Algebraic and model-theoretic methods in constraint satisfaction

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Outline

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Part I: CSPs

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Part II: pp definitions / polymorphism clones / ω-categoricity

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Part III: pp interpretations / topological clones

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Part V: Canonical functions / Ramsey theory

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Model theory, Universal algebra, Ramsey theory, Topological dynamics
$\rightarrow$ Theoretical computer science

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Part I: CSPs

Part II: pp definitions / polymorphism clones / ω-categoricity

Part III: pp interpretations / topological clones

Part IV: The tractability conjecture

Part V: Canonical functions / Ramsey theory

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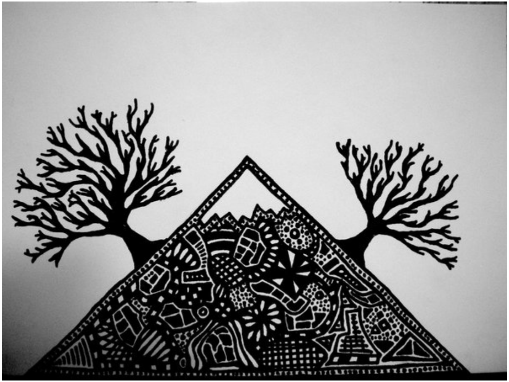
Building new dimension out of two smaller

Most statements in this presentation are imprecise / false.



Part I

CSPs

Constraint Satisfaction Problems (CSPs)

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Irrelevant whether Γ is finite or infinite.

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Computational problem: Graph-SAT(Ψ)

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- A finite set W of variables (vertices), and
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Computational complexity depends on Ψ. Always in NP.

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Γ_Ψ is a reduct of G, i.e.,
a structure with a first-order definition in G (without parameters).

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An instance

- $W = \{w_1, \ldots, w_m\}$
- $\phi_1, \ldots, \phi_n$

of Graph-SAT(Ψ) has a positive solution $\leftrightarrow$
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Graph-SAT problems $\leftrightarrow$ CSPs of reducts of the random graph.

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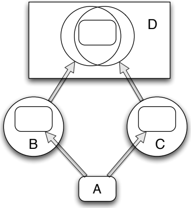

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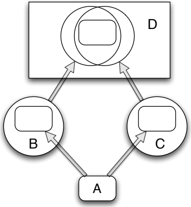

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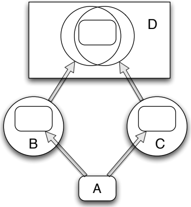

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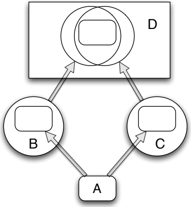

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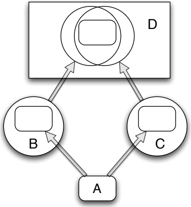

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Asks whether a given conjunction using $\psi_1, \ldots, \psi_n$
is satisfiable in some member of $\mathcal{C}$.

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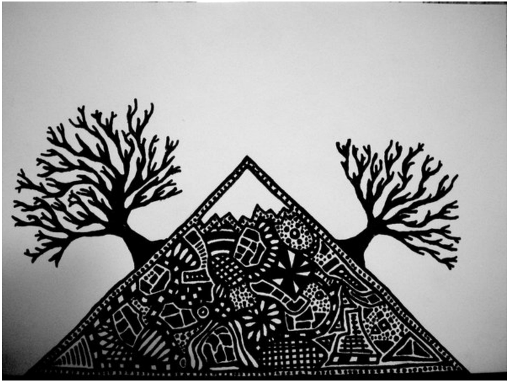
Then $\mathcal{C}$-SAT is always in P or NP-complete.



Part II

pp definitions / polymorphism clones / ω-categoricity

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Observation (Bulatov + Krokhin + Jeavons '00)

Expanding Γ by pp definable relations
increases the complexity of the CSP by at most polynomial-time.

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Observe: Pol(Γ) $\supseteq$ End(Γ) $\supseteq$ Aut(Γ).

