## Reconstructing structures from their abstract clones

#### Michael Pinsker

Technische Universität Wien / Université Diderot - Paris 7 Funded by FWF grant I836-N23

Special Session on Universal algebra and Constraint Satisfaction ASL 2014 North American Annual Meeting, Boulder

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- The topology of algebras

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- The topology of algebras
- Reconstruction the topology of function clones



Part I

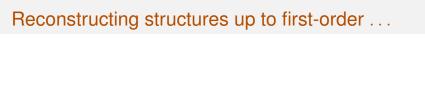
Reconstructing structures from their automorphism groups and polymorphism clones







countable,  $\omega$ -categorical





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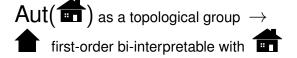
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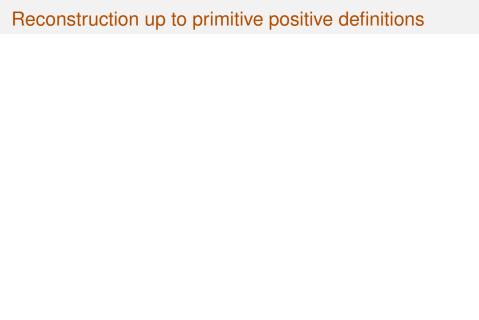
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**Observe:**  $Pol(\Delta) \supseteq End(\Delta) \supseteq Aut(\Delta)$ .



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#### Definition (Constraint Satisfaction Problem)

 $\mathsf{CSP}(\Delta)$  is the computational problem to decide whether a given primitive positive  $\tau$ -sentence holds in  $\Delta$ .

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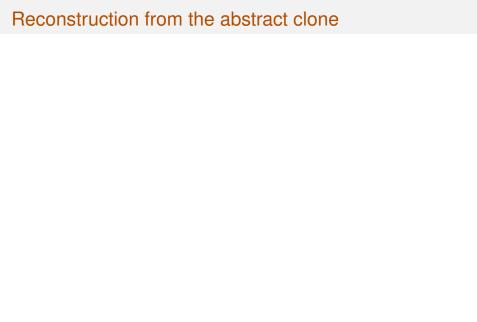
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Part II

The topology of algebras

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Structural conclusions about  $\mathfrak A$  from abstract clone  $Clo(\mathfrak A)$ : Varieties.

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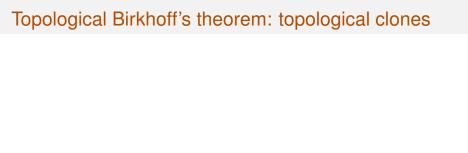
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#### Theorem (Birkhoff 1935)

Let  $\mathfrak{A}$ ,  $\mathfrak{B}$  be finite.

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Part III
Reconstructing the topology

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**Fact.** For groups (3)  $\implies$  (2).

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- $\blacksquare$  (N; =) (Dixon+Neumann+Thomas'86)
- $\blacksquare$  ( $\mathbb{Q}$ ; <) and the atomless Boolean algebra (Truss'89)
- the random graph (Hodges+Hodkinson+Lascar+Shelah'93)
- the random  $K_n$ -free graphs (Herwig'98)

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- the random k-hypergraphs the Henson digraphs (Barbina+MacPherson '07).



Part IV
Negative results

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Important in constraint satisfaction:

"main reason" for NP-hardness of the CSP of a structure.

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- Involves non-principal ultrafilter: unfair in the CSP context.
- Also has a continuous homomorphism to 1.

# Automatic homeomorphicity + reconstruction

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Theorem (Bodirsky + MP + Pongrácz '13)

There exists a closed oligomorphic clone **C** and  $\xi \colon \mathbf{C} \to \mathbf{C}$  such that:

- $\blacksquare$   $\xi$  is an isomorphism;
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#### Theorem (Evans + Hewitt '90)

There exists an closed oligomorphic group which does not have reconstruction.



Part V
Positive results



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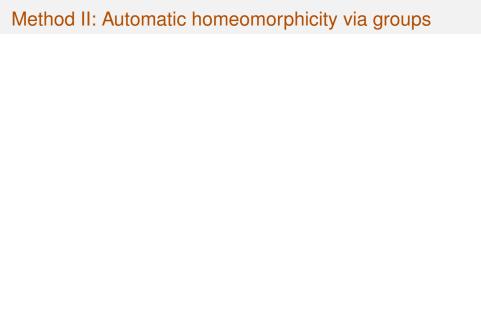
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#### Theorem (Bodirsky + MP + Pongrácz '13)

Any closed subclone of  ${\bf O}$  containing  $\omega^\omega$  has automatic continuity and automatic homeomorphicity.



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#### Show:

- the closure of  $G_C$  in O has reconstruction;
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### Theorem (Bodirsky + MP + Pongrácz '13)

Let G be the random graph.

The following have automatic homeomorphicity:

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### Theorem (Bodirsky + MP + Pongrácz '13)

Let *G* be the random graph.

The following have automatic homeomorphicity:

■ End(G);

Show:

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### Theorem (Bodirsky + MP + Pongrácz '13)

Let G be the random graph.

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Show:

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### Theorem (Bodirsky + MP + Pongrácz '13)

Let G be the random graph.

The following have automatic homeomorphicity:

■ End(G);

Show:

- Pol(*G*);
- All minimal tractable clones containing Aut(G).



**Part VI**Open problems

Which oligomorphic closed subclones of O have automatic homeomorphicity?

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- In ZF?
- Which topological clones are closed subclones of O?









Thank you!