Algebraic and model-theoretic methods in constraint satisfaction

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Doc-Course, Charles University Prague

2014

Outline

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Part I: CSPs / dividing the world /
pp definitions, polymorphism clones, ω-categoricity

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Model theory, Universal algebra, Ramsey theory, Topological dynamics
$\rightarrow$ Theoretical computer science

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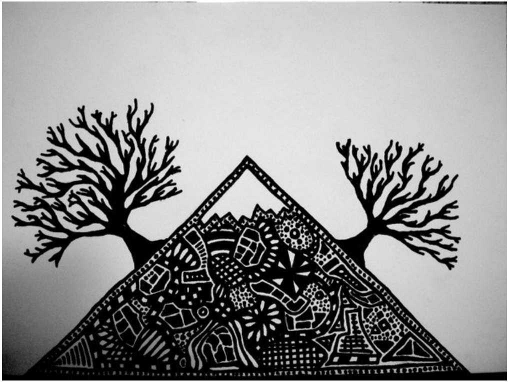
Building new dimension out of two smaller

Most statements in this presentation are imprecise / false.



Part I:

CSPs / pp definitions / polymorphism clones / ω-categoricity

Constraint Satisfaction Problems (CSPs)

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Irrelevant whether Γ is finite or infinite. But language finite.

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HOM(Γ) and CSP(Γ) are equivalent.

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Question: Is $(D; E)$ acyclic?

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Question: Is there a linear order on the variables such that for each triple (x, y, z) either $x < y < z$ or $z < y < x$?

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Input: A finite system of equations using $=, +, \cdot, 1$

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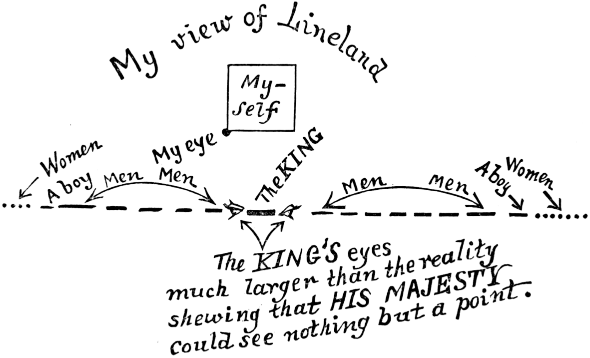
Is a CSP: template K_n



Dividing the world

The Graph Satisfiability Problem

Let E be a binary relation symbol.

(Imagine: edge relation of an undirected graph.)

Let Ψ be a finite set of quantifier-free $\{E\}$-formulas.

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INPUT:

- A finite set W of variables (vertices), and
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Graph-SAT(Ψ_2) is in P.

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Γ_Ψ is a *reduct of* the random graph, i.e.,
a structure with a first-order definition in G.

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An instance

- $W = \{w_1, \ldots, w_m\}$
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Graph-SAT problems $\leftrightarrow$ CSPs of reducts of the random graph.

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Boolean-SAT: Example

$\Gamma = (\{0,1\}; \{(1,0,0),(0,1,0),(0,0,1)\})$

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Given Ψ, we can decide in which class the problem falls.

- **Boolean-SAT:** Schaefer ('78)
- **Temp-SAT:** Bodirsky+Kára ('07)
- **Graph-SAT:** Bodirsky+MP ('10) (*Schaefer's theorem for graphs*)

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Graph-SAT(Ψ): Is there a finite graph such that... (constraints)

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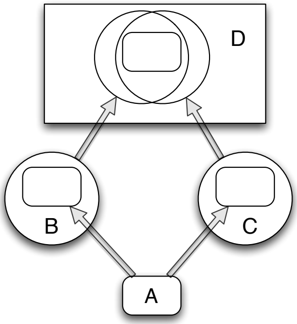
The classes of finite graphs and linear orders are *amalgamation classes*.



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Homogeneous digraphs classified by Cherlin.

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Note: This type of CSP cannot be modeled by finite templates.

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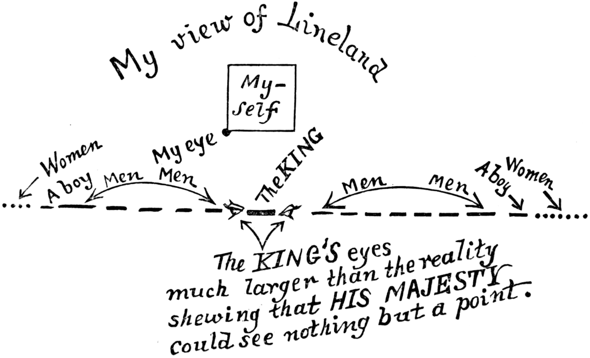

pp definitions, polymorphism clones, ω-categoricity

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Observation (Bulatov+Krokhin+Jeavons '00)

Expanding Γ by pp definable relations
increases the complexity of the CSP by at most polynomial-time.

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Observe: $\mathrm{Pol}(\Gamma) \supseteq \mathrm{End}(\Gamma) \supseteq \mathrm{Aut}(\Gamma)$.

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Let Γ be a countable ω-categorical structure.

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Corollary

Let Γ be ω-categorical.

If Pol(Γ) $\subseteq$ Pol(Γ'),
then CSP(Γ') is polynomial-time reducible to CSP(Γ).

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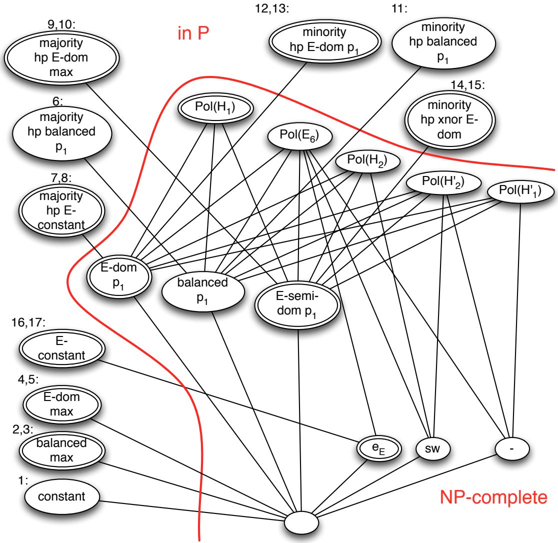
- $\mathrm{Aut}(\Gamma)$ *is oligomorphic;*
- Γ *is ω-categorical: the only countable model of its theory.*

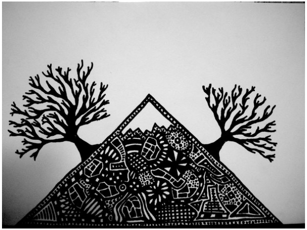
Proof of the preservation theorem

Blackboard

Graph-SAT classification



Until the moment when I placed my mouth in his World,
he had not heard anything
except confused sounds beating against –
what I called his side,
but what he called his INSIDE or STOMACH.



Part II: November 3rd