Constraint satisfaction with homogeneous templates

Applications of model theory and Ramsey theory
in theoretical computer science

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MALOA Training Workshop, 2011

Outline

Constraint satisfaction problems (CSPs)

Constraint satisfaction - general

An *instance* of a *constraint satisfaction problem (CSP)* consists of:

- A finite set of variables
- Constraints for these variables

(the kind of constraints and possible values of the variables being defined by the CSP)

A *solution* to the CSP is an assignment of values to the variables such that all constraints are satisfied.

Examples:

- Sudokus
- Equations

We will consider a special kind of CSPs, namely Homomorphism problems.

Interested in the *complexity* of solving the problem.

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Examples:

- Graphs $G = (V; E)$, Digraphs
- Vertex-colored graphs
- Graphs with different types of edges
- Hypergraphs
- Databases
- Mathematical structures: $(\mathbb{N}; \neq)$, $(\mathbb{Q}; <, \leq, \neq, =)$, $(\mathbb{R}; \{(x, y)\ x^2 + y^2 \leq 1\})$, $\dots$

Homomorphisms

Let Δ and Γ be structures with the same relational signature τ.

Definition

A function $f : \Delta \to \Gamma$ is called homomorphism iff for each k-ary relation symbol R of τ

$$(a_1, \ldots, a_k) \in R^\Delta \;\to\; (f(a_1), \ldots, f(a_k)) \in R^\Gamma \; .$$

The Constraint Satisfaction Problem

Let Γ be a structure with a finite relational signature τ.
Γ also called the template.

Definition

CSP(Γ) is the computational problem to decide whether a given finite τ-structure Δ homomorphically maps to Γ.

Note: Γ need not be finite.

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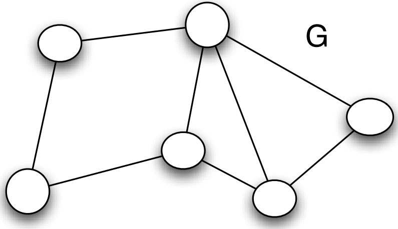
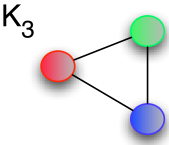
Example: 3-colorability is CSP(K_3)





Examples of CSPs

Positive 1-in-3-3SAT

Input: A set of triples of variables (x, y, z)

Question: Is there a 0/1-assignment to the variables such that in each clause exactly one variable is true?

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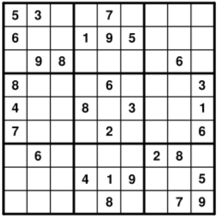
Sudokus



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A template for **Sudokus**:

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and $P_i = \{i\}$ for all $1 \leq i \leq 9$.

- Every Sudoku can be formulated as an instance of $\mathrm{CSP}(\Gamma_S)$
- Not all instances of $\mathrm{CSP}(\Gamma_S)$ correspond to a Sudoku.

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Which problems can be formulated as CSPs?

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$\mathcal{C}$ is closed under disjoint unions iff whenever $A, B \in \mathcal{C}$ then $A \mathbin{\dot\cup} B \in \mathcal{C}$.

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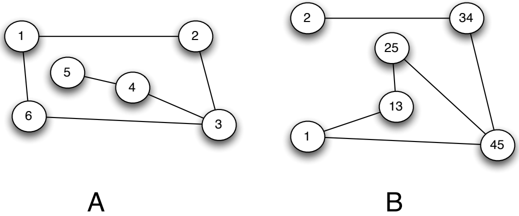

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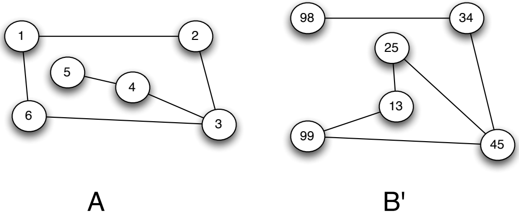

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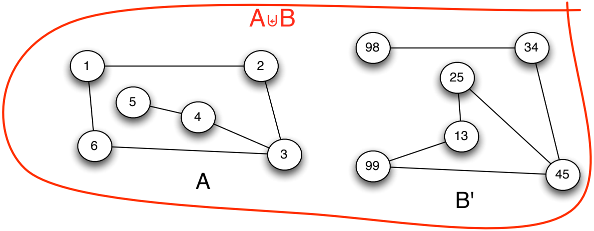

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Fundamental Lemma

Lemma (Feder ’93)

$\mathcal{C} =$CSP(Γ) for some relational structure Γ if and only if $\mathcal{C}$ is closed under disjoint unions and inverse homomorphisms.

