Reducts of homogeneous structures with the Ramsey property

Michael Pinsker

ÉLM Université Denis-Diderot Paris 7

Algebra TU Wien

Einstein Institute of Mathematics Jerusalem

March 8, 2010

Outline

Reducts of homogeneous structures

Let Γ be a countable relational structure in a finite language

Reducts of homogeneous structures

Let Γ be a countable relational structure in a finite language which is *homogeneous*, i.e.,

For all $A, B \subseteq \Gamma$ finite, for all isomorphisms $i : A \to B$ there exists $\alpha \in \mathrm{Aut}(\Gamma)$ extending i.

Reducts of homogeneous structures

Let Γ be a countable relational structure in a finite language which is *homogeneous*, i.e.,

For all $A, B \subseteq \Gamma$ finite, for all isomorphisms $i : A \to B$ there exists $\alpha \in \mathrm{Aut}(\Gamma)$ extending i.

Γ is the Fraïssé limit of its *age*, i.e., its class of finite induced substructures.

Reducts of homogeneous structures

Let Γ be a countable relational structure in a finite language which is *homogeneous*, i.e.,

For all $A, B \subseteq \Gamma$ finite, for all isomorphisms $i : A \to B$ there exists $\alpha \in \mathrm{Aut}(\Gamma)$ extending i.

Γ is the Fraïssé limit of its *age*, i.e., its class of finite induced substructures.

Definition

A *reduct* of Γ is a structure with a first-order (f.o.) definition in Γ.

Reducts of homogeneous structures

Let Γ be a countable relational structure in a finite language which is *homogeneous*, i.e.,

For all $A, B \subseteq \Gamma$ finite, for all isomorphisms $i : A \to B$ there exists $\alpha \in \mathrm{Aut}(\Gamma)$ extending i.

Γ is the Fraïssé limit of its *age*, i.e., its class of finite induced substructures.

Definition

A *reduct* of Γ is a structure with a first-order (f.o.) definition in Γ.

Problem

Classify the reducts of Γ.

Possible classifications

Consider two reducts Δ, Δ' of Γ *equivalent* iff Δ is also a reduct of Δ' and vice-versa.

Possible classifications

Consider two reducts Δ, Δ' of Γ *equivalent* iff Δ is also a reduct of Δ' and vice-versa.

We say that Δ and Δ' are *first-order interdefinable*.

Possible classifications

Consider two reducts Δ, Δ' of Γ *equivalent* iff Δ is also a reduct of Δ' and vice-versa.

We say that Δ and Δ' are *first-order interdefinable*.

"Δ is a reduct of Δ'" is a *quasiorder* on relational structures over the same domain.

Possible classifications

Consider two reducts Δ, Δ' of Γ *equivalent* iff Δ is also a reduct of Δ' and vice-versa.

We say that Δ and Δ' are *first-order interdefinable*.

"Δ is a reduct of Δ'" is a *quasiorder* on relational structures over the same domain.

This quasiorder, factored by f.o.-interdefinability, becomes a *complete lattice*.

Possible classifications

Consider two reducts Δ, Δ' of Γ *equivalent* iff Δ is also a reduct of Δ' and vice-versa.

We say that Δ and Δ' are *first-order interdefinable*.

"Δ is a reduct of Δ'" is a *quasiorder* on relational structures over the same domain.

This quasiorder, factored by f.o.-interdefinability, becomes a *complete lattice*.

Finer classifications of reducts, e.g. up to

Possible classifications

Consider two reducts Δ, Δ' of Γ *equivalent* iff Δ is also a reduct of Δ' and vice-versa.

We say that Δ and Δ' are *first-order interdefinable*.

"Δ is a reduct of Δ'" is a *quasiorder* on relational structures over the same domain.

This quasiorder, factored by f.o.-interdefinability, becomes a *complete lattice*.

Finer classifications of reducts, e.g. up to

- Existential interdefinability

Possible classifications

Consider two reducts Δ, Δ' of Γ *equivalent* iff Δ is also a reduct of Δ' and vice-versa.

We say that Δ and Δ' are *first-order interdefinable*.

"Δ is a reduct of Δ'" is a *quasiorder* on relational structures over the same domain.

This quasiorder, factored by f.o.-interdefinability, becomes a *complete lattice*.

Finer classifications of reducts, e.g. up to

- Existential interdefinability
- Existential positive interdefinability

Possible classifications

Consider two reducts Δ, Δ' of Γ *equivalent* iff Δ is also a reduct of Δ' and vice-versa.

We say that Δ and Δ' are *first-order interdefinable*.

"Δ is a reduct of Δ'" is a *quasiorder* on relational structures over the same domain.

This quasiorder, factored by f.o.-interdefinability, becomes a *complete lattice*.

Finer classifications of reducts, e.g. up to

- Existential interdefinability
- Existential positive interdefinability
- Primitive positive interdefinability

Example: The Random graph

Denote by $G = (V; E)$ the *random graph*, i.e.,
the unique countably infinite graph which is

Example: The Random graph

Denote by $G = (V; E)$ the *random graph*, i.e.,
the unique countably infinite graph which is

- homogeneous

Example: The Random graph

Denote by $G = (V; E)$ the *random graph*, i.e.,
the unique countably infinite graph which is

- homogeneous
- $\aleph_0$-universal.

