Reconstructing the topology of clones

Michael Pinsker

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> Workshop on Homogeneous Structures HIM, 2013

Part I: Reconstructing structures from their automorphism groups and polymorphism clones

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- Part II: The topology of algebras

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- Part VI: Perspectives & Open problems



Part I

Reconstructing structures from their automorphism groups and polymorphism clones





countable

Reconstructing the topology of clones



countable, ω -categorical

Reconstructing the topology of clones

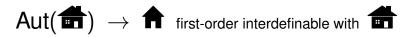
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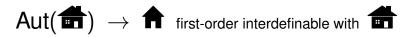
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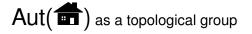


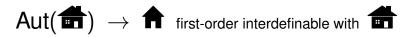


Theorem (Ryll-Nardzewski)

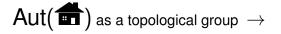


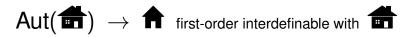
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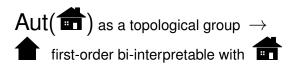


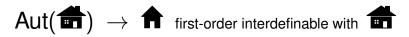
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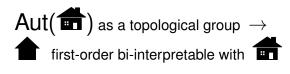
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Can we reconstruct the topological structure of closed oligomorphic permutation groups from their algebraic structure?

Reconstructing the topology of clones

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Observe: $Pol(\Delta) \supseteq End(\Delta) \supseteq Aut(\Delta)$.

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Confer Manuel Bodirsky's talk.

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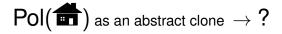
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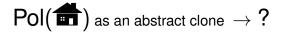
Theorem (Bodirsky + MP '12)

Let Δ , Γ be ω -categorical structures. Then: Pol(Δ) \cong^{τ} Pol(Γ) iff Δ , Γ are primitive positive bi-interpretable.

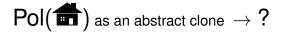
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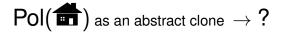




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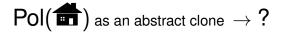
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Part II

The topology of algebras

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Structural conclusions about *finite* \mathfrak{A} from variety of \mathfrak{A} (i.e., from abstract clone $Clo(\mathfrak{A})$).

Reconstructing the topology of clones

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Theorem (Birkhoff 1935)

Let $\mathfrak{A}, \mathfrak{B}$ be finite.

 \mathfrak{B} is in HSP^{fin}(\mathfrak{A}) \leftrightarrow

the natural homomorphism from $Clo(\mathfrak{A})$ to $Clo(\mathfrak{B})$ exists.

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Part III

Reconstruction notions

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Similarly for closed subgroups of \boldsymbol{S}_∞ and closed submonoids of $\boldsymbol{O}^{(1)}.$

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Unclear for monoids and clones.

Reconstructing the topology of clones

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- the random K_n -free graphs (Herwig'98)
- ω-categorical ω-stable structures
 (Hodges+Hodkinson+Lascar+Shelah'93)

Groups: Rubin's forall-exists interpretations

Reconstructing the topology of clones

Michael Pinsker

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- the random graph
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 all homogeneous countable graphs
 various ω-categorical semilinear orders
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 (Rubin '94)
- the random k-hypergraphs the random K_n-free graphs the Henson digraphs (Barbina+MacPherson '07).



Part IV

Negative results

Reconstructing the topology of clones

Reconstructing the topology of clones

Not the right notion:

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Recall: 1 is the clone of projections on a set of at least two elements.

Important in constraint satisfaction:

"main reason" for NP-hardness of the CSP of a structure.

Automatic continuity to 1

Reconstructing the topology of clones

There exists an oligomorphic closed subclone of **O** with a discontinuous homomorphism to the projection clone **1**.

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Moreover, this clone also has a continuous homomorphism to 1.

Automatic homeomorphicity

Reconstructing the topology of clones

Automatic homeomorphicity

Theorem (Bodirsky + MP + Pongrácz '13)

There exists an oligomorphic closed submonoid **M** of $O^{(1)}$ and $\xi: \mathbf{M} \to \mathbf{M}$ such that:

- ξ is an isomorphism;
- **\blacksquare** ξ fixes the invertibles of **M** pointwise;
- ξ is not continuous.

In particular **M** does not have automatic homeomorphicity.

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Theorem (Evans + Hewitt '90)

There exists an oligomorphic closed subgroup ${\bm G}$ of ${\bm S}_\infty$ which does not have reconstruction.

Reconstruction

Reconstructing the topology of clones

Problem

Find an oligomorphic closed subclone of ${\bf O}$ without reconstruction.

Reconstructing the topology of clones



Part V Positive results

Reconstructing the topology of clones



Let **C** be a closed subclone of **O** whose group \mathbf{G}_{C} of invertibles has reconstruction.

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For constraint satisfaction:

Can assume that **C** is a model-complete core: G_C is dense in $C^{(1)}$.

Reconstructing the topology of clones

Theorem (Bodirsky + MP + Pongrácz '13)

Let Δ be homogeneous in a finite relational language without algebraicity (\leftrightarrow strong amalgamation).

If Aut(Δ) has automatic continuity, then its closure in **O**⁽¹⁾ has automatic homeomorphicity.

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Examples:

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Reconstructing the topology of clones

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Proposition

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Reconstructing the topology of clones

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Fact

There are closed oligomorphic subclones of **O** without transitive action.

Reconstructing the topology of clones

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■
$$g^i(x_1,...,x_n) = \alpha^i(f_U(\beta^i(x_1),...,\beta^i(x_n)))$$
 and
■ $(\alpha^i)_{i\in\omega}$ and $(\beta^i_1)_{i\in I},...,(\beta^i_n)_{i\in\omega}$ converge.

Reconstructing the topology of clones

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H H

Many more...?

Reconstructing the topology of clones



Part VI

Perspectives & Open problems

Reconstructing the topology of clones

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Perspectives

Reconstructing the topology of clones

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Perspectives

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Gate covering?

Reconstructing the topology of clones

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- Is there a model of ZF where all homomorphisms from oligomorphic closed subclones of O to the projection clone 1 are continuous?



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Thank you!

Reconstructing the topology of clones

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