Proof. It remains to show the ‘if’-part of the statement.

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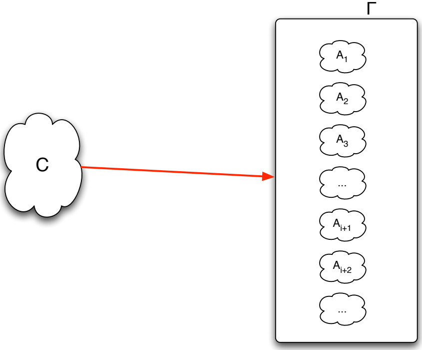

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Examples of CSPs

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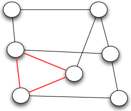

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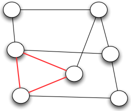

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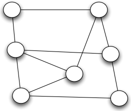

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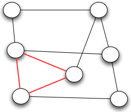

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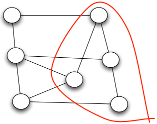

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Exercises

Prove that the following two problems can be formulated as CSPs with an infinite template. For each problem, give two proofs: one using the previous lemma, and one by direct construction of the template.

Acylic Bipartition:

Input: A digraph G

Question: Can we partition $V(G) = V_1 \uplus V_2$ such that $G[V_1]$ and $G[V_2]$ are acyclic?

Cyclic Embedding:

Input: A digraph G

Question: Can we map $V(G)$ to the plane such that all arcs in $E(G)$ have the origin on the left side?

Logical Perspective on CSPs

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Let τ be a relational signature.

Definition

A primitive positive τ-formula is a first-order τ-formula of the form

$$\exists x_1, \ldots, x_n . \psi_1 \wedge \cdots \wedge \psi_m \ ,$$

where $\psi_1, \ldots, \psi_m$ are atomic formulas, i.e., formulas of the form $x = y$ or of the form $R(x_{i_1}, \ldots, x_{i_k})$ for $R \in \tau$.

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Alternative definition of CSP(Γ):

Input: a primitive positive sentence Φ

Question: Is Φ true in Γ?

The Canonical Query

Let Δ be a finite τ-structure with domain D.

Definition

The canonical query $\Phi(\Delta)$ of Δ is the primitive positive formula with existentially quantified variables D that contains a conjunct $R(a_1, \ldots, a_n)$ iff $(a_1, \ldots, a_n) \in R^\Delta$.

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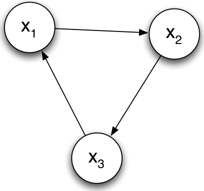

$$\Phi(\Delta) := \exists x_1, x_2, x_3.\ E(x_1, x_2) \wedge E(x_2, x_3) \wedge E(x_3, x_1)$$

Homomorphisms vs Logic

Let Γ be a τ-structure.

Lemma

For any finite τ-structure Δ the following are equivalent.

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Proof. trivial.

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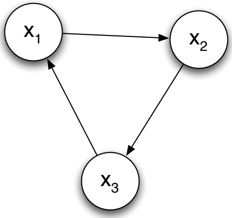

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- CSP(Γ) might be undecidable: CSP$(\mathbb{Z}; \{(x, y, z) \in \mathbb{Z}^3 \mid x + y = z\}, \{(x, y, z) \in \mathbb{Z}^3 \mid x * y = z\}, \{1\})$

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is polynomial-time equivalent to the problem of deciding whether a given polynomial equation has an integer solution (solving *diophantine* equations; ‘Hilberts 10th problem’). This problem was shown to be undecidable by Matiyasevich in 1970.

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Theorem

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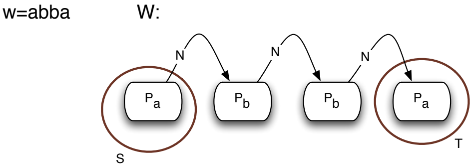
For every $L \subseteq \{a, b\}^*$ there is a relational structure Γ such that L is polynomial-time equivalent to CSP(Γ).

