Making the Infinite Finite : Polymorphisms on Ramsey structures

Michael Pinsker

Université Denis Diderot - Paris 7 (60%)
Technische Universität Wien (30%)
Hebrew University of Jerusalem (10%)

Workshop on Algebra and CSPs
Fields Institute, Toronto, 2011

Outline

- **Part I**

  The global picture

- **Part II**

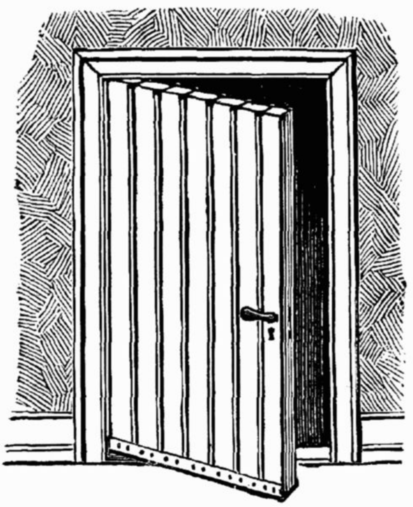
  Infinite template CSPs are natural
  Homogeneous structures

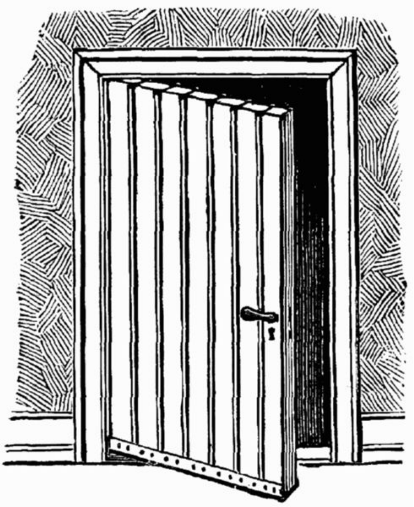
- **Part III**

  Infinite polymorphisms $\to$ finite polymorphisms
  Ramsey theory

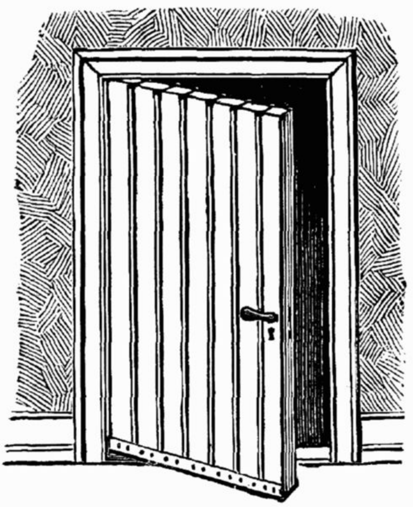
- **Part IV**

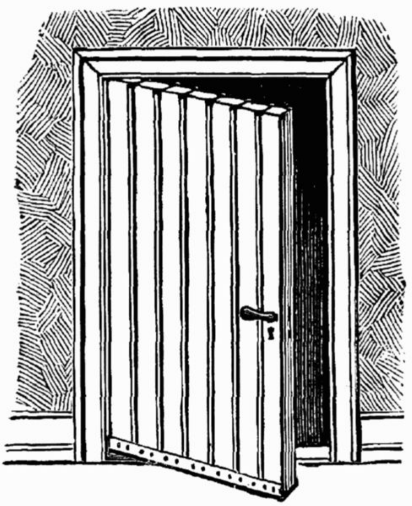
  The past and the future



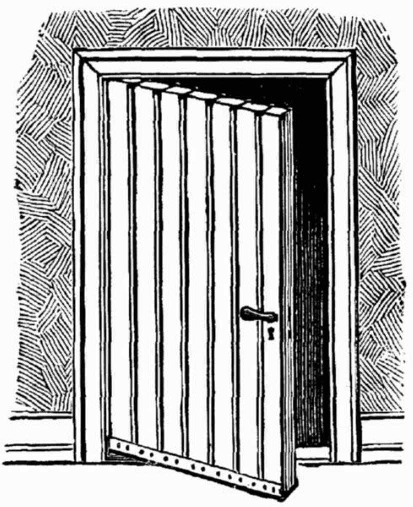


“I liked the doors ... I do not know what they mean, and they confused me, but they look nice.”