Example: The Random graph

Denote by $G = (V; E)$ the *random graph*, i.e.,
the unique countably infinite graph which is

- homogeneous
- $\aleph_0$-universal.

Set $R^{(k)} := \{(x_1, \ldots, x_k) \subseteq V^k : x_i$ distinct, number of edges odd$\}$.

Example: The Random graph

Denote by $G = (V; E)$ the *random graph*, i.e.,
the unique countably infinite graph which is

- homogeneous
- $\aleph_0$-universal.

Set $R^{(k)} := \{(x_1, \ldots, x_k) \subseteq V^k : x_i \text{ distinct, number of edges odd}\}$.

Theorem (S. Thomas 1991)

Let Γ be a reduct of G. Then:

Example: The Random graph

Denote by $G = (V; E)$ the *random graph*, i.e.,
the unique countably infinite graph which is

- homogeneous
- $\aleph_0$-universal.

Set $R^{(k)} := \{(x_1, \ldots, x_k) \subseteq V^k : x_i \text{ distinct, number of edges odd}\}$.

Theorem (S. Thomas 1991)

Let Γ be a reduct of G. Then:

1. Γ is first-order interdefinable with $(V; E)$, or

Example: The Random graph

Denote by $G = (V; E)$ the *random graph*, i.e.,
the unique countably infinite graph which is

- homogeneous
- $\aleph_0$-universal.

Set $R^{(k)} := \{(x_1, \ldots, x_k) \subseteq V^k : x_i \text{ distinct, number of edges odd}\}$.

Theorem (S. Thomas 1991)

Let Γ be a reduct of G. Then:

1. Γ is first-order interdefinable with $(V; E)$, or
2. Γ is first-order interdefinable with $(V; R^{(3)})$, or

Example: The Random graph

Denote by $G = (V; E)$ the *random graph*, i.e., the unique countably infinite graph which is

- homogeneous
- $\aleph_0$-universal.

Set $R^{(k)} := \{(x_1, \ldots, x_k) \subseteq V^k : x_i$ distinct, number of edges odd$\}$.

Theorem (S. Thomas 1991)

Let Γ be a reduct of G. Then:

1. Γ is first-order interdefinable with $(V; E)$, or
2. Γ is first-order interdefinable with $(V; R^{(3)})$, or
3. Γ is first-order interdefinable with $(V; R^{(4)})$, or

Example: The Random graph

Denote by $G = (V; E)$ the *random graph*, i.e.,
the unique countably infinite graph which is

- homogeneous
- $\aleph_0$-universal.

Set $R^{(k)} := \{(x_1, \ldots, x_k) \subseteq V^k : x_i \text{ distinct, number of edges odd}\}$.

Theorem (S. Thomas 1991)

Let Γ be a reduct of G. Then:

1. Γ is first-order interdefinable with $(V; E)$, or
2. Γ is first-order interdefinable with $(V; R^{(3)})$, or
3. Γ is first-order interdefinable with $(V; R^{(4)})$, or
4. Γ is first-order interdefinable with $(V; R^{(5)})$, or

Example: The Random graph

Denote by $G = (V; E)$ the *random graph*, i.e.,
the unique countably infinite graph which is

- homogeneous
- $\aleph_0$-universal.

Set $R^{(k)} := \{(x_1, \ldots, x_k) \subseteq V^k : x_i$ distinct, number of edges odd$\}$.

Theorem (S. Thomas 1991)

Let Γ be a reduct of G. Then:

1. Γ is first-order interdefinable with $(V; E)$, or
2. Γ is first-order interdefinable with $(V; R^{(3)})$, or
3. Γ is first-order interdefinable with $(V; R^{(4)})$, or
4. Γ is first-order interdefinable with $(V; R^{(5)})$, or
5. Γ is first-order interdefinable with $(V; =)$.

Further examples

Further examples

Example (Cameron 1976)

$(\mathbb{Q}; <)$ has 5 reducts, up to f.o.-interdefinability.

Further examples

Example (Cameron 1976)

$(\mathbb{Q}; <)$ has 5 reducts, up to f.o.-interdefinability.

Example (Junker, Ziegler 2008)

$(\mathbb{Q}; <, 0)$ has 116 reducts, up to f.o.-interdefinability.

Further examples

Example (Cameron 1976)

$(\mathbb{Q}; <)$ has 5 reducts, up to f.o.-interdefinability.

Example (Junker, Ziegler 2008)

$(\mathbb{Q}; <, 0)$ has 116 reducts, up to f.o.-interdefinability.

Example (Thomas 1991)

The homogeneous K_n-free graph has 2 reducts, up to f.o.-interdefinability.

Further examples

Example (Cameron 1976)

$(\mathbb{Q}; <)$ has 5 reducts, up to f.o.-interdefinability.

Example (Junker, Ziegler 2008)

$(\mathbb{Q}; <, 0)$ has 116 reducts, up to f.o.-interdefinability.

Example (Thomas 1991)

The homogeneous K_n-free graph has 2 reducts, up to f.o.-interdefinability.