Reminder: Turing reduction. Write $L_1 \leq_t^p L_2$ if there is a deterministic polynomial-time Turing machine that decides L_1 with an oracle for L_2.

L_1 and L_2 are polynomial-time (Turing) equivalent if $L_1 \leq_t^p L_2$ and $L_2 \leq_t^p L_1$.

Proof

Proof idea. Encode words w from $\{a, b\}^*$ by structures W with signature $\{N, P_a, P_b, S, T\}$ as follows.



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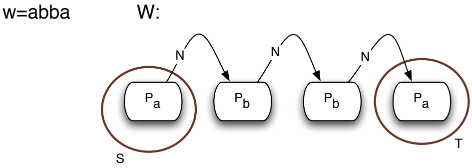

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Let Γ be the disjoint union over all structures in $\{W \mid w \in L\} \cup \mathcal{X}$.

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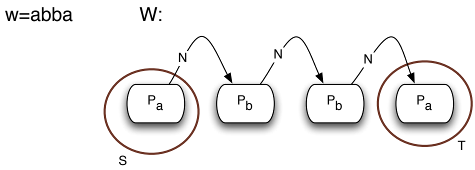

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Claim: L is polynomial-time Turing equivalent to $\mathrm{CSP}(\Gamma)$.

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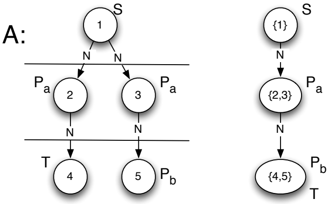

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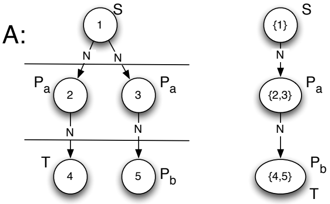

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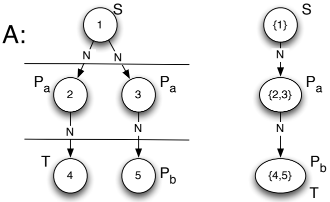

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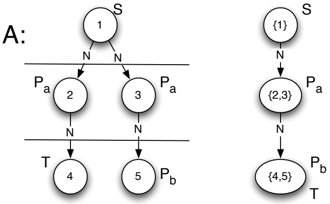

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otherwise have the word-structure of a word $w \in \{a, b\}^*$, and accept iff $w \in L$.

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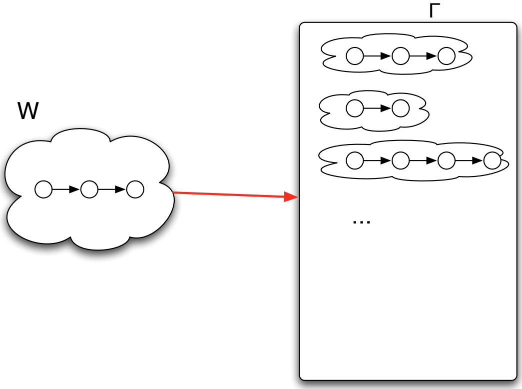

Tractability

One of the main questions in this course:

Which CSPs are tractable (=can be solved efficiently)?

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Criticism:

- Is worst-case complexity really the right concept (rather than e.g. ‘average-case complexity’)?
- Is a $O(n^{100})$ algorithm really better than an $O(1.01^n)$ algorithm?

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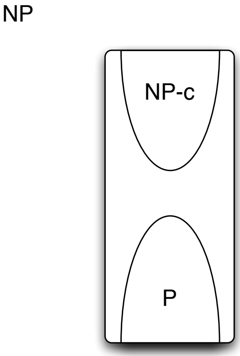
- ‘Classical’ complexity theory is mathematically rich, deep, and beautiful

Computational Complexity



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NP: class of computational problems decidable in non-deterministic polynomial time.

