The 42 reducts of the random ordered graph

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Part I: The setting of The Answer

Part II: The 42 reducts of the random ordered graph

Part III: The effect of The Answer

Part IV: The question to The Answer
Part I: The setting of The Answer
Homogeneous structures

Let $\Delta$ be a countable structure.
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**Definition**

$\Delta$ is **homogeneous** $\iff$

every isomorphism between finitely generated substructures of $\Delta$
extends to an automorphism of $\Delta$. 

*Examples*

- Order of the rationals $(\mathbb{Q}; <)$
- Random graph $(V; E)$
- Free Boolean algebra with $\aleph_0$ generators
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Fraïssé limits

Let $C$ be a class of finitely generated structures in a countable language, closed under isomorphism.

**Theorem (Fraïssé)**

Assume $C$ is closed under substructures has joint embeddings:

for all $B, C \in C$ there is $D \in C$ containing isomorphic copies of $B, C$ has amalgamation:

for all $A, B, C \in C$ and embeddings $e_B: A \to B$ and $e_C: A \to C$ there is $D \in C$ and embeddings $f_B: B \to D$ and $f_C: C \to D$ such that $f_B \circ e_B = f_C \circ e_C$.

Then there exists a unique countable homogeneous structure $\Delta$ whose age (=substructures up to iso) equals $C$. 

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Examples

- (Finite) linear orders $\leftrightarrow (\mathbb{Q}; <)$
- Undirected graphs $\leftrightarrow$ random graph $(V; E)$
- Boolean algebras $\leftrightarrow$ random (= free) Boolean algebra
- Lattices $\leftrightarrow$ random lattice
- Distributive lattices $\leftrightarrow$ random distributive lattice
- Partial orders $\leftrightarrow$ random partial order
- Tournaments $\leftrightarrow$ random tournament
- Linearly ordered graphs $\leftrightarrow$ random ordered graph $(D; <, E)$
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Reducts

Let $\Delta$ be a structure.

Definition
A reduct of $\Delta$ is a structure on the same domain whose relations and functions are first-order definable in $\Delta$ (without parameters).

Examples
- $\left(\mathbb{Q}; <\right)$: reduct $\left(\mathbb{Q}; \text{Between}(x, y, z)\right)$
- Random graph $\left(V; E\right)$: reduct $\left(V; K_3(x, y, z)\right)$
- Random poset $\left(P; \leq\right)$: reduct $\left(P; \bot(x, y)\right)$

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Understand the reducts of homogeneous structures.
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**Problem**

Understand the reducts of homogeneous structures.
Motivation

Why reducts?

Understand it itself:
- its first-order theory
- its symmetries (via connection with permutation groups)

Understand the age $C$ of $\Delta$:
- uniform group actions on $C$ (via permutation groups - combinatorics of $C$)
- Constraint Satisfaction Problems related to $C$: Graph-SAT, Poset-SAT, ...
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Reducts up to first-order equivalence

For reducts $\Gamma, \Gamma'$ of $\Delta$ set $\Gamma \leq \Gamma'$ iff $\Gamma$ is a reduct of $\Gamma'$.

Quasiorder.

Consider reducts $\Gamma, \Gamma'$ equivalent iff $\Gamma \leq \Gamma'$ and $\Gamma' \leq \Gamma$.

Factoring out we get a complete lattice.

Multiple choice: Equivalent or not?

$\left( \mathbb{Q}; \prec \right)$ and $\left( \mathbb{Q}; \succ \right)$

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random poset $\left( \mathcal{P}; \leq \right)$ and $\left( \mathcal{P}; \perp \right)$

random graph $\left( \mathcal{V}; E \right)$ and $\left( \mathcal{V}; K_3 \right)$

Question How many inequivalent reducts? 42

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How many inequivalent reducts?
Examples

- \( Q; < \) : 5 (Cameron '76)
- Random graph \((V; E)\) : 5 (Thomas '91)
- Random \( k \)-hypergraph: \( k + 1 \) (Thomas '96)
- Random tournament: 5 (Bennett '97)
- \( Q; <, 0 \) : 116 (Junker+Ziegler '08)
- Random partial order: 5 (Pach+MP+Pongrácz+Szabó '11)

Conjecture (Thomas '91)

Homogeneous structures in finite relational language have finitely many reducts.
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A permutation group is closed:

it contains all permutations which it can interpolate on finite subsets.