Welcome to the insane world of MP's talks



Welcome to the ~~insane world of MP's talks~~

madhouse of infinity

Part I

Cloning is fun

The organizers of the workshop

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Because most participants are [...] you can assume basic knowledge of algebra and CSP over a finite set, namely

- pp-definitions, polymorphisms, the Galois correspondence
- the complexity of the CSP depends only on the variety generated by the polymorphism algebra, wlog idempotent
- the dichotomy conjecture

”

Cloning finite sheep

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Let Γ be a finite structure.

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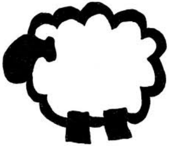

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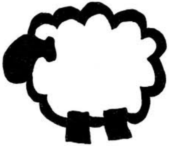
Let Γ be a finite structure. Let $\mathrm{Pol}\,(\Gamma)$ be its polymorphism clone.



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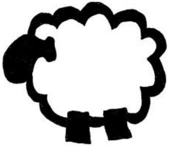

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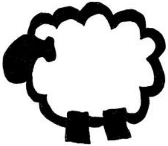

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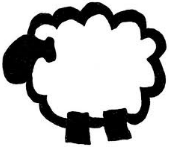

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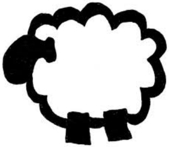

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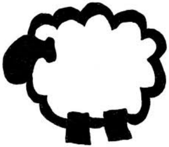

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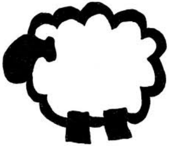

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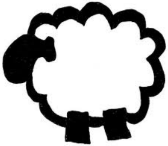

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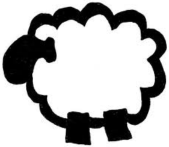

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Equations $\rightarrow$ in P
No equations $\rightarrow$ NP-complete

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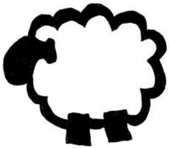

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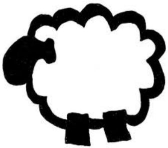

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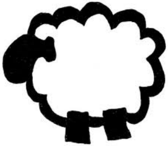

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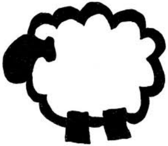

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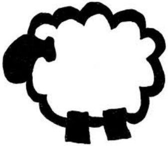

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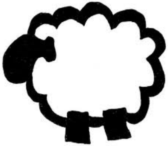

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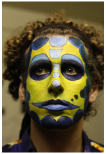

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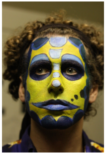

$\mathfrak{A}(\mathrm{Pol}(\Gamma))$

Abstractions seem possible.
Reduction to the finite?

Science fiction

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Wanted: Reduction of a certain class of infinite CSPs to finite CSPs.
This involves:

- Model theory
  (pp-definability, homogeneous templates Γ)
- Ramsey theory
  (analyzing polymorphisms, make them finite for algorithms)
- Topological dynamics
  (topological automorphism groups and clones)
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  (automatic continuity: topological clones vs. abstract clones)
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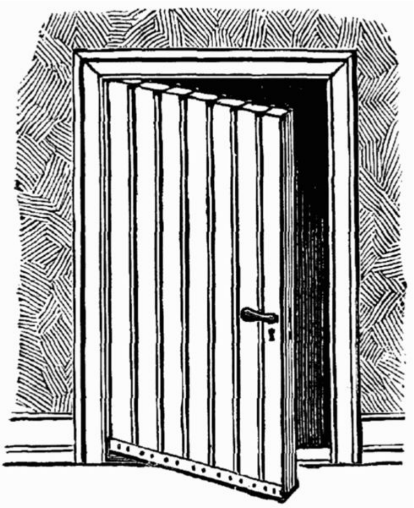
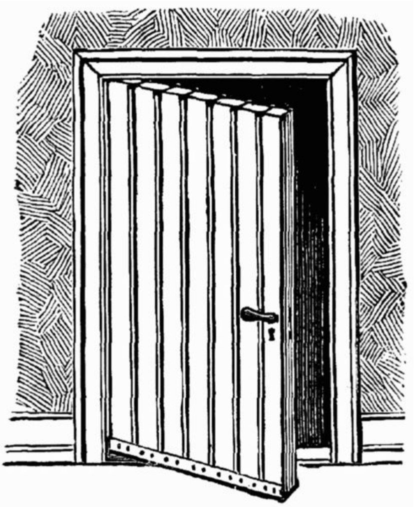
It might never work out. But imagine it does...