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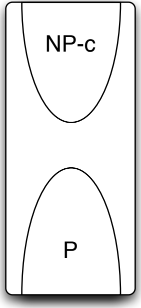

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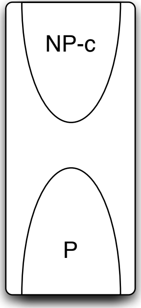

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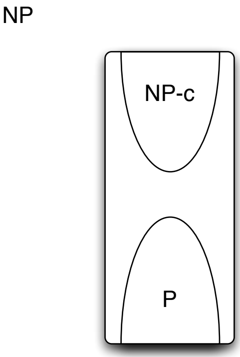

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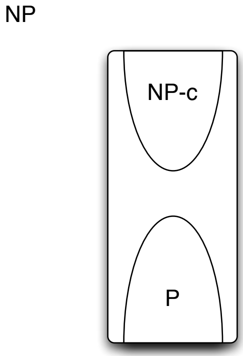
Ladner 1975: Unless P=NP, there are NP-intermediate problems: problems in NP that are neither in P nor NP-c.



The Quest for Tractable CSPs

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Theorem (Feder, Vardi ’93)

For every finite Γ, there is a directed graph H such that CSP(Γ) and CSP(H) are polynomial-time equivalent.

The algebraic approach

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We therefore identify such structures and call them *pp-interdefinable* or *pp-equivalent*.

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A *clone* is a set of finitary operations on D which

- contains the projections and
- is closed under composition.

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The operators Pol and Inv define a *Galois connection*, i.e.,

- Pol and Inv are antitone, and
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Structures which do not pp-define hard relations have polymorphisms violating them.

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Sample application: If Γ has a polymorphism which generates a tractable polymorphism, then CSP(Γ) is tractable.

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Tractability conjecture

For all structures Γ on a finite domain which are a core,

- either there is a poymorphism $f(x_1, x_2, x_3, x_4)$ satisfying $f(y, y, x, x) = f(x, x, x, y) = f(y, x, y, x)$, and CSP($\Gamma$) is tractable,
- or CSP(Γ) is NP-complete.

ω-categorical templates

Ryll-Nardzewski

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Thus, a relation R has a fo definition from Γ iff
it is preserved by all automorphisms of Γ.

Ryll-Nardzewski generalized

Theorem (Bodirsky and Nešetřil) '02

Let Γ be ω-categorical.

A relation *R* has a primitive positive (pp) definition from Γ iff it is preserved by all polymorphisms of Γ.

In other words, $\mathrm{Inv}\,\mathrm{Pol}(\Gamma) = \langle\Gamma\rangle_{pp}$.

Ryll-Nardzewski generalized

Theorem (Bodirsky and Nešetřil) ’02

Let Γ be ω-categorical.

A relation *R* has a primitive positive (pp) definition from Γ iff it is preserved by all polymorphisms of Γ.

In other words, $\mathrm{Inv}\,\mathrm{Pol}(\Gamma) = \langle \Gamma \rangle_{pp}$.

The corresponding first-order statement:

$\mathrm{Inv}\,\mathrm{Aut}(\Gamma) = \langle \Gamma \rangle_{fo}$

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Graph-SAT(Ψ_2) is in P.

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Γ_Ψ is a *reduct of* the random graph, i.e., a structure with a first-order definition in G.

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Classifying the complexity of all Graph-SAT problems is the same as classifying the complexity of CSPs of all reducts of the random graph.

Temporal constraints

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(Imagine: linear order relation.)

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Three classification theorems

All problems Boolean-SAT(Ψ), Graph-SAT(Ψ), and Temp-SAT(Ψ) are either in P or NP-complete.

Given Ψ, we can decide in which class the problem falls.

Boolean-SAT(Ψ): Schaefer (1978).

Temp-SAT(Ψ): Bodirsky and Kara (2007).

Graph-SAT(Ψ): Bodirsky and Pinsker (2010).

Remark: Complexity of CSPs for 3-element domains classified by Bulatov in ’03.

Ramsey structures

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\text{betw}(x, y, z) :=& \{(x, y, z) \in \mathbb{Q}^3 : x < y < z \text{ or } z < y < x\} \\
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Let Γ be a reduct of $(\mathbb{Q}; <)$. Then:

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Theorem (Junker, Ziegler '08)

$(\mathbb{Q}; <, 0)$ has 116 reducts, up to f.o.-interdefinability.

Thomas' conjecture

Conjecture (Thomas '91)

Let Γ be homogeneous in a finite language.

