Reducts of Ramsey structures: the canonical approach

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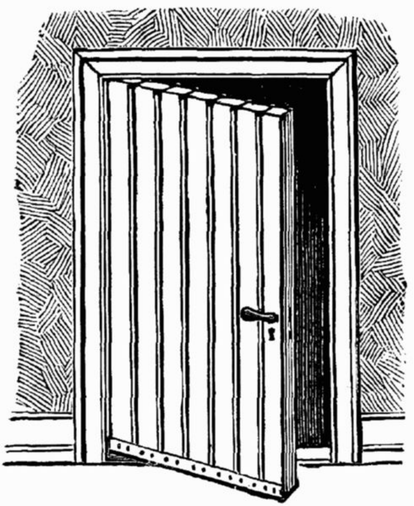
Outline

1. Reductions of homogeneous structures
   - First-order interdefinability
   - Finer classifications
   - Examples

2. Functions on homogeneous structures
   - Groups, monoids, clones
   - Canonical functions on Ramsey structures
   - The climbing up theorem

3. Reducts of the random graph

4. What we can do and what we cannot do
   - Decidability of interdefinability



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Problem

Classify the reducts of Δ.

We call Δ the *base structure*.

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We factor this quasiorder
by the equivalence relation of fo-interdefinability,
and obtain a complete lattice.

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Equivalence relation, quasiorder: Transitivity!
Not for all fragments of first-order logic.

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In fact:

The lattice corresponding to fo-definability is a factor of the lattice corresponding to ep-definability is a factor of the lattice corresponding to pp-definability.

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STOP!

In practice helps also for fo.

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For our method, we will need even "more" than homogeneity in a finite language:

The Ramsey property

Example: The dense linear order

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Denote by $(\mathbb{Q}; <)$ be the order of the rationals, and set

$$\begin{aligned}
\text{betw}(x, y, z) :=& \{(x, y, z) \in \mathbb{Q}^3 : x < y < z \text{ or } z < y < x\} \\
\text{cycl}(x, y, z) :=& \{(x, y, z) \in \mathbb{Q}^3 : x < y < z \text{ or } z < x < y \\
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5. Γ is first-order interdefinable with $(\mathbb{Q}; =)$.

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Let $G = (V; E)$ be the random graph, and set for all $k \geq 2$

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5. Γ is first-order interdefinable with $(V; =)$.

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The homogeneous universal k-graph has $2^k + 1$ reducts up to fo-interdefinability.

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Theorem (Junker, Ziegler '08)

$(\mathbb{Q}; <, 0)$ has 116 reducts up to fo-interdefinability.

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Depressing fact (Horváth, Pongrácz, P. '11)

The random graph with a constant has too many reducts up to fo-interdefinability.

Thomas' conjecture

Conjecture (Thomas '91)

Let Δ be homogeneous in a finite language.

Then Δ has finitely many reducts up to fo-interdefinability.

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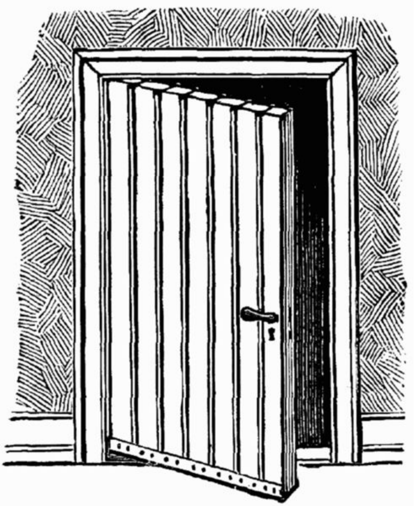

Functions on homogeneous structures

Permutation groups

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Theorem (Ryll-Nardzewski)

Let Δ be ω-categorical.

The mapping

$$\Gamma \mapsto \mathrm{Aut}(\Gamma)$$

is a one-to-one correspondence between

the *first-order closed* reducts of Δ and

the *closed permutation groups* containing $\mathrm{Aut}(\Delta)$.

first order closed $=$ contains all fo-definable relations

Monoids

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Theorem (follows from the Homomorphism preservation thm)

Let Δ be ω-categorical.

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$$\Gamma \mapsto \mathrm{End}(\Gamma)$$

is a one-to-one correspondence between
the *existential positive closed* reducts of Δ and
the *closed transformation monoids* containing $\mathrm{Aut}(\Delta)$.

A monoid of functions from Δ to Δ is *closed* iff
it is closed in the Baire space Δ^{Δ}.

Clones

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Theorem (Bodirsky, Nešetřil '03)

Let Δ be ω-categorical. Then

$$\Gamma \mapsto \mathrm{Pol}(\Gamma)$$

is a one-to-one correspondence between
the *primitive positive closed* reducts of Δ and
the *closed clones* containing $\mathrm{Aut}(\Delta)$.

A clone is a set of finitary operations on Δ which

- contains all projections $\pi_i^n(x_1, \ldots, x_n) = x_i$, and
- is closed under composition.