Example (Thomas 1996)

The homogeneous k-graph has $2^k + 1$ reducts, up to f.o.-interdefinability.

Thomas' conjecture

Conjecture (Thomas 1991)

Γ has always finitely many reducts up to f.o. interdefinability.

Finer classifications

Finer classifications

A formula is *existential* iff
it is of the form $\exists x_1, \ldots, x_n.\psi$, where ψ is quantifier-free.

Finer classifications

A formula is *existential* iff
it is of the form $\exists x_1, \ldots, x_n.\psi$, where ψ is quantifier-free.

A formula is *existential positive* iff
it is existential and does not contain negations.

Finer classifications

A formula is *existential* iff
it is of the form $\exists x_1, \ldots, x_n.\psi$, where ψ is quantifier-free.

A formula is *existential positive* iff
it is existential and does not contain negations.

A formula is *primitive positive* iff
it is of the form $\exists x_1, \ldots, x_n.\psi_1 \wedge \ldots \wedge \psi_m$, where the ψ_i are atomic.

Finer classifications

A formula is *existential* iff
it is of the form $\exists x_1, \ldots, x_n.\psi$, where ψ is quantifier-free.

A formula is *existential positive* iff
it is existential and does not contain negations.

A formula is *primitive positive* iff
it is of the form $\exists x_1, \ldots, x_n.\psi_1 \wedge \ldots \wedge \psi_m$, where the ψ_i are atomic.

Theorem (Bodirsky, Chen, P. 2008)

For the structure $\Gamma := (X; =)$, there exist:

Finer classifications

A formula is *existential* iff
it is of the form $\exists x_1, \ldots, x_n.\psi$, where ψ is quantifier-free.

A formula is *existential positive* iff
it is existential and does not contain negations.

A formula is *primitive positive* iff
it is of the form $\exists x_1, \ldots, x_n.\psi_1 \wedge \ldots \wedge \psi_m$, where the ψ_i are atomic.

Theorem (Bodirsky, Chen, P. 2008)

For the structure $\Gamma := (X; =)$, there exist:

- 1 reduct up to first order / existential interdefinability

Finer classifications

A formula is *existential* iff
it is of the form $\exists x_1, \ldots, x_n . \psi$, where ψ is quantifier-free.

A formula is *existential positive* iff
it is existential and does not contain negations.

A formula is *primitive positive* iff
it is of the form $\exists x_1, \ldots, x_n . \psi_1 \wedge \ldots \wedge \psi_m$, where the ψ_i are atomic.

Theorem (Bodirsky, Chen, P. 2008)

For the structure $\Gamma := (X; =)$, there exist:

- 1 reduct up to first order / existential interdefinability
- $\aleph_0$ reducts up to existential positive interdefinability

Finer classifications

A formula is *existential* iff
it is of the form $\exists x_1, \ldots, x_n.\psi$, where ψ is quantifier-free.

A formula is *existential positive* iff
it is existential and does not contain negations.

A formula is *primitive positive* iff
it is of the form $\exists x_1, \ldots, x_n.\psi_1 \wedge \ldots \wedge \psi_m$, where the ψ_i are atomic.

Theorem (Bodirsky, Chen, P. 2008)

For the structure $\Gamma := (X; =)$, there exist:

- 1 reduct up to first order / existential interdefinability
- $\aleph_0$ reducts up to existential positive interdefinability
- $2^{\aleph_0}$ reducts up to primitive positive interdefinability

Groups, Monoids, Clones

Theorem

Groups, Monoids, Clones

Theorem

- The mapping $\Delta \mapsto \mathrm{Aut}(\Delta)$ is a one-to-one correspondence between the first-order closed reducts of Γ and the closed supergroups of $\mathrm{Aut}(\Gamma)$.

Groups, Monoids, Clones

Theorem

- The mapping $\Delta \mapsto \mathrm{Aut}(\Delta)$ is a one-to-one correspondence between the first-order closed reducts of Γ and the closed supergroups of $\mathrm{Aut}(\Gamma)$.
- The mapping $\Delta \mapsto \mathrm{End}(\Delta)$ is a one-to-one correspondence between the existential positive closed reducts of Γ and the closed supermonoids of $\mathrm{Aut}(\Gamma)$.

Groups, Monoids, Clones

Theorem

- The mapping $\Delta \mapsto \mathrm{Aut}(\Delta)$ is a one-to-one correspondence between the first-order closed reducts of Γ and the closed supergroups of $\mathrm{Aut}(\Gamma)$.
- The mapping $\Delta \mapsto \mathrm{End}(\Delta)$ is a one-to-one correspondence between the existential positive closed reducts of Γ and the closed supermonoids of $\mathrm{Aut}(\Gamma)$.
- The mapping $\Delta \mapsto \mathrm{Pol}(\Delta)$ is a one-to-one correspondence between the primitive positive closed reducts of Γ and the closed superclones of $\mathrm{Aut}(\Gamma)$.

