Clones on Ramsey structures and Schaefer's theorem for graphs

An opera in a prologue, three acts, and an epilogue

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Synopsis

- **Prologue:** The graph satisfiability problem
  (Bodirsky, Pinsker)
- **Act I:** Reducts of homogeneous structures
  (Cameron, Thomas)
- **Act II:** The Ramsey property
  (Nešetřil, Rödl)
- **Act III:** Topological dynamics
  (Kechris, Pestov, Todorcevic, Tsankov)
- **Epilogue:** Schaefer's theorem for graphs

Prologue

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Theorem (Schaefer '78)

Boolean-SAT(Ψ) is either in P or NP-complete, for all Ψ.

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Γ_Ψ is a *reduct of* the random graph, i.e.,
a structure with a first-order definition in G.

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Let's study the reducts of the random graph!

Act I

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Problem

Classify the reducts of Γ.

Possible classifications

Consider two reducts Δ, Δ' of Γ *equivalent* iff Δ is also a reduct of Δ' and vice-versa.

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Theorem (Junker, Ziegler '08)

$(\mathbb{Q}; <, 0)$ has 116 reducts, up to f.o.-interdefinability.

Thomas' conjecture

Conjecture (Thomas '91)

Let Γ be homogeneous in a finite language.

Then Γ has finitely many reducts up to f.o.-interdefinability.

Act II

The Ramsey property

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Groups, Monoids, Clones

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Clone. . . set of finitary operations which contains all projections and which is closed under composition

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How to find all reducts up to . . .-interdefinability?

Climb up the lattice!

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$f : V \to V$ is *canonical on* $F \subseteq V$ iff its restriction to F is canonical.

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Conclusion: Every finite graph has a copy in G on which f is canonical.

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The structure $(V; E, c, d)$ has similar Ramsey properties as $(V; E)$.

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Theorem (Thomas '96)

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Theorem (Bodirsky, P. '09)

Let $f : V^n \to V$, $f \notin \mathrm{Aut}(G)$.
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A class $\mathcal{C}$ of structures of the same signature is called a *Ramsey class* iff
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We would like to fix $c_1, \ldots, c_n$ witnessing $f \notin \mathrm{Aut}(\Gamma)$,
and have canonical behavior on $(\Gamma, c_1, \ldots, c_n)$.

Act III

Topological dynamics

Adding constants to Ramsey classes

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An ordered homogeneous structure Δ is Ramsey iff
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Corollary

If Γ is ordered Ramsey, then so is $(\Gamma, c_1, \ldots, c_n)$.

Minimal monoids above Ramsey structures

Thus:

If Γ is ordered Ramsey, $f : \Gamma \to \Gamma$, and $c_1, \ldots, c_n \in \Gamma$, then f generates a function canonical for $(\Gamma, c_1, \ldots, c_n)$ which behaves like f on $\{c_1, \ldots, c_n\}$.

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- Every closed clone above $\mathrm{Aut}(\Gamma)$ contains a minimal one.

Epilogue

Schaefer’s theorem for graphs

The Graph Satisfiability Problem

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Let Ψ be a finite set of graph formulas.

Computational problem: Graph-SAT(Ψ)

INPUT:

- A set W of variables (vertices), and
- statements $\phi_1, \ldots, \phi_n$ about the elements of W, where each ϕ_i is taken from Ψ.

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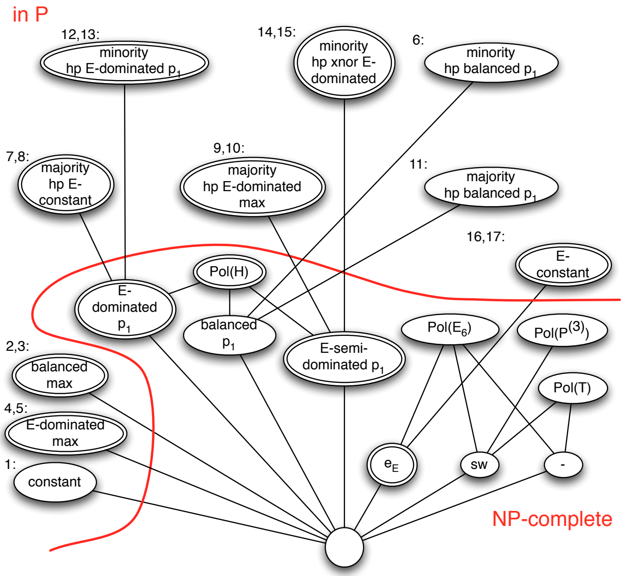
Theorem (Bodirsky, P. '10)

Graph-SAT(Ψ) is either in P or NP-complete, for all Ψ.

Classification



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Problem (universal algebra)

Can a clone containing the automorphism group of an ordered Ramsey structure Γ have infinitely many minimal superclones?

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Prove: If Γ is ordered Ramsey, then so is $(\Gamma, c_1, \ldots, c_n)$.

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Application

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Open problems II

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If Γ is not ω-categorical, does it always have infinitely many reducts?

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Application

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Determine the reducts of the countable atomless Boolean algebra.

Application

Determine the reducts of the random partial order.