Pol(Γ) is the clone of all homomorphisms from finite powers of Γ to Γ.

A clone C is closed if for each $n \geq 1$, the set of n-ary operations in C is a closed subset of the Baire space Δ^{Δ^n}.

Groups, Monoids, Clones

For ω-categorical Δ:

Reducts up to **fo-interdefinability** $\leftrightarrow$
closed **permutation groups** $\supseteq \mathrm{Aut}(\Delta)$;

Reducts up to **ep-interdefinability** $\leftrightarrow$
closed **monoids** $\supseteq \mathrm{Aut}(\Delta)$

Reducts up to **pp-interdefinability** $\leftrightarrow$
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5. The full symmetric group S_V.

Climb up the lattice!

Canonical functions between structures

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Let Δ, Λ be structures.

Definition

$f : \Delta \to \Lambda$ is *canonical* iff
for all tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type in Δ
$(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type in Λ.

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Then $f : G \to G$ is canonical iff

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Possible types: edge, non-edge, point.

Examples of canonical functions

General examples.

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Canonical functions induce functions on types.

If the structures Δ, Λ are homogeneous in a finite language, then there are just finitely many canonical behaviors for $f : \Delta \to \Lambda$.

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Definition

A class $\mathcal{C}$ of τ-structures is called a *Ramsey class* iff for all $H, P \in \mathcal{C}$ there exists S in $\mathcal{C}$ such that $S \to (H)^P$.

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on which f is canonical.

Why? Let $t_1, \ldots, t_n$ be all tuples in F of length at most $|F|$.

For each t_i: color all tuples in Δ of the same type as t_i according to the type of $f(t_i)$ in Λ.

Ramsey property implies that all colorings are constant on a copy of F, even simultaneously.

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Note:

- only finitely many different behaviors of canonical functions.
- g, g' same behavior $\to$ generate one another (with $\mathrm{Aut}(\Delta)$).

Topological dynamics

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Theorem (Kechris, Pestov, Todorcevic '05)

Let Δ be ordered homogeneous. Then:

Δ is Ramsey if and only if

Aut(Δ) is *extremely amenable*, i.e.,
any continuous action of Aut(Δ) on any compact Hausdorff space X has a fixed point.

Fixed point: $x \in X$ such that $gx = x$ for all $g \in \mathrm{Aut}(\Delta)$.

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g is canonical: $B(g \circ \alpha^{-1}) = g$ for all $\alpha \in \mathrm{Aut}(\Delta, c_1, \ldots, c_n)$. $\square$

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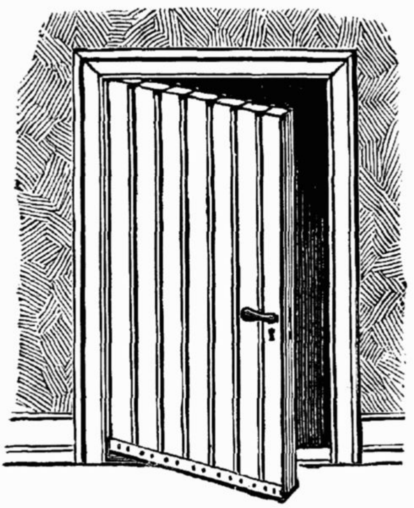
Non-trivial: arity bound!



Reducts of the random graph

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Conclusion: If $f : G \to G$ is any function, and $c_1, \ldots, c_n \in V$,
then f with $\mathrm{Aut}(G)$ generates a function which

- agrees with f on $\{c_1, \ldots, c_n\}$
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Theorem (Thomas '96)

Let $f : G \to G$ a function
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Generalized to *minimal closed clones* (14) by Bodirsky, P. 2010.

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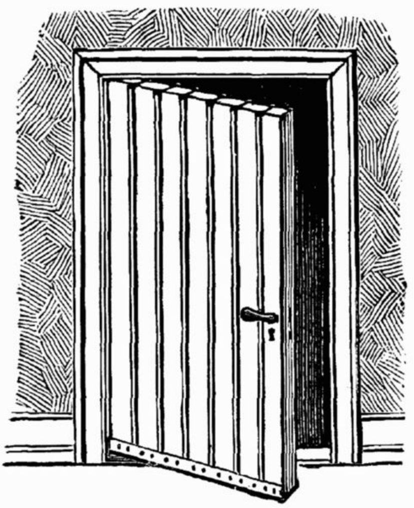

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Theorem (Bodirsky, P., Tsankov '10)

Let Δ be

- ordered
- homogeneous
- Ramsey
- with finite language
- *finitely bounded*.

Then the following problem is decidable:

INPUT: Two finite language reducts Γ, Γ' of Δ.
QUESTION: Are Γ, Γ' pp (ep-) interdefinable?

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- Is the ordered Ramsey context really a proper special case of the homogeneous in a finite language context?
- Is fo-interdefinability decidable?

References

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