(We pass on to the next part.)

Part II

Do infinite sheep exist?

Digraph acyclicity

Input: A finite directed graph $(V; E)$

Question: Is $(V; E)$ acyclic?

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Input: A finite set of triples of variables (x, y, z)

Question: Is there a weak linear order on the variables such that for each triple either $x < y < z$ or $z < y < x$?

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More infinite sheep in nature

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Is a CSP: template is the homogeneous universal K_n-free graph

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Klagenfurt sheep

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Let E be a binary relation symbol.

(Imagine: edge relation of an undirected graph.)

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INPUT:

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Question

For which Ψ is Graph-SAT(Ψ) tractable?

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Graph-SAT(Ψ_2) is in P.

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- $W = \{w_1, \ldots, w_m\}$
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Classifying the complexity of all Graph-SAT problems is the same as classifying the complexity of CSPs of all reducts of the random graph.

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So Boolean-SAT(Ψ) and CSP(Γ_Ψ) are one and the same problem.

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For which Ψ is Temp-SAT(Ψ) tractable?

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Temp-SAT(Ψ) and CSP(Γ_Ψ) are one and the same problem.

Could have used any infinite linear order, but $(\mathbb{Q}; <)$ is homogeneous.

Three classification theorems

All problems Boolean-SAT(Ψ), Graph-SAT(Ψ), and Temp-SAT(Ψ) are either in P or NP-complete.

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- **Boolean-SAT:** Schaefer (1978)
- **Temp-SAT:** Bodirsky and Kára (2007)
- **Graph-SAT:** Bodirsky and MP (2010)

Homogeneous structures

Graph-SAT(Ψ): Is there a finite graph such that... (constraints)

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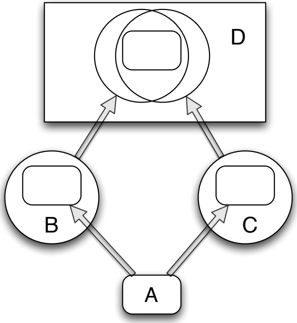
The classes of finite graphs and linear orders are *amalgamation classes*.



Fraïssé's theorem

Theorem (Fraïssé)

