Permutations on the random permutation

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A relational structure $\Delta$ is homogeneous iff every isomorphism between finite substructures of $\Delta$ extends to an automorphism of $\Delta$.

**Examples**

- $(\mathbb{Q}; <)$
- random graph
- random poset
Homogeneous structures

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Fraïssé’s Theorem

Theorem (Fraïssé)

Let $C$ be a class of finite relational structures which
▶ is closed under isomorphism
▶ is closed under taking induced substructures
▶ has countably many members up to isomorphism
▶ has the amalgamation property: for all $A, B, C \in C$ and embeddings $f: A \to B$, $g: A \to C$ there exist $D \in C$ and embeddings $f': B \to D$, $g': C \to D$ such that $f' \circ f = g' \circ g$.

Then there exists a unique (up to isomorphism) countable homogeneous structure $\Delta$ whose age is $C$.

Such a structure is called the Fraïssé limit of $C$. 

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Reducts

| Preliminaries | The random permutation | Ramsey structures and canonical functions | Constraint satisfaction problems |

**Definition**
A reduct of a relational structure $\mathcal{R}$ is a structure on the same domain whose relations are first-order definable in $\mathcal{R}$ without parameters.

**Example:** reducts of $(\mathbb{Q}; <)$
- $(\mathbb{Q}; =)$
- $(\mathbb{Q}; \text{Btw})$
- $(\mathbb{Q}; \text{Cyc})$
- $(\mathbb{Q}; \text{Sep})$

**Problem**
Classify the reducts of a homogeneous structure up to first-order interdefinability, existential-positive interdefinability, etc.
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- Conjecture (Simon Thomas, 1991): If $\Delta$ is a countable relational structure which is homogeneous in a finite language, then $\Delta$ has only finitely many reducts, up to first-order interdefinability.
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- classifying computational complexity of constraint satisfaction problems
Closed groups
A permutation group $G \leq \text{Sym}(X)$ is **closed** iff $h \in G$ whenever for all finite $A \subseteq X$ there exists $g \in G$ which agrees with $h$ on $A$. 

Theorem (Corollary of Ryll-Nardzewski, Engeler, Svenonius)

If $\Delta$ is homogeneous in a finite relational language, then

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\{\text{reducts of } \Delta\} / \sim \rightarrow \{\text{closed supergroups of } \text{Aut}(\Delta)\}
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is an antiisomorphism.
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Examples

Let $\rightarrow$ be a permutation of $\mathbb{Q}$ which reverses $<$. Let $\leftrightarrow$ be a permutation of $\mathbb{Q}$ which reverses $<$ between $(-\infty, \pi)$ and $(\pi, \infty)$, for some irrational $\pi$, and preserves $<$ otherwise.

Then $\overset{\rightarrow}{\text{Aut}}(\mathbb{Q}; \text{Btw}) = \langle \text{Aut}(\mathbb{Q}; <) \cup \{\rightarrow\} \rangle$

$\overset{\rightarrow}{\text{Aut}}(\mathbb{Q}; \text{Cyc}) = \langle \text{Aut}(\mathbb{Q}; <) \cup \{\leftrightarrow\} \rangle$

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Any permutation on a finite set $A$ may be regarded as

- a bijection $A \rightarrow A$
- a relational structure $(A; <_1, <_2)$
The random permutation

Definition

The random permutation, \(\Pi = (D; \leq_1, \leq_2)\), is the Fraïssé limit of the class of all finite permutations.

Equivalently, \(\Pi\) is the unique (up to isomorphism) countable structure with two linear orders which is homogeneous and contains all finite permutations. \(\Pi\) appears with probability 1 in the random process that constructs both orders independently.

Question (Cameron, 2002)

What are the closed supergroups of \(\text{Aut}(\Pi)\)?
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Then $(D; <_1, <_2) \cong \Pi$. 

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**Preliminaries**

- The random permutation
- Ramsey structures and canonical functions
- Constraint satisfaction problems
The closed supergroups of Aut(Π)
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Theorem (Linman and Pinsker, 2014)
There are precisely 39 closed supergroups of Aut(Π).
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Each closed supergroup either contains Aut(D; <ι) for some \( i \in \{1, 2\} \), or is generated by permutations which are compositions of the following:
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- sw: switches the orders <_1 and <_2
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- $(\text{id}_t)$
- $(\text{id})$
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 Ramsey structures and canonical functions
 Constraint satisfaction problems

Aut(\(\Pi\))
\langle (\text{revrev}) \rangle
\langle \text{sw} \rangle
\langle (\text{idrev}) \rangle
\langle (\text{idt}) \rangle
\langle \text{sw} \circ (\text{idrev}) \rangle
\langle (\text{revid}) \rangle
\langle (\text{t} \circ \text{id}) \rangle
\langle (\text{id}) \rangle

Aut(D; \langle 1 \rangle)

Aut(D; \langle 2 \rangle)

Sym(D)
Asymmetry in the roles of \((\text{id}_{\text{rev}})\) and \((\text{id}_t)\)

While \(\leftrightarrow\) and \(\circlearrowright\) appear to play symmetric roles as generators of closed supergroups of \(\text{Aut}(\mathbb{Q}; <)\), the corresponding permutations \((\text{id}_{\text{rev}})\) and \((\text{id}_t)\) of \(D\) do not.
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There is a group consisting of all permutations which either preserve or reverse both orders simultaneously, but no corresponding simultaneous action of turns:
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While \(\leftrightarrow\) and \(\circ\) appear to play symmetric roles as generators of closed supergroups of \(\text{Aut}(\mathbb{Q}; <)\), the corresponding permutations \((\text{id}_{\text{rev}})\) and \((\text{id}_{t})\) of \(D\) do not.

