Constraint Satisfaction over Infinite Structures

Michael Pinsker

Technische Universität Wien / Université Diderot - Paris 7

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Γ is called the template of the CSP.

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Theorem (Bodirsky + MP '10)

Graph-SAT problems are always in P or NP-complete.

Graph-SAT: Examples

Example. Let Γ have precisely the relation

$$\begin{aligned} R(x,y,z) :\leftrightarrow & (E(x,y) \wedge \neg E(y,z) \wedge \neg E(x,z)) \\ & \vee (\neg E(x,y) \wedge E(y,z) \wedge \neg E(x,z)) \\ & \vee (\neg E(x,y) \wedge \neg E(y,z) \wedge E(x,z)) \, . \end{aligned}$$

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Note: This type of CSP cannot be modeled by finite templates.

Polymorphism clones

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Observation

Expanding Γ by pp definable relations
increases the complexity of the CSP only by polynomial-time.

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Observe: Pol(Γ) $\supseteq$ End(Γ) $\supseteq$ Aut(Γ).

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Corollary

Let Γ be ω-categorical.

If $\mathrm{Pol}(\Gamma) = \mathrm{Pol}(\Gamma')$,
then CSP(Γ) and CSP(Γ') are polynomial-time equivalent.

Topological clones

Finite structures and abstract clones

Theorem (Bulatov + Jeavons + Krokhin '05)

Let Γ be finite. The complexity of CSP(Γ) only depends on the algebraic structure of Pol(Γ), i.e., on Pol(Γ) viewed as an abstract clone.

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Reason: combination of

- Birkhoff's HSP^{fin} theorem
- the fact that when Γ' has a pp interpretation in Γ,
  then CSP(Γ') is polynomial-time reducible to CSP(Γ).

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For countable ω-categorical Γ, the complexity of $\mathrm{CSP}(\Gamma)$ only depends on $\mathrm{Pol}(\Gamma)$ viewed as an topological clone.

A hardness criterion

Let **1** be the clone of projections on any set of size ≥ 2.

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Corollary

Let Γ be ω-categorical.

If Pol(Γ) has a continuous homomorphism to **1**,
then CSP(Γ) is NP-hard.

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This is a property of the topological clone $\mathrm{Pol}(\Gamma)$.

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We are interested in topological clones
whose unary invertible elements are dense in the unary sort.

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Conjecture (Bulatov + Jeavons + Krokhin '05; Barto + Kozik '10)

Let Γ be finite, and $\mathrm{Pol}(\Gamma)$ be idempotent. Then

- either $\mathrm{Pol}(\Gamma)$ has a homomorphism to $\mathbf{1}$ (and $\mathrm{CSP}(\Gamma)$ is NP-hard),
- or $\mathrm{Pol}(\Gamma)$ contains a cyclic operation f of arity $n > 1$, i.e.,

$$f(x_1, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$$

and $\mathrm{CSP}(\Gamma)$ *is in P*.

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Theorem (Bodirsky + Kara '08, reformulated)

Let Γ be a reduct of $(\mathbb{Q}; <)$. Then one of the following holds.

- there is an expansion Γ' of the model-complete core of Γ by finitely many constants such that $\mathrm{Pol}(\Gamma')$ has a continuous homomorphism to $\mathbf{1}$ (and $\mathrm{CSP}(\Gamma)$ is NP-hard);

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- Using Ramsey theory we find canonical (='nice') such polymorphisms.
- These canonical polymorphisms are essentially finite functions.

  So they allow for combinatorial analysis and algorithmic use, and "should" satisfy equations.

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Let Δ be a structure.

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Examples on the random graph

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- binary edge-max or edge-min.

Ramsey structures

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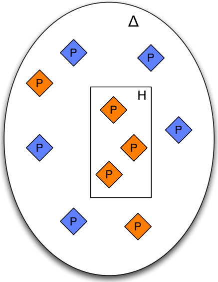

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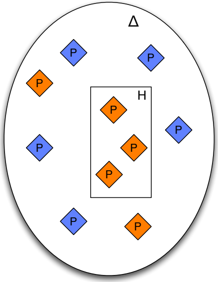

Theorem (Nešetřil-Rödl)

The random ordered graph is Ramsey.

Canonizing functions on Ramsey structures

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Proof: Via topological dynamics (Kechris + Pestov + Todorcevic '05).

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Canonical functions of same behavior generate one another.

Conclusion: Violation of hard relations
(and thus non-existence of a continuous homomorphism to $\mathbf{1}$)
is witnessed by canonical functions.

The infinite tractability conjecture

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Conjecture (Bodirsky + MP '11)

Let Γ be a reduct of a finitely bounded homogeneous structure.

Then CSP(Γ) is in P or NP-complete.

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One of the following holds.

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- $\mathrm{Pol}(\Gamma)$ satisfies a non-trivial equation, and $\mathrm{CSP}(\Gamma)$ is tractable.

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- If $\mathrm{Pol}(\Gamma)$ has a homomorphism to $\mathbf{1}$, does it also have a continuous homomorphism?