- If $\mathcal{C}$ is a countable class of structures closed under substructures which has amalgamation, then there exists a unique homogeneous structure with age $\mathcal{C}$.
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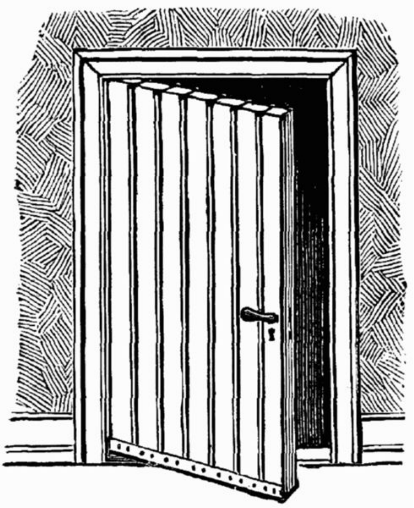
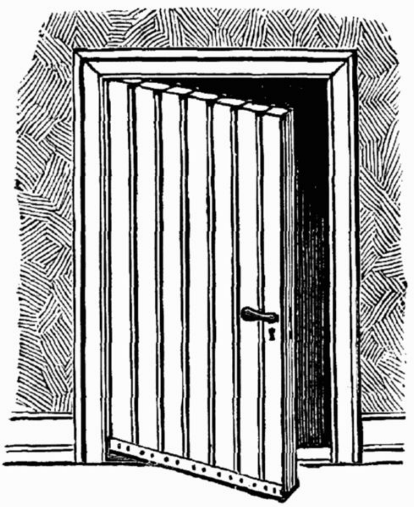
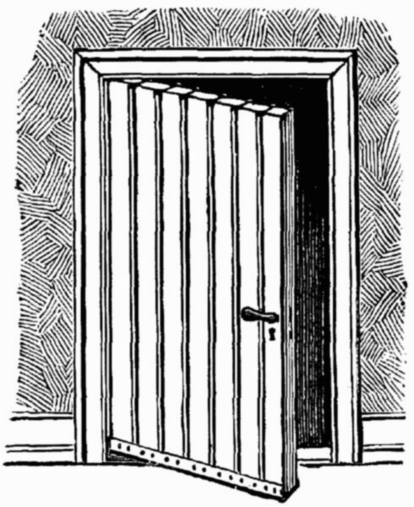
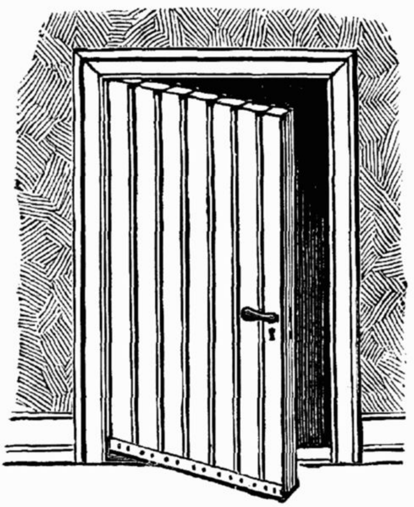
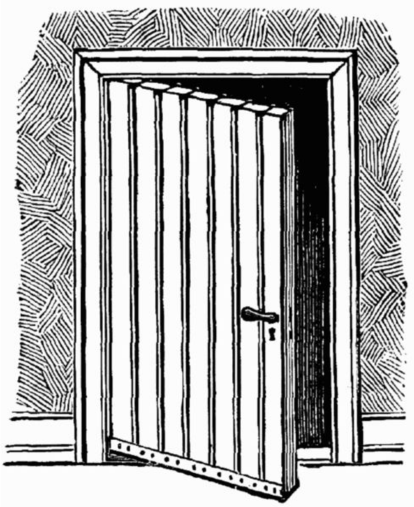
Homogeneous digraphs classified by Cherlin.











Part III

Making the infinite finite

Reducts of homogeneous structures

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We factor this quasiorder
by the equivalence relation of pp-interdefinability,
and obtain a complete lattice.

Reducts and closed clones

Problem

Classify the reducts of Δ up to pp-interdefinability.

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Theorem (Bodirsky, Nešetřil '03)

Let Δ be ω-categorical (e.g., homogeneous in a finite language). Then

$$\Gamma \mapsto \mathrm{Pol}(\Gamma)$$

is a one-to-one correspondence between
the *primitive positive closed* reducts of Δ and
the *closed clones* containing $\mathrm{Aut}(\Delta)$.

Richard's friends

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Use Ramsey theory to make them finite.

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Conclusion: Every finite graph has a copy in G on which f is canonical.

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The following are all possibilities of canonical functions:

- Turning everything into edges (e_E)

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A canonical function $f : G \to G$ induces a function $f' : \{E, N, =\} \to \{E, N, =\}$ (i.e., a function on the 2-types of G).

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Problem: Keeping some information on f when canonizing.

Adding constants

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By topological closure, f generates a function which:

- behaves like f on $\{c_1, \ldots, c_n\}$, and
- is canonical as a function from $(V; E, c_1, \ldots, c_n)$ to $(V; E)$.

The minimal clones on the random graph

Theorem (Bodirsky, MP '10)

Let f be a finitary operation on G which “is” not an automorphism.

Then f generates one of the following:

- A constant operation
- e_E
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More involved argument: Extend G by a random dense linear order.

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Definition

A class $\mathcal{C}$ of τ-structures is called a *Ramsey class* iff
for all $H, P \in \mathcal{C}$ there exists S in $\mathcal{C}$ such that $S \to (H)^P$.

Canonical functions on Ramsey structures

Let Δ now be an arbitrary structure.

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$f : \Delta \to \Delta$ is *canonical* iff
for all tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type
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- ω-categorical,

then all finite substructures of Δ have a copy in Δ
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We would like to fix $c_1, \ldots, c_n \in \Delta$ witnessing that f does something interesting (e.g., violate a certain relation), and have canonical behavior of f as a function from $(\Delta, c_1, \ldots, c_n)$ to Δ.