Groups, Monoids, Clones

Theorem

- The mapping $\Delta \mapsto \mathrm{Aut}(\Delta)$ is a one-to-one correspondence between the first-order closed reducts of Γ and the closed supergroups of $\mathrm{Aut}(\Gamma)$.
- The mapping $\Delta \mapsto \mathrm{End}(\Delta)$ is a one-to-one correspondence between the existential positive closed reducts of Γ and the closed supermonoids of $\mathrm{Aut}(\Gamma)$.
- The mapping $\Delta \mapsto \mathrm{Pol}(\Delta)$ is a one-to-one correspondence between the primitive positive closed reducts of Γ and the closed superclones of $\mathrm{Aut}(\Gamma)$.

$\mathrm{Pol}(\Delta)\ldots$ Polymorphisms of Δ, i.e.,
all homomorphisms from finite powers of Δ to Δ

Groups, Monoids, Clones

Theorem

- The mapping $\Delta \mapsto \mathrm{Aut}(\Delta)$ is a one-to-one correspondence between the first-order closed reducts of Γ and the closed supergroups of $\mathrm{Aut}(\Gamma)$.
- The mapping $\Delta \mapsto \mathrm{End}(\Delta)$ is a one-to-one correspondence between the existential positive closed reducts of Γ and the closed supermonoids of $\mathrm{Aut}(\Gamma)$.
- The mapping $\Delta \mapsto \mathrm{Pol}(\Delta)$ is a one-to-one correspondence between the primitive positive closed reducts of Γ and the closed superclones of $\mathrm{Aut}(\Gamma)$.

$\mathrm{Pol}(\Delta)\ldots$ Polymorphisms of Δ, i.e.,
all homomorphisms from finite powers of Δ to Δ

Clone... set of finitary operations which contains all projections and which is closed under composition

The reducts of the random graph, revisited

Let $G := (V; E)$ be the random graph.

The reducts of the random graph, revisited

Let $G := (V; E)$ be the random graph.

Let $\bar{G}$ be the graph that arises by switching edges and non-edges.

The reducts of the random graph, revisited

Let $G := (V; E)$ be the random graph.

Let $\bar{G}$ be the graph that arises by switching edges and non-edges.

Let $- : V \to V$ be an isomorphism between G and $\bar{G}$.

The reducts of the random graph, revisited

Let $G := (V; E)$ be the random graph.

Let $\bar{G}$ be the graph that arises by switching edges and non-edges.

Let $- : V \to V$ be an isomorphism between G and $\bar{G}$.

For $c \in V$, let G_c be the graph that arises by switching all edges and non-edges from c.

The reducts of the random graph, revisited

Let $G := (V; E)$ be the random graph.

Let $\bar{G}$ be the graph that arises by switching edges and non-edges.

Let $- : V \to V$ be an isomorphism between G and $\bar{G}$.

For $c \in V$, let G_c be the graph that arises by switching all edges and non-edges from c.

Let $\mathrm{sw}_c : V \to V$ be an isomorphism between G and G_c.

The reducts of the random graph, revisited

Let $G := (V; E)$ be the random graph.

Let $\bar{G}$ be the graph that arises by switching edges and non-edges.

Let $- : V \to V$ be an isomorphism between G and $\bar{G}$.

For $c \in V$, let G_c be the graph that arises by switching all edges and non-edges from c.

Let $\mathrm{sw}_c : V \to V$ be an isomorphism between G and G_c.

Theorem (Thomas 1991)

The closed groups containing $\mathrm{Aut}(G)$ are the following:

The reducts of the random graph, revisited

Let $G := (V; E)$ be the random graph.

Let $\bar{G}$ be the graph that arises by switching edges and non-edges.

Let $- : V \to V$ be an isomorphism between G and $\bar{G}$.

For $c \in V$, let G_c be the graph that arises by switching all edges and non-edges from c.

Let $\mathrm{sw}_c : V \to V$ be an isomorphism between G and G_c.

Theorem (Thomas 1991)

The closed groups containing $\mathrm{Aut}(G)$ are the following:

- $\mathrm{Aut}(G)$

The reducts of the random graph, revisited

Let $G := (V; E)$ be the random graph.

Let $\bar{G}$ be the graph that arises by switching edges and non-edges.

Let $- : V \to V$ be an isomorphism between G and $\bar{G}$.

For $c \in V$, let G_c be the graph that arises by switching all edges and non-edges from c.

Let $\mathrm{sw}_c : V \to V$ be an isomorphism between G and G_c.

Theorem (Thomas 1991)

The closed groups containing $\mathrm{Aut}(G)$ are the following:

- $\mathrm{Aut}(G)$
- $\langle \{-\} \cup \mathrm{Aut}(G) \rangle$

The reducts of the random graph, revisited

Let $G := (V; E)$ be the random graph.

Let $\bar{G}$ be the graph that arises by switching edges and non-edges.

Let $- : V \to V$ be an isomorphism between G and $\bar{G}$.

For $c \in V$, let G_c be the graph that arises by switching all edges and non-edges from c.

Let $\mathrm{sw}_c : V \to V$ be an isomorphism between G and G_c.

Theorem (Thomas 1991)

The closed groups containing $\mathrm{Aut}(G)$ are the following:

- $\mathrm{Aut}(G)$
- $\langle \{-\} \cup \mathrm{Aut}(G) \rangle$
- $\langle \{\mathrm{sw}_c\} \cup \mathrm{Aut}(G) \rangle$

The reducts of the random graph, revisited

Let $G := (V; E)$ be the random graph.

