O}$, set $Fix(f) := \{x \in X : f(x, \ldots, x) = x\}$.

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Theorem (Rosenberg 76; Marchenkov 81; Goldstern & Shelah 02)

The clone lattice has $2^{2^{|X|}}$ dual atoms.

Dual atomicity of the clone lattice

Problem (Gavrilov 59)

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Method

So-called *creatures* which measure the growth of functions.

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Intuition

Unary operations: Pigeonhole principle.
Higher arity operations: “Real” partition properties.

More on T_1, T_2

Theorem (2004)

T_1 has a nice (finite) generating system and is well-understood.

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Theorem (Goldstern 04)

T_2 is a complete co-analytic set.
In particular, it is not countably generated over $\mathcal{O}^{(1)}$.

Monoidal intervals

Fix a transformation monoid $\mathcal{M} \subseteq \mathcal{O}^{(1)}$.

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This set is in interval of the clone lattice.

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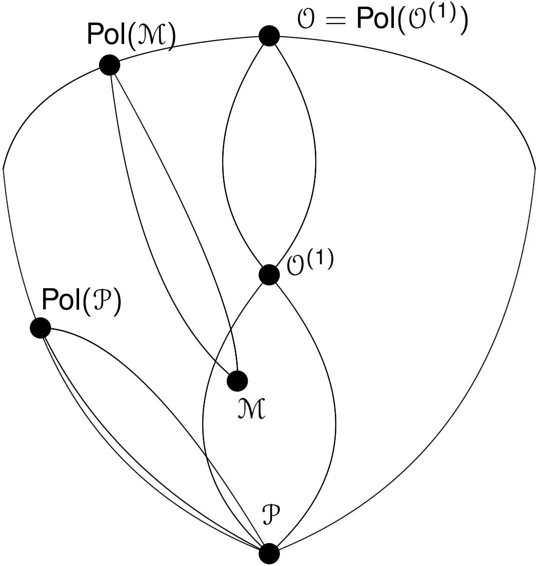
Problem

What do they look like?
What cardinalities can they have?

Monoidal intervals



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Theorem (Abraham, Goldstern, P. 07)

It is consistent with ZFC that there exists no algebraic lattice of cardinality λ (with $\leq 2^{|X|}$ compact elements).
It is also consistent that there exists a monoidal interval of size λ.

Clones $\to$ Model theory

Reducts of relational structures

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- Thomas '91: 5 reducts of the random graph.

First order definability and permutations

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Let $\mathcal{G}$ be a set of permutations.
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Corollary

The reducts of Γ correspond to the closed groups containing $\text{Aut}(\Gamma)$.

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Given a structure Γ, determine its reducts *up to primitive positive interdefinability*.

Formulas of the form $\exists x_1, \ldots, x_n \; \phi_1 \wedge \ldots \wedge \phi_m$,
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Theorem (Bodirsky & Nešetřil '06)

Let Γ be ω-categorical. Then $\mathrm{Inv}\,\mathrm{Pol}(\Gamma) = pp(\Gamma)$.

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Theorem (Bodirsky & Nešetřil '06)

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Corollary

The reducts of Γ, up to pp-interdefinability,
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The reducts of equality

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Reducts of the simplest structure, $\Gamma := (X, =)$.

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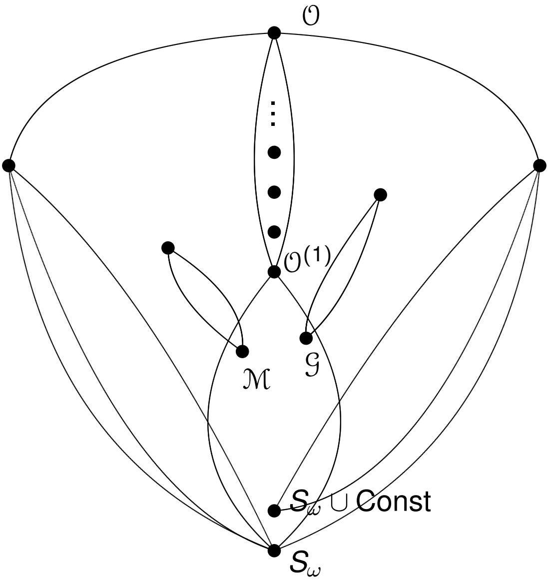
Theorem (Bodirsky, Chen, P. 2008)

There are uncountably many reducts of $(X, =)$ up to pp-interdefinability.

The reducts of equality



Future projects

Problem

Determine the reducts of other nice ω-categorical structures, such as

- The random graph
- The unbounded dense linear order.

Connection to CSP

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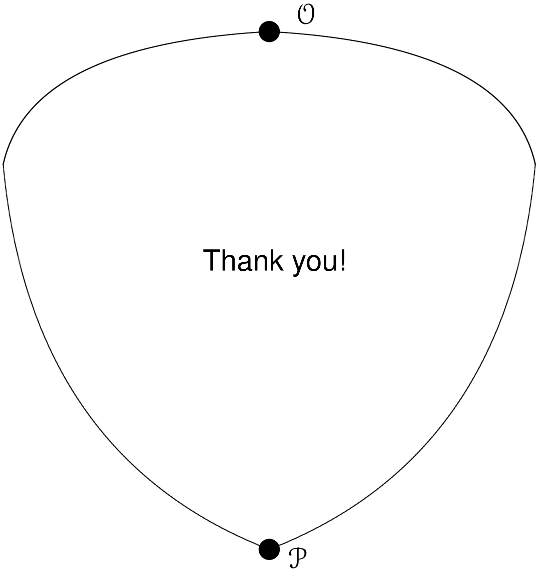
Consequence

For ω-categorical Γ, the Galois connection Inv-Pol can be used.
This is called the "algebraic approach" to CSP.



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