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Why don't you just do it?

Adding constants to Ramsey structures

Problem

If Δ is Ramsey, is $(\Delta, c_1, \ldots, c_n)$ still Ramsey?

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Theorem (Kechris, Pestov, Todorcevic '05)

An ordered homogeneous structure is Ramsey iff
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Corollary

If Δ is ordered, homogeneous, and Ramsey, then so is $(\Delta, c_1, \ldots, c_n)$.

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Proposition

If Δ is ordered Ramsey homogeneous finite language, $f : \Delta^k \to \Delta$, and $c_1, \ldots, c_n \in \Delta$, then f generates a function which

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Any element of the fixed point is canonical. $\square$

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  (Arity bound!)

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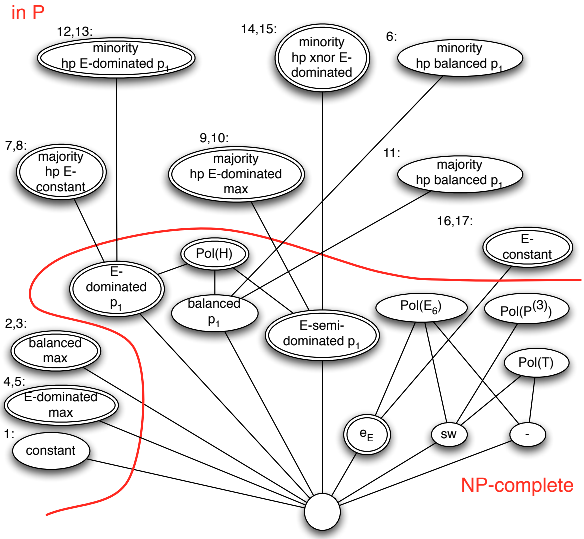
  (Arity bound!)
- Every closed superclone of $\mathrm{Pol}(\Gamma)$ contains a minimal closed superclone of $\mathrm{Pol}(\Gamma)$.

The Graph-SAT dichotomy visualized



The Graph-SAT dichotomy in more detail

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Let Γ be a reduct of the random graph. Then:

- Either Γ has one out of 17 canonical polymorphisms, and CSP(Γ) is tractable,
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Theorem (Bodirsky, MP '10)

Let Γ be a reduct of the random graph. Then:

- Either Γ pp-defines one out of 4 hard relations, and CSP(Γ) is NP-complete,
- or CSP(Γ) is tractable.

Theorem

The following 17 distinct clones are precisely the minimal tractable local clones containing Aut(G):

1. The clone generated by a constant operation.
2. The clone generated by a balanced binary injection of type max.
3. The clone generated by a balanced binary injection of type min.
4. The clone generated by an E-dominated binary injection of type max.
5. The clone generated by an N-dominated binary injection of type min.
6. The clone generated by a function of type majority which is hyperplanely balanced and of type projection.
7. The clone generated by a function of type majority which is hyperplanely E-constant.
8. The clone generated by a function of type majority which is hyperplanely N-constant.
9. The clone generated by a function of type majority which is hyperplanely of type max and E-dominated.
10. The clone generated by a function of type majority which is hyperplanely of type min and N-dominated.

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Meta-Problem of Graph-SAT(Ψ)

INPUT: A finite set Ψ of graph formulas.

QUESTION: Is Graph-SAT(Ψ) in P?

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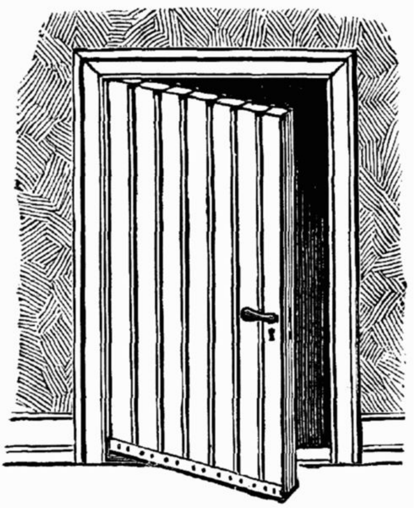
Theorem (Bodirsky, MP '10)

The Meta-Problem of Graph-SAT(Ψ) is decidable.