Then Γ has finitely many reducts up to f.o.-interdefinability.

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Groups, Monoids, Clones

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Let Γ be ω-categorical.

- The mapping $\Delta \mapsto \text{Aut}(\Delta)$ is a one-to-one correspondence between the first-order closed reducts of Γ and the closed supergroups of $\text{Aut}(\Gamma)$.

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5. The full symmetric group S_V.

How to find all reducts up to …-interdefinability?

Climb up the lattice!

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$f : V \to V$ is *canonical on* $F \subseteq V$ iff its restriction to F is canonical.

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Same for non-edges.

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Conclusion: Every finite graph has a copy in G on which f is canonical.

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The structure $(V; E, c, d)$ has similar Ramsey properties as $(V; E)$.

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Theorem (Bodirsky, P. ’09)

Let $f : V^n \to V,\ f \notin \mathrm{Aut}(G)$.
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Definition

A class $\mathcal{C}$ of structures of the same signature τ is called a *Ramsey class* iff
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Let H be a finite structure in the age of Γ.
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We would like to fix $c_1, \ldots, c_n$ witnessing $f \notin \mathrm{Aut}(\Gamma)$, and have canonical behavior on $(\Gamma, c_1, \ldots, c_n)$.

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Theorem (Kechris, Pestov, Todorcevic '05)

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Easy observation (Tsankov '10)

Every open subgroup of an extremely amenable group is extremely amenable.

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If Γ is Ramsey, is $(\Gamma, c_1, \ldots, c_n)$ still Ramsey?

Theorem (Kechris, Pestov, Todorcevic '05)

An ordered homogeneous structure Δ is Ramsey iff its automorphism group is *extremely amenable*, i.e., it has a fixed point whenever it acts on a compact topological space.

Easy observation (Tsankov '10)

Every open subgroup of an extremely amenable group is extremely amenable.

Corollary

If Γ is ordered, homogeneous, and Ramsey, then so is $(\Gamma, c_1, \ldots, c_n)$.

Minimal monoids above Ramsey structures

Thus:

If Γ is ordered Ramsey, $f : \Gamma \to \Gamma$, and $c_1, \ldots, c_n \in \Gamma$, then f generates a function canonical for $(\Gamma, c_1, \ldots, c_n)$ which behaves like f on $\{c_1, \ldots, c_n\}$.

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Theorem (Bodirsky, P., Tsankov '10)

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- There are finitely many minimal closed supermonoids of $\mathrm{End}(\Gamma)$.
- Every closed supermonoid of $\mathrm{End}(\Gamma)$ contains a minimal closed supermonoid of $\mathrm{End}(\Gamma)$.

Minimal clones above Ramsey structures

Going to products of Γ, we get:

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Theorem (Bodirsky, P., Tsankov '10)

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- There are finitely many minimal closed clones containing $\mathrm{Pol}(\Gamma)$. (Arity bound: $|S_2(\Gamma)|$.)
- Every closed clone above $\mathrm{Pol}(\Gamma)$ contains a minimal one.

Schaefer's theorem for graphs

Theorem (Bodirsky, P. '10)

Let Γ be a reduct of the random graph. Then CSP(Γ) is either in P or NP-complete.

Method: Prove hardness for certain relations, and tractability for certain polymorphisms.

If a reduct of G does not pp define any of the hard relations, then it has polymorphisms violating them.

These polymorphisms can be assumed to be canonical.

Thus they can easily be handled, and one can show that they produce one of the tractable polymorphisms.

The theorem in more detail

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Theorem

Let Γ be a reduct of the random graph. Then:

- Either Γ has one out of 17 canonical polymorphisms, and CSP(Γ) is tractable,
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- Either Γ pp-defines one out of 4 hard relations, and CSP(Γ) is NP-complete,
- or CSP(Γ) is tractable.