Let $\bar{G}$ be the graph that arises by switching edges and non-edges.

Let $- : V \to V$ be an isomorphism between G and $\bar{G}$.

For $c \in V$, let G_c be the graph that arises by switching all edges and non-edges from c.

Let $\mathrm{sw}_c : V \to V$ be an isomorphism between G and G_c.

Theorem (Thomas 1991)

The closed groups containing $\mathrm{Aut}(G)$ are the following:

- $\mathrm{Aut}(G)$
- $\langle \{-\} \cup \mathrm{Aut}(G) \rangle$
- $\langle \{\mathrm{sw}_c\} \cup \mathrm{Aut}(G) \rangle$
- $\langle \{-, \mathrm{sw}_c\} \cup \mathrm{Aut}(G) \rangle$

The reducts of the random graph, revisited

Let $G := (V; E)$ be the random graph.

Let $\bar{G}$ be the graph that arises by switching edges and non-edges.

Let $- : V \to V$ be an isomorphism between G and $\bar{G}$.

For $c \in V$, let G_c be the graph that arises by switching all edges and non-edges from c.

Let $\mathrm{sw}_c : V \to V$ be an isomorphism between G and G_c.

Theorem (Thomas 1991)

The closed groups containing $\mathrm{Aut}(G)$ are the following:

- $\mathrm{Aut}(G)$
- $\langle\{-\} \cup \mathrm{Aut}(G)\rangle$
- $\langle\{\mathrm{sw}_c\} \cup \mathrm{Aut}(G)\rangle$
- $\langle\{-, \mathrm{sw}_c\} \cup \mathrm{Aut}(G)\rangle$
- The full symmetric group S_V.

Proving Thomas' theorem

Proving Thomas' theorem

Step 1.
Let $\alpha \in S_V \setminus \mathrm{Aut}(G)$. Then α and $\mathrm{Aut}(G)$ generate $-$ or sw_c

Proving Thomas' theorem

Step 1.
Let $\alpha \in S_V \setminus \mathrm{Aut}(G)$. Then α and $\mathrm{Aut}(G)$ generate $-$ or sw_c

"$-$ and sw_c are the *minimal* permutations over $\mathrm{Aut}(G)$."

Proving Thomas' theorem

Step 1.
Let $\alpha \in S_V \setminus \text{Aut}(G)$. Then α and Aut(G) generate $-$ or sw_c

"$-$ and sw_c are the *minimal* permutations over Aut(G)."

Step 2.
Let $\alpha \in S_V \setminus \langle \{-\} \cup \text{Aut}(G) \rangle$. Then $\alpha, -$ and Aut(G) generate sw_c.

Proving Thomas' theorem

Step 1.
Let $\alpha \in S_V \setminus \mathrm{Aut}(G)$. Then α and $\mathrm{Aut}(G)$ generate $-$ or sw_c

"$-$ and sw_c are the *minimal* permutations over $\mathrm{Aut}(G)$."

Step 2.
Let $\alpha \in S_V \setminus \langle\{-\} \cup \mathrm{Aut}(G)\rangle$. Then $\alpha, -$ and $\mathrm{Aut}(G)$ generate sw_c.

Step 3.
Let $\alpha \in S_V \setminus \langle\{\mathrm{sw}_c\} \cup \mathrm{Aut}(G)\rangle$. Then α, sw_c and $\mathrm{Aut}(G)$ generate $-$.

Step 1.
Let $\alpha \in S_V \setminus \mathrm{Aut}(G)$. Then α and $\mathrm{Aut}(G)$ generate $-$ or sw_c

"$-$ and sw_c are the *minimal* permutations over $\mathrm{Aut}(G)$."

Step 2.
Let $\alpha \in S_V \setminus \langle \{-\} \cup \mathrm{Aut}(G) \rangle$. Then $\alpha, -$ and $\mathrm{Aut}(G)$ generate sw_c.

Step 3.
Let $\alpha \in S_V \setminus \langle \{\mathrm{sw}_c\} \cup \mathrm{Aut}(G) \rangle$. Then α, sw_c and $\mathrm{Aut}(G)$ generate $-$.

Step 4.
Let $\alpha \in S_V \setminus \langle \{-, \mathrm{sw}_c\} \cup \mathrm{Aut}(G) \rangle$. Then $\alpha, -, \mathrm{sw}_c$ and $\mathrm{Aut}(G)$ generate S_V.

Canonical functions on the Random graph

Definition. $f : V \to V$ is *canonical* iff

Canonical functions on the Random graph

Definition. $f : V \to V$ is *canonical* iff
for all $x, y, u, v \in V$,
if (x, y) and (u, v) have the same type,

Canonical functions on the Random graph

Definition. $f : V \to V$ is *canonical* iff
for all $x, y, u, v \in V$,
if (x, y) and (u, v) have the same type,
then so do $(f(x), f(y))$ and $(f((u), f(v))$.

Canonical functions on the Random graph

Definition. $f : V \to V$ is *canonical* iff
for all $x, y, u, v \in V$,
if (x, y) and (u, v) have the same type,
then so do $(f(x), f(y))$ and $(f((u), f(v))$.

Examples.