Theorem (Corollary of Ryll-Nardzewski, Engeler, Svenonius)

Let $\Delta$ be homogeneous in a finite relational language.

Then the mapping $\Gamma \mapsto \text{Aut}(\Gamma)$ is an anti-isomorphism from the lattice of reducts to the lattice of closed supergroups of $\text{Aut}(\Delta)$.
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Let \(\leftrightarrow\) be any permutation of \(\mathbb{Q}\) which reverses the order.

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The closed supergroups of \(\text{Aut}(\mathbb{Q}; <)\) are precisely:
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- \( \langle \{\circ\} \cup \text{Aut}(\mathbb{Q}; <) \rangle \)
- \( \langle \{\leftrightarrow, \circ\} \cup \text{Aut}(\mathbb{Q}; <) \rangle \)
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Let \(\leftrightarrow\) be any permutation of \(\mathbb{Q}\) which reverses the order.

Let \(\circlearrowleft\) be any permutation of \(\mathbb{Q}\) which for some irrational \(\pi\) puts \((-\infty; \pi)\) behind \((\pi; \infty)\) and preserves the order otherwise.

**Theorem (Cameron ’76)**

The closed supergroups of \(\text{Aut}(\mathbb{Q}; <)\) are precisely:

- \(\text{Aut}(\mathbb{Q}; <)\)
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The random graph \((V; E)\)

Let \(-\) be any permutation of \(V\) which switches edges and non-edges.

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For \(k \geq 1\), let \(R(k)\) consist of the \(k\)-tuples of distinct elements of \(V\) which induce an odd number of edges.

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The random graph \((V; E)\)

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Part II: The 42 reducts of the random ordered graph
The random ordered graph

**Definition**

The random ordered graph \((D; <, E)\) is the unique countable linearly ordered graph which
- contains all finite linearly ordered graphs
- is homogeneous.
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**Observation**
- $(D; <)$ is the order of the rationals
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This is because the two structures are superposed *freely*, i.e., in all possible ways.
Strong amalgamation

A class \( C \) has strong amalgamation if for all \( A, B, C \in C \) and embeddings \( e_B: A \to B \) and \( e_C: A \to C \), there is \( D \in C \) and embeddings \( f_B: B \to D \) and \( f_C: C \to D \) such that
\[
f_B \circ e_B = f_C \circ e_C
\]
and
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f_B[B] \cap f_C[C] = f_B[e_B[A]].
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Strong amalgamation

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A class $\mathcal{C}$ has strong amalgamation $\iff$
for all $A, B, C \in \mathcal{C}$ and embeddings $e_B : A \to B$ and $e_C : A \to C$
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Mixing

Let $\tau_1, \tau_2$ be disjoint languages. Let $C_1, C_2$ be Fraïssé classes in those languages, $\Delta_1, \Delta_2$ be their limits.

Free superposition

Assume that $C_1, C_2$ have strong amalgamation. Then the class $C_1$ of $\tau_1 \cup \tau_2$-structures whose $\tau_i$-reduct is in $C_i$ is a Fraïssé class and the $\tau_i$-reduct of its limit is isomorphic to $\Delta_i$.
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- the $\tau_i$-reduct of its limit is isomorphic to $\Delta_i$. 
Every reduct of \((D;\prec)\) is a reduct of the random ordered graph.

Every reduct of \((D;E)\) is a reduct of the random ordered graph.

If \((D;R)\) is a reduct of \((D;\prec)\) and \((D;S)\) is a reduct of \((D;E)\) then \((D;R,S)\) is a reduct of the random ordered graph.

Corresponds to intersecting the groups \(\text{Aut}(D;R)\) and \(\text{Aut}(D;S)\).

Yields distinct reducts because of free superposition.

Examples
- Keeping the order while flipping the graph relation.
- Reversing the order while keeping the graph relation.

Lemma The random ordered graph has at least 25 reducts.
Every reduct of \((D; <)\) is a reduct of the random ordered graph.
Trivial reducts of the random ordered graph

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The random ordered graph has at least 25 reducts.
Similarities between the order and the graph reducts

The following permutations yield new non-trivial reducts. Reversing the order and simultaneously flipping the graph relation for an irrational \( \pi \), put \((−\infty, \pi)\) behind \((\pi, \infty)\) whilst flipping the graph relation between these parts.