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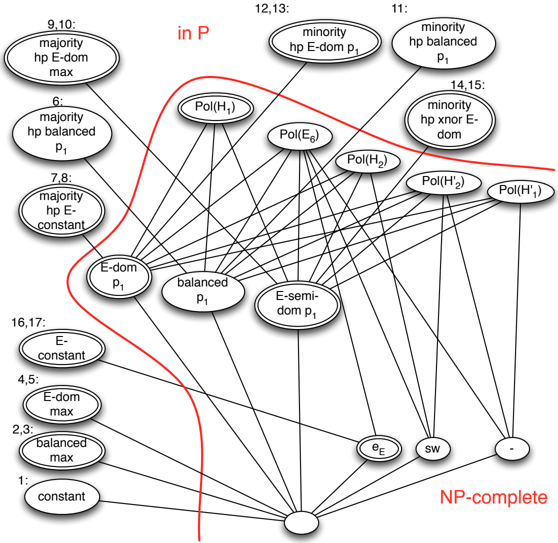
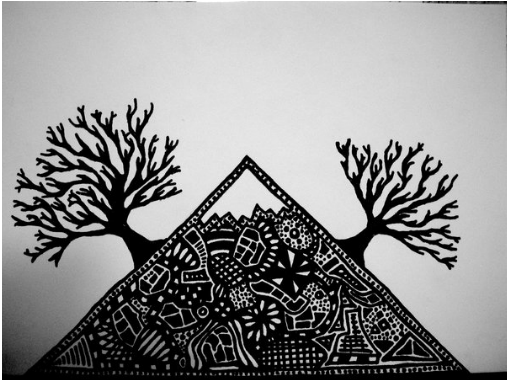
Corollary

Let Γ be ω-categorical or finite.

If Pol(Γ) $\subseteq$ Pol(Γ'),
then CSP(Γ') is polynomial-time reducible to CSP(Γ).

Graph-SAT classification





Part III

pp interpretations / topological clones

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Like topological groups / monoids: topological clones.

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Function clones carry natural structure:

- algebraic (composition / equations)
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Set of all finitary functions $\bigcup_n D^{D^n}\ldots$ sum space.

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Theorem (Bodirsky + MP '12)

Let Δ, Γ be ω-categorical or finite. TFAE:

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When Δ has a pp interpretation in Γ, then $\mathrm{CSP}(\Delta)$ is polynomial-time reducible to $\mathrm{CSP}(\Gamma)$.

Corollary

Let Δ, Γ be ω-categorical or finite. If $\mathrm{Pol}(\Delta) \cong \mathrm{Pol}(\Gamma)$, then $\mathrm{CSP}(\Delta)$ and $\mathrm{CSP}(\Gamma)$ are polynomial-time equivalent.

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Corollary

Let Γ be ω-categorical. TFAE:

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- there exists a continuous clone homomorphism $\xi\colon$ Pol(Γ) $\to$ **1**;
- all finite structures have a pp interpretation in Γ.

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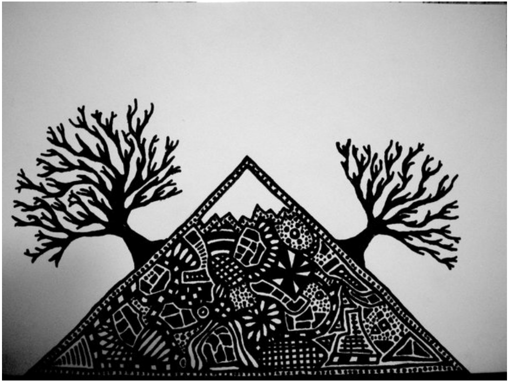
So the Betweenness problem is NP-hard.



Part IV

The tractability conjecture

The finite tractability conjecture

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Conjecture

(Feder + Vardi ’93; Bulatov + Jeavons + Krokhin ’05; Barto + Kozik ’10)

Let Γ be finite. Then:

- either $\mathrm{Pol}(\Gamma)$ has a homomorphism to $\mathbf{1}$ (and $\mathrm{CSP}(\Gamma)$ is NP-complete),
- or $\mathrm{Pol}(\Gamma)$ contains a cyclic operation f of arity $n > 1$, i.e.,

$$f(x_1, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$$

and $\mathrm{CSP}(\Gamma)$ *is in P*.

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Theorem (Bodirsky + Kara '08, reformulated)

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- either $\mathrm{Pol}(\Gamma)$ has a continuous homomorphism to **1** (and $\mathrm{CSP}(\Gamma)$ is NP-hard);
- or there are $f(x_1, x_2) \in \mathrm{Pol}(\Gamma)$ and $\alpha, \beta \in \mathrm{Aut}(\mathbb{Q}; <)$ such that

$$f(x_1, x_2) = \alpha(f(\beta x_2, \beta x_1))$$

  and $\mathrm{CSP}(\Gamma)$ is in P.

Reducts of the random graph

Theorem (Bodirsky + MP '11, reformulated)

Let Γ be a reduct of the random graph. Then:

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Infinite tractability conjecture

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Fact: There are homogeneous digraphs with undecidable CSP.

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Examples: $(\mathbb{Q};<)$ and the random graph.

Fact: The CSP of any reduct of a finitely bounded structure is in NP.

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Let Γ be a reduct of a finitely bounded homogeneous structure.

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- or $\mathrm{Pol}(\Gamma)$ satisfies a non-trivial equation, and $\mathrm{CSP}(\Gamma)$ is in P.