Canonical functions on the Random graph

Definition. $f : V \to V$ is *canonical* iff
for all $x, y, u, v \in V$,
if (x, y) and (u, v) have the same type,
then so do $(f(x), f(y))$ and $(f((u), f(v))$.

Examples.
The identity is canonical.

Canonical functions on the Random graph

Definition. $f : V \to V$ is *canonical* iff
for all $x, y, u, v \in V$,
if (x, y) and (u, v) have the same type,
then so do $(f(x), f(y))$ and $(f((u), f(v))$.

Examples.
The identity is canonical.
$-$ is canonical on V.

Canonical functions on the Random graph

Definition. $f : V \to V$ is *canonical* iff
for all $x, y, u, v \in V$,
if (x, y) and (u, v) have the same type,
then so do $(f(x), f(y))$ and $(f((u), f(v))$.

Examples.

The identity is canonical.

$-$ is canonical on V.

sw_c is canonical on any $F \subseteq V \setminus \{c\}$.

Canonical functions on the Random graph

Definition. $f : V \to V$ is *canonical* iff
for all $x, y, u, v \in V$,
if (x, y) and (u, v) have the same type,
then so do $(f(x), f(y))$ and $(f((u), f(v))$.

Examples.

The identity is canonical.

$-$ is canonical on V.

sw_c is canonical on any $F \subseteq V \setminus \{c\}$.

$f : V \to V$ is *canonical on* $F \subseteq V$ iff its restriction to F is canonical.

Finding canonical behaviour

The class of finite graphs has the following Ramsey property:

Finding canonical behaviour

The class of finite graphs has the following Ramsey property:

For all graphs H
there exists a graph S such that

Finding canonical behaviour

The class of finite graphs has the following Ramsey property:

For all graphs H
there exists a graph S such that
if the edges of S are colored with 2 colors,

Finding canonical behaviour

The class of finite graphs has the following Ramsey property:

For all graphs H
there exists a graph S such that
if the edges of S are colored with 2 colors,
then there is a copy of H in S
on which the coloring is constant.

Finding canonical behaviour

The class of finite graphs has the following Ramsey property:

For all graphs H
there exists a graph S such that
if the edges of S are colored with 2 colors,
then there is a copy of H in S
on which the coloring is constant.

Given $f : V \to V$, color an edge according to the type of its image (3 possibilities).
Same for non-edges.

Finding canonical behaviour

The class of finite graphs has the following Ramsey property:

For all graphs H
there exists a graph S such that
if the edges of S are colored with 2 colors,
then there is a copy of H in S
on which the coloring is constant.

Given $f : V \to V$, color an edge according to the type of its image (3 possibilities).
Same for non-edges.

Conclusion: Every finite graph has a copy in G on which f is canonical.

Patterns in functions on the Random graph

Being canonical means:

Patterns in functions on the Random graph

Being canonical means:

Turning everything into edges (e_E), or

Patterns in functions on the Random graph

Being canonical means:

Turning everything into edges (e_E), or
turning everything into non-edges (e_N), or

Patterns in functions on the Random graph

Being canonical means:

Turning everything into edges (e_E), or
turning everything into non-edges (e_N), or
behaving like $-$, or

Patterns in functions on the Random graph

Being canonical means:

Turning everything into edges (e_E), or
turning everything into non-edges (e_N), or
behaving like $-$, or
being constant, or

Patterns in functions on the Random graph

Being canonical means:

Turning everything into edges (e_E), or
turning everything into non-edges (e_N), or
behaving like $-$, or
being constant, or
behaving like the identity.

Patterns in functions on the Random graph

Being canonical means:

Turning everything into edges (e_E), or
turning everything into non-edges (e_N), or
behaving like $-$, or
being constant, or
behaving like the identity.

Let $f : V \to V$.
If $f \notin \mathrm{Aut}(G)$, then there are $c, d \in V$ witnessing this.

Patterns in functions on the Random graph

Being canonical means:

Turning everything into edges (e_E), or
turning everything into non-edges (e_N), or
behaving like $-$, or
being constant, or
behaving like the identity.

Let $f : V \to V$.
If $f \notin \mathrm{Aut}(G)$, then there are $c, d \in V$ witnessing this.

The structure $(V; E, c, d)$ has similar Ramsey properties as $(V; E)$:

The subsets of elements of the same type contain the Random graph or have just one element.

An improved version of Thomas' theorem

Theorem (Thomas 1996)

Let $f : V \to V$, $f \notin \mathrm{Aut}(G)$.
Then f generates one of the following:

- A constant operation
- e_E
- e_N
- $-$
- sw_c

An improved version of Thomas' theorem

Theorem (Thomas 1996)

Let $f : V \to V$, $f \notin \mathrm{Aut}(G)$.
Then f generates one of the following:

- A constant operation
- e_E
- e_N
- $-$
- sw_c

We thus know the *minimal closed monoids* containing $\mathrm{Aut}(G)$.

Ramsey classes

Ramsey classes

Let N, H, P be graphs.

$$N \to (H)^P$$

means:

Ramsey classes

Let N, H, P be graphs.

$$N \to (H)^P$$

means:

For all partitions of the copies of P in N into *good* and *bad*

there exists a copy of H in N

such that the copies of P in H are all good or all bad.