No other combination of this kind!

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The random ordered graph has at least 27 reducts.
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Hello random tournament!

Definition
A tournament is a digraph with precisely one edge between any two vertices.

Theorem (Bennett '97)
The random tournament has 5 reducts.

Observation
Set $T(x, y)$ iff $x < y \land E(x, y)$ or $x > y \land N(x, y)$.

Then $(D; T)$ is the random tournament.

Lemma
The random ordered graph has at least 32 reducts.
Hello random tournament!

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<table>
<thead>
<tr>
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Part III: The effect of The Answer
Discussion

We have learnt from the result: similarities between the symmetries of the order and the graph nonetheless their combination yields an asymmetry we cannot calculate the reducts of a superposed structure from its factors.

On a technical level: our Ramsey-theoretic method is quite efficient (first classification of free superposition) improved it to reduce work to the join irreducible elements our method is not sporadic (same for order, graph, tournament)
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Ramsey structures

Definition (Ramsey structure $\Delta$)

For all finite substructures $P$, $H$ of $\Delta$:

Whenever we color the copies of $P$ in $\Delta$ with 2 colors then there is a monochromatic copy of $H$ in $\Delta$.

Theorem (Nešetřil-Rödl)

The random ordered graph is Ramsey.
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Michael Pinsker
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Ramsey structures

Definition (Ramsey structure $\Delta$)
For all finite substructures $P, H$ of $\Delta$:
Whenever we color the copies of $P$ in $\Delta$ with 2 colors then there is a monochromatic copy of $H$ in $\Delta$.

Theorem (Nešetřil-Rödl)
The random ordered graph is Ramsey.
Canonical functions

Definition

Let $\Delta, \Lambda$ be structures. $f: \Delta \to \Lambda$ is canonical iff for all tuples $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ of the same type in $\Delta$ $(f(x_1), \ldots, f(x_n))$ and $(f(y_1), \ldots, f(y_n))$ have the same type in $\Lambda$.

Examples on $(D; <, E)$

- self-embeddings reversing $<$, preserving edges and non-edges
- preserving $<$, flipping edges and non-edges
- preserving $<$, send to clique
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Magical proposition (Bodirsky+MP+Tsankov '11)

Let \( \Delta \) is ordered Ramsey homogeneous finite language \( f: \Delta \to \Delta \) and \( c_1, \ldots, c_n \in \Delta \). Then the closed monoid generated by \( \{f\} \cup \text{Aut}(\Delta) \) contains a function \( g \) which is canonical as a function from \((\Delta, c_1, \ldots, c_n)\) to \( \Delta \) is identical with \( f \) on \( \{c_1, \ldots, c_n\} \).

Note: only finitely many different behaviors of canonical functions. \( g, g' \) same behavior \( \rightarrow \) generate one another (with \( \text{Aut}(\Delta) \)).
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Canonizing functions on Ramsey structures

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Note:
- only finitely many different behaviors of canonical functions.
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\begin{align*}
\text{Aut}(E, <) & \quad \text{(sw, turn)} \\
\text{Aut}(E, \text{sep}) & \quad \text{Aut}(E) \\
\text{Aut}(<) & \quad \text{Aut}(\text{sep}) \\
\text{Aut}(R(5)) & \quad \text{Aut}(R(5), <) \\
\text{Aut}(T) & \quad (\text{id, turn}) \\
\text{Aut}(E, <) & \quad (\text{id, lr}) \\
\text{Aut}(E, \text{sep}) & \quad (\text{sw, turn}) \\
\text{Aut}(E) & \quad (\text{sw, id}) \\
\text{Aut}(<) & \quad (\text{id, id}) \\
\end{align*}
Part IV: The Question to The Answer
The Question

Problem

Suppose that $\Delta_1$, $\Delta_2$ have finitely many reducts. Does their free superposition have finitely many reducts?

Problem

Suppose that $\Delta$ is homogeneous in a finite relational language. Does it have a finite homogeneous extension which is Ramsey?
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Suppose that $\Delta$ is homogeneous in a finite relational language. Does it have a finite homogeneous extension which is Ramsey?
“The Answer to the Great Question . . .
Of Life, the Universe and Everything . . . Is . . . Forty-two,”
said Deep Thought, with infinite majesty and calm.

Douglas Adams