Part IV

The past and the future

The Past: What we can do

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- Climb up the clone lattice

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Theorem (Bodirsky, MP, Tsankov '10)

Let Δ be

- ordered Ramsey
- homogeneous
- with finite language
- finitely bounded.

Then the following problem is decidable:

INPUT: Two finite language reducts Γ_1, Γ_2 of Δ.
QUESTION: Is Γ_1 primitive positive definable in Γ_2?

The Future

1

The Future

Generalize setting of method

Is every structure Δ which is

- homogeneous
- with finite language
- finitely bounded

a reduct of a structure Δ' which is

- ordered Ramsey
- homogeneous
- with finite language
- finitely bounded.

?

2

The Future

Apply method

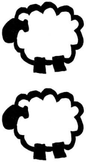
- Random partial order
- Random tournament
- Random K_n-free graph
- Atomless Boolean algebra
- Random lattice

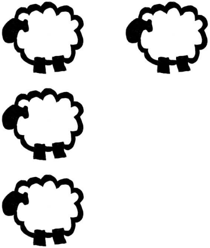
3

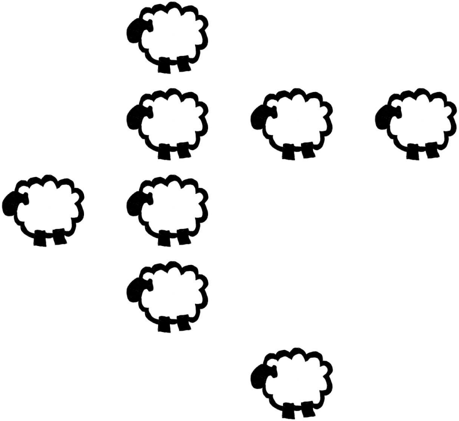
The Future

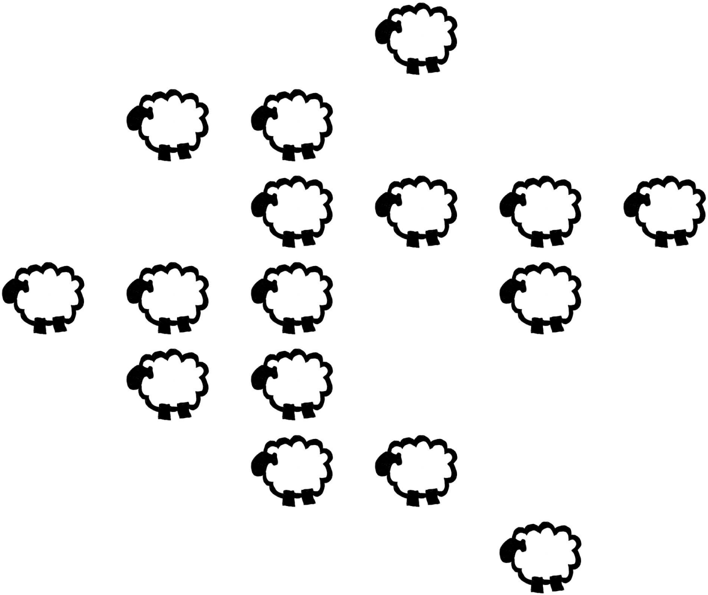
Develop method

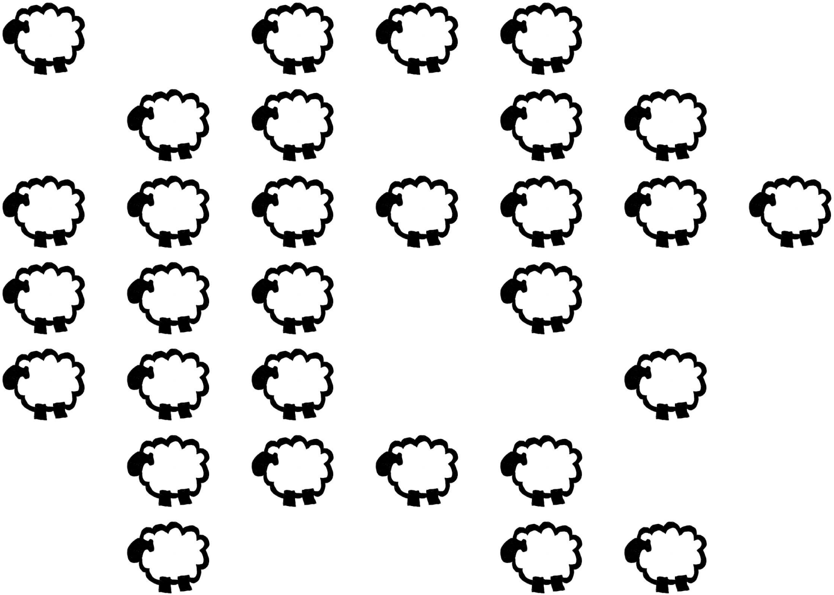
Abstract cloning $\rightarrow$ Manuel's talk

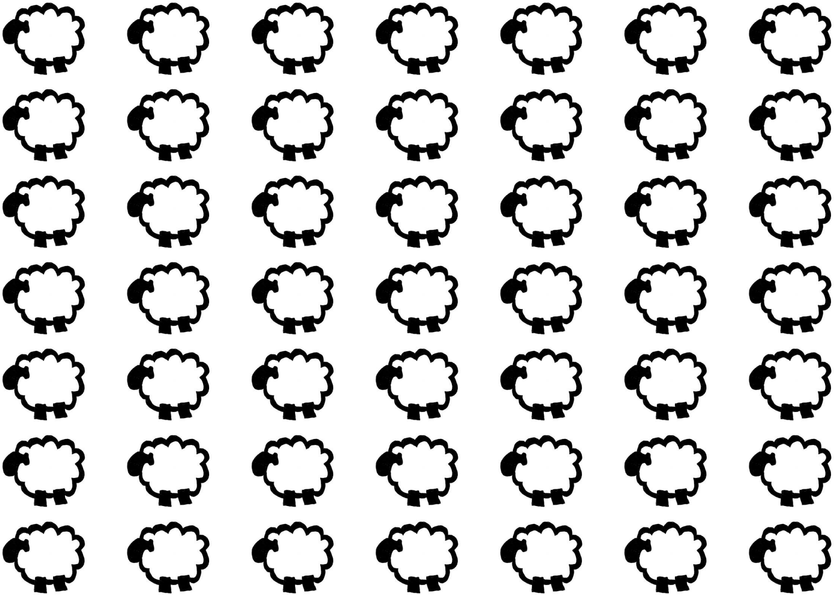


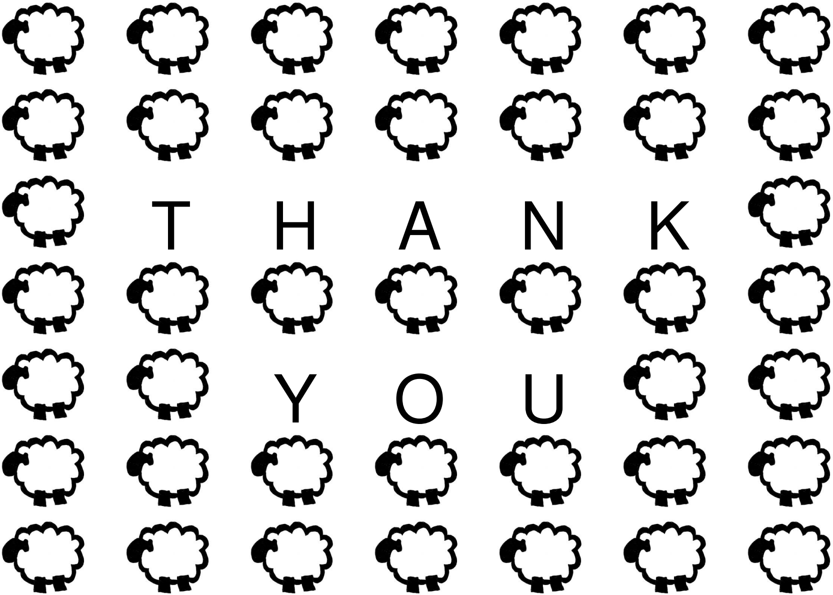











THANK
YOU