Ramsey classes

Let N, H, P be graphs.

$$N \to (H)^P$$

means:

For all partitions of the copies of P in N into *good* and *bad*

there exists a copy of H in N

such that the copies of P in H are all good or all bad.

Definition

A class $\mathcal{C}$ of structures of the same signature is called a *Ramsey class* iff
for all $H, P \in \mathcal{C}$ there is N in $\mathcal{C}$ such that $N \to (H)^P$.

Canonical functions on Ramsey structures

Let Γ be Ramsey (i.e., its age is a Ramsey class).

Let n be the maximum of the arities of its relations.

Canonical functions on Ramsey structures

Let Γ be Ramsey (i.e., its age is a Ramsey class).

Let n be the maximum of the arities of its relations.

Definition

$f : \Gamma \to \Gamma$ is *canonical* iff
for all n-tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type
$(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type too.

Canonical functions on Ramsey structures

Let Γ be Ramsey (i.e., its age is a Ramsey class).

Let n be the maximum of the arities of its relations.

Definition

$f : \Gamma \to \Gamma$ is *canonical* iff
for all n-tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type
$(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type too.

Observation. Let H be a finite structure in the age of Γ.
Then there is a copy of H in Γ on which f is canonical.

Canonical functions on Ramsey structures

Let Γ be Ramsey (i.e., its age is a Ramsey class).

Let n be the maximum of the arities of its relations.

Definition

$f : \Gamma \to \Gamma$ is *canonical* iff
for all n-tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type
$(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type too.

Observation. Let H be a finite structure in the age of Γ.
Then there is a copy of H in Γ on which f is canonical.

Thus: Any $f : V \to V$ generates a canonical function,

Canonical functions on Ramsey structures

Let Γ be Ramsey (i.e., its age is a Ramsey class).

Let n be the maximum of the arities of its relations.

Definition

$f : \Gamma \to \Gamma$ is *canonical* iff
for all n-tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type
$(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type too.

Observation. Let H be a finite structure in the age of Γ.
Then there is a copy of H in Γ on which f is canonical.

Thus: Any $f : V \to V$ generates a canonical function,
but it could be the identity.

Canonical functions on Ramsey structures

Let Γ be Ramsey (i.e., its age is a Ramsey class).

Let n be the maximum of the arities of its relations.

Definition

$f : \Gamma \to \Gamma$ is *canonical* iff
for all n-tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type
$(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type too.

Observation. Let H be a finite structure in the age of Γ.
Then there is a copy of H in Γ on which f is canonical.

Thus: Any $f : V \to V$ generates a canonical function,
but it could be the identity.

We would like to fix $c_1, \ldots, c_n$ witnessing $f \notin \mathrm{Aut}(\Gamma)$,
and have canonical behavior on $(\Gamma, c_1, \ldots, c_n)$.

Adding constants to Ramsey classes

Problem

If Γ is Ramsey, is $(\Gamma, c_1, \ldots, c_n)$ still Ramsey?

Adding constants to Ramsey classes

Problem

If Γ is Ramsey, is $(\Gamma, c_1, \ldots, c_n)$ still Ramsey?

Theorem (Kechris, Pestov, Todorcevic 2005)

An ordered homogeneous structure Δ is Ramsey iff
its automorphism group is *extremely amenable*, i.e.,
it has a fixed point whenever it acts on a compact topological space.

Adding constants to Ramsey classes

Problem

If Γ is Ramsey, is $(\Gamma, c_1, \ldots, c_n)$ still Ramsey?

Theorem (Kechris, Pestov, Todorcevic 2005)

An ordered homogeneous structure Δ is Ramsey iff
its automorphism group is *extremely amenable*, i.e.,
it has a fixed point whenever it acts on a compact topological space.

Theorem (Tsankov 2010)

Every open subgroup of an extremely amenable group is extremely amenable.

Adding constants to Ramsey classes

Problem

If Γ is Ramsey, is $(\Gamma, c_1, \ldots, c_n)$ still Ramsey?

Theorem (Kechris, Pestov, Todorcevic 2005)

An ordered homogeneous structure Δ is Ramsey iff
its automorphism group is *extremely amenable*, i.e.,
it has a fixed point whenever it acts on a compact topological space.

Theorem (Tsankov 2010)

Every open subgroup of an extremely amenable group is extremely amenable.

Corollary

If Γ is ordered Ramsey, then so is $(\Gamma, c_1, \ldots, c_n)$.

Minimal reducts of Ramsey structures

Thus:

If Γ is ordered Ramsey, $f : \Gamma \to \Gamma$, and $c_1, \ldots, c_n \in \Gamma$, then f generates a function canonical for $(\Gamma, c_1, \ldots, c_n)$ which behaves like f on $\{c_1, \ldots, c_n\}$.

Minimal reducts of Ramsey structures

Thus:

If Γ is ordered Ramsey, $f : \Gamma \to \Gamma$, and $c_1, \ldots, c_n \in \Gamma$,
then f generates a function canonical for $(\Gamma, c_1, \ldots, c_n)$
which behaves like f on $\{c_1, \ldots, c_n\}$.