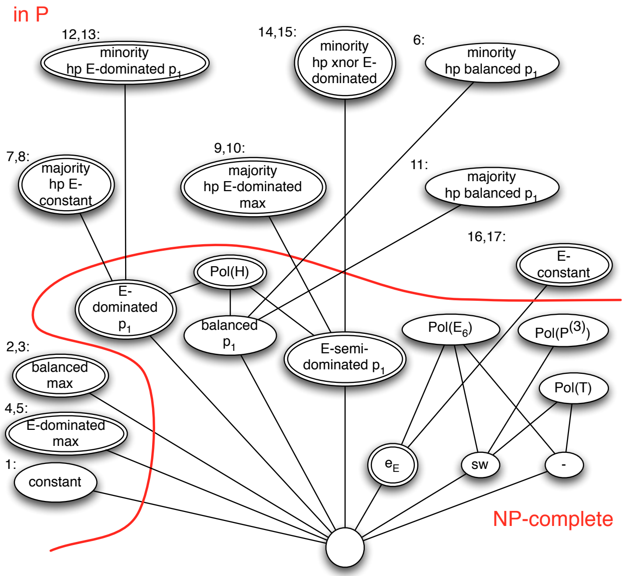
Examples of tractable polymorphisms

Theorem

The following 17 distinct clones are precisely the minimal tractable local clones containing $\mathrm{Aut}(G)$:

1. The clone generated by a constant operation.
2. The clone generated by a balanced binary injection of type max.
3. The clone generated by a balanced binary injection of type min.
4. The clone generated by an *E*-dominated binary injection of type max.
5. The clone generated by an *N*-dominated binary injection of type min.
6. The clone generated by a function of type majority which is hyperplanely balanced and of type projection.
7. The clone generated by a function of type majority which is hyperplanely *E*-constant.
8. The clone generated by a function of type majority which is hyperplanely *N*-constant.
9. The clone generated by a function of type majority which is

Classification



The Meta Problem

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Meta-Problem of Graph-SAT(Ψ)

INPUT: A finite set Ψ of graph formulas.

QUESTION: Is Graph-SAT(Ψ) in P?

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Theorem

The Meta-Problem of Graph-SAT(Ψ) is decidable.

Temp-SAT

Theorem (Bodirsky and Kara ’08)

Let Γ be a reduct of the order of the rationals. Then Γ either has one out of 9 binary canonical polymorphisms, and CSP(Γ) is in P, or CSP(Γ) is NP-complete.

Method: Prove hardness for certain relations, and tractability for certain polymorphisms.

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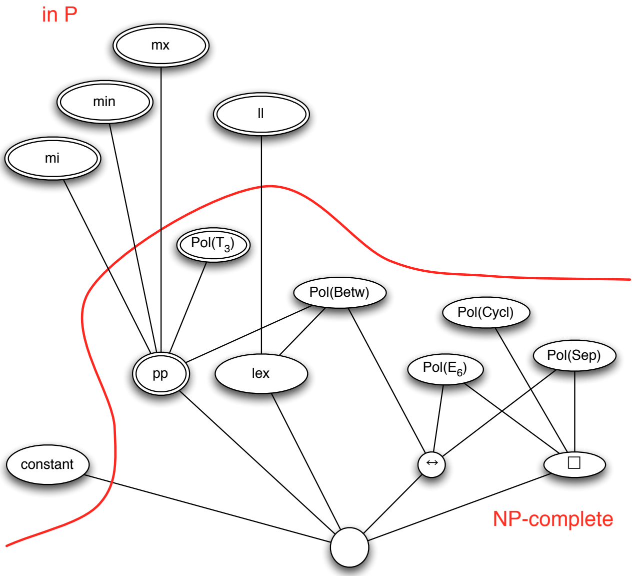

Boolean-SAT

Theorem (Schaefer '78)

Let Γ be a structure on a Boolean domain. Then Γ either has one of the polymorphisms listed below, and CSP(Γ) is in P, or CSP(Γ) is NP-complete.

- Constant
- Max
- Min
- Majority
- Minority

Proof: Any operation which depends on at least two variables generates Max, Min, Majority, or Minority. If all polymorphisms of Γ depend on at most one variable, and no polymorphism is constant, then the polymorphisms preserve

$\{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1)\}$,

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- CSPs model many real computational problems from theoretical computer science.
- Universal algebra useful in classifications since $\mathrm{Pol}(\Gamma)$ captures complexity.
- Many real computational problems are infinite domain CSPs.
- On infinite domains, add model theory + Ramsey theory to study polymorphisms. They then become functions on types, hence finite.