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- Identify relations R such that $\mathrm{Pol}(V; R)$ has a continuous homomorphism to $\mathbf{1}$ (hard relations).

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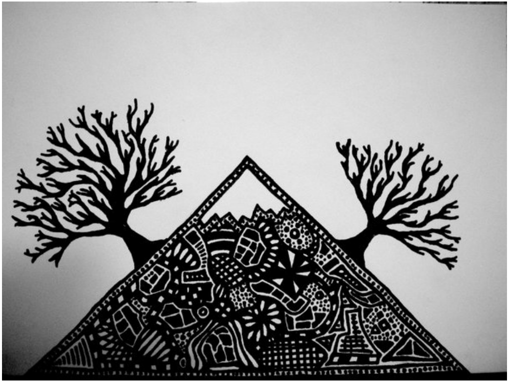
  So they allow for combinatorial analysis and algorithmic use, and "should" satisfy equations.



Part V

Canonical functions / Ramsey theory

Canonical functions

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Definition

Let Δ be a structure.

$f : \Delta^n \to \Delta$ is canonical iff
for all tuples $t_1, \ldots, t_n$ of the same length
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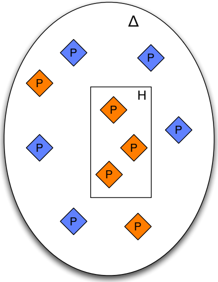

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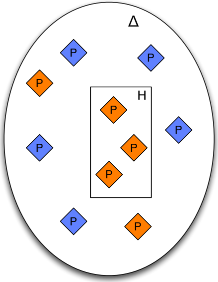

Theorem (Nešetřil + Rödl)

The random ordered graph is Ramsey.

Canonizing functions on Ramsey structures

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Proposition (Bodirsky + MP + Tsankov '11)

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Using canonical functions

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Conclusion: Violation of hard relations
(and thus non-existence of a continuous homomorphism to **1**)
is witnessed by canonical functions.

Complexity of CSP for reducts of the random graph

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Let Γ be a reduct of the random graph. Then:

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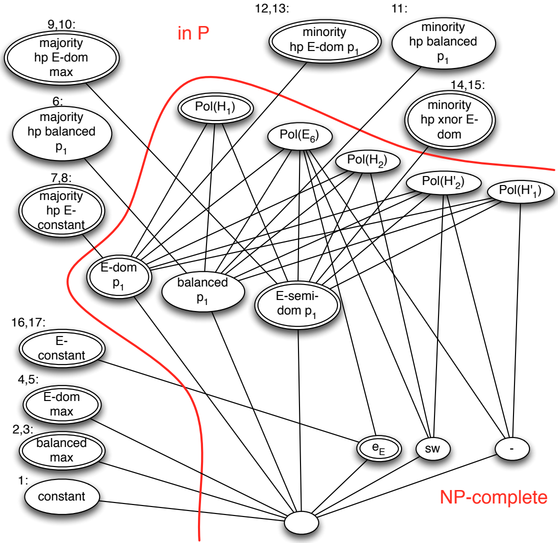
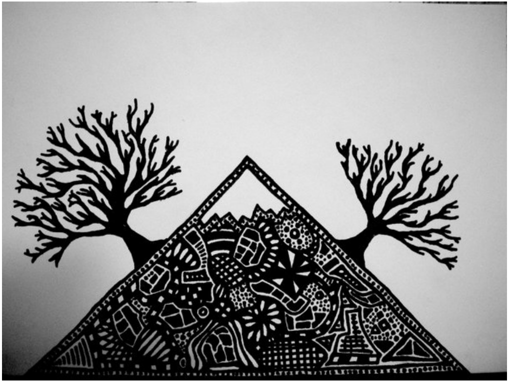
Graph-SAT classification





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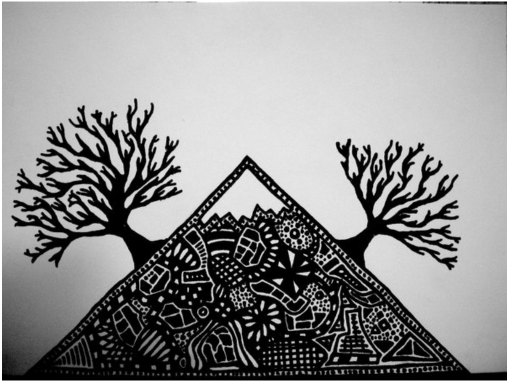
- Clarify relationship between canonical functions and their finite counterparts (algorithmic / equational).

- Let Δ be homogeneous in a finite relational language. Does $\mathrm{Aut}(\Delta)$ have finitely many closed supergroups?

  (S. Thomas, *Reducts of the random graph*)



Thank you