Theorem

Let Γ be ordered Ramsey. Then:

Minimal reducts of Ramsey structures

Thus:

If Γ is ordered Ramsey, $f : \Gamma \to \Gamma$, and $c_1, \ldots, c_n \in \Gamma$,
then f generates a function canonical for $(\Gamma, c_1, \ldots, c_n)$
which behaves like f on $\{c_1, \ldots, c_n\}$.

Theorem

Let Γ be ordered Ramsey. Then:

- There are finitely many minimal closed supergroups of $\mathrm{Aut}(\Gamma)$.

Minimal reducts of Ramsey structures

Thus:

If Γ is ordered Ramsey, $f : \Gamma \to \Gamma$, and $c_1, \ldots, c_n \in \Gamma$, then f generates a function canonical for $(\Gamma, c_1, \ldots, c_n)$ which behaves like f on $\{c_1, \ldots, c_n\}$.

Theorem

Let Γ be ordered Ramsey. Then:

- There are finitely many minimal closed supergroups of $\mathrm{Aut}(\Gamma)$.
- Every closed supergroup of $\mathrm{Aut}(\Gamma)$ contains a minimal closed supergroup of $\mathrm{Aut}(\Gamma)$.

Minimal reducts of Ramsey structures

Thus:

If Γ is ordered Ramsey, $f : \Gamma \to \Gamma$, and $c_1, \ldots, c_n \in \Gamma$,
then f generates a function canonical for $(\Gamma, c_1, \ldots, c_n)$
which behaves like f on $\{c_1, \ldots, c_n\}$.

Theorem

Let Γ be ordered Ramsey. Then:

- There are finitely many minimal closed supergroups of $\mathrm{Aut}(\Gamma)$.
- Every closed supergroup of $\mathrm{Aut}(\Gamma)$ contains a minimal closed supergroup of $\mathrm{Aut}(\Gamma)$.

Same is true for closed monoids.

Minimal reducts of Ramsey structures

Thus:

If Γ is ordered Ramsey, $f : \Gamma \to \Gamma$, and $c_1, \ldots, c_n \in \Gamma$, then f generates a function canonical for $(\Gamma, c_1, \ldots, c_n)$ which behaves like f on $\{c_1, \ldots, c_n\}$.

Theorem

Let Γ be ordered Ramsey. Then:

- There are finitely many minimal closed supergroups of $\mathrm{Aut}(\Gamma)$.
- Every closed supergroup of $\mathrm{Aut}(\Gamma)$ contains a minimal closed supergroup of $\mathrm{Aut}(\Gamma)$.

Same is true for closed monoids.

There are infinitely many closed supermonoids of $\mathrm{Aut}(\Gamma)$.

Primitive positive definitions

How about the minimal closed clones containing $\mathrm{Aut}(G)$?

= Reducts closed under primitive positive definitions

Primitive positive definitions

How about the minimal closed clones containing Aut(G)?

= Reducts closed under primitive positive definitions

Theorem (Bodirsky, Chen, P. 2008)

There are 2 minimal closed clones containing the automorphism group of $(X, =)$.

Primitive positive definitions

How about the minimal closed clones containing $\mathrm{Aut}(G)$?

= Reducts closed under primitive positive definitions

Theorem (Bodirsky, Chen, P. 2008)

There are 2 minimal closed clones containing the automorphism group of $(X, =)$.

Theorem (Bodirsky, P. 2009)

There are 14 minimal closed clones containing the automorphism group of the random graph.

Primitive positive definitions

How about the minimal closed clones containing Aut(G)?

= Reducts closed under primitive positive definitions

Theorem (Bodirsky, Chen, P. 2008)

There are 2 minimal closed clones containing the automorphism group of $(X, =)$.

Theorem (Bodirsky, P. 2009)

There are 14 minimal closed clones containing the automorphism group of the random graph.

Since arities of canonical functions are unbounded, there might be infinitely many minimal clones.

Open problems

Problem

If Γ is Ramsey, does it only have finitely many reducts up to f.o.-interdefinability?

Open problems

Problem

If Γ is Ramsey, does it only have finitely many reducts up to f.o.-interdefinability?

Problem

Is there a Ramsey Γ with an infinite number of minimal closed clones?

Open problems

Problem

If Γ is Ramsey, does it only have finitely many reducts up to f.o.-interdefinability?

Problem

Is there a Ramsey Γ with an infinite number of minimal closed clones?

Problem

Determine the reducts of the countable atomless Boolean algebra.

Open problems

Problem

If Γ is Ramsey, does it only have finitely many reducts up to f.o.-interdefinability?

Problem

Is there a Ramsey Γ with an infinite number of minimal closed clones?

Problem

Determine the reducts of the countable atomless Boolean algebra.

Problem

Determine the reducts of the random partial order.

Open problems

Problem

If Γ is Ramsey, does it only have finitely many reducts up to f.o.-interdefinability?

Problem

Is there a Ramsey Γ with an infinite number of minimal closed clones?

Problem

Determine the reducts of the countable atomless Boolean algebra.

Problem

Determine the reducts of the random partial order.

Problem (Junker, Ziegler)

If Γ is not ω-categorical, does it always have infinitely many reducts?

Thank you