There is a group consisting of all permutations which either preserve or reverse both orders simultaneously, but no corresponding simultaneous action of turns:

\[
\langle (\text{rev}) \rangle = \langle (\text{id}) \circ (\text{rev}) \rangle \subsetneq \langle (\text{id}) , (\text{rev}) \rangle
\]

\[
\langle (\text{id}_{t}) \circ (\text{id}) \rangle = \langle (\text{id}_{t}) , (\text{id}) \rangle
\]
Closed transformation monoids
A first-order formula is called **existential-positive** iff it is of the form

$$\exists x_1, \ldots, x_n \psi_1 \land \cdots \land \psi_m,$$

where each $$\psi_i$$ is a disjunction of atomic formulas.
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Theorem (Bodirsky and Pinsker, 2012)
If $\Delta$ is countable and $\omega$-categorical, then

$$\{\text{reducts of } \Delta\}/\sim \rightarrow \{\text{closed monoids containing } \text{Aut}(\Delta)\}$$

$$\Gamma/\sim \mapsto \text{End}(\Gamma)$$

is an antiisomorphism.
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Let $\mathcal{M}$ be a closed transformation monoid containing $\text{Aut}(\Pi)$. Then one of the following holds.

- $\mathcal{M}$ has a constant operation.
- The permutations in $\mathcal{M}$ form a group which is a dense subset of $\mathcal{M}$ in $D^D$.  

In other words, if $\Gamma$ is a reduct of $\Pi$, either $\Gamma$ has a constant endomorphism or all endomorphisms of $\Gamma$ can be interpolated on finite sets by automorphisms of $\Gamma$. 
Closed transformation monoids containing Aut(Π)

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In other words, if \( Γ \) is a reduct of \( Π \), either \( Γ \) has a constant endomorphism or all endomorphisms of \( Γ \) can be interpolated on finite sets by automorphisms of \( Γ \).
Definition
A structure is model-complete iff every embedding between models of its theory preserves all first-order formulas.

Lemma (Bodirsky and Pinsker, 2012)
A countable $\omega$-categorical structure $\Delta$ is model-complete iff $\text{Aut}(\Delta)$ is dense in $\text{Emb}(\Delta)$.

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All reducts of $\Pi$ are model-complete.
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All reducts of $\Pi$ are model-complete.
Ramsey structures

A structure $\Delta$ is a Ramsey structure iff for all finite $P$, $H \subseteq \Delta$ and all colorings of the copies of $P$ in $\Delta$ with finitely many colors, there is a copy of $H$ in $\Delta$ on which the coloring is constant.

Theorem (Böttcher and Foniok, 2011)
The random permutation is a Ramsey structure.
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Let \( a \) be an \( n \)-tuple of elements in a structure \( \Delta \). The type of \( a \) in \( \Delta \) is the set of first-order formulas with free variables \( x_1, \ldots, x_n \) that hold for \( a \) in \( \Delta \).

Definition
Let \( \Delta, \Gamma \) be structures. A function \( f : \Delta \rightarrow \Gamma \) is canonical iff it sends \( n \)-tuples of the same type in \( \Delta \) to \( n \)-tuples of the same type in \( \Gamma \).

Examples
▶ embeddings
▶ constant functions
▶ \((\text{id}, \text{rev})\) and \((\text{id}, \text{sw})\) are canonical from \( \Pi \) to \( \Pi \)
▶ \((\text{id}, \text{t})\) is canonical from \( (\Pi, c) \) to \( \Pi \)
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Canonical functions

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Let $a$ be an $n$-tuple of elements in a structure $\Delta$. The type of $a$ in $\Delta$ is the set of first-order formulas with free variables $x_1, \ldots, x_n$ that hold for $a$ in $\Delta$.

Definition
Let $\Delta, \Gamma$ be structures. A function $f : \Delta \to \Gamma$ is canonical iff it sends $n$-tuples of the same type in $\Delta$ to $n$-tuples of the same type in $\Gamma$.

Examples
- embeddings
- constant functions
- $(id_{rev})$ and $sw$ are canonical from $\Pi$ to $\Pi$
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- $(\text{id}_{\text{rev}})$ and $\text{sw}$ are canonical from $\Pi$ to $\Pi$
- $(\text{id}_t)$ is canonical from $(\Pi, c)$ to $\Pi$
We say that \( F \subseteq D \) generates a function \( g : D \rightarrow D \) iff for all finite \( A \subseteq D \) there exist \( f_1, \ldots, f_n \in F \) such that \( f_1 \circ \cdots \circ f_n \) agrees with \( g \) on \( A \).

Theorem (Bodirsky, Pinsker, Tsankov, 2011)

Let \( \Delta \) be a structure which is ordered Ramsey and homogeneous in a finite relational language. Let \( c_1, \ldots, c_n \in \Delta \) and \( f : \Delta \rightarrow \Delta \) be a function. Then \( \{f\} \cup \text{Aut}(\Delta) \) generates a function which is canonical as a function \((\Delta, c_1, \ldots, c_n) \rightarrow \Delta\) agrees with \( f \) on \( \{c_1, \ldots, c_n\} \).
We say that $\mathcal{F} \subseteq D^D$ generates a function $g : D \to D$ iff for all finite $A \subseteq D$ there exist $f_1, \ldots, f_n \in \mathcal{F}$ such that $f_1 \circ \cdots \circ f_n$ agrees with $g$ on $A$. 
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- is canonical as a function $(\Delta, c_1, \ldots, c_n) \to \Delta$
- agrees with $f$ on $\{c_1, \ldots, c_n\}$
**Clones**

- **Definition**
  - Let $A$ be a set. A clone on $A$ is a set of finitary operations on $A$ which is closed under composition and contains all projections.

- **Examples**
  - The projection clone
  - The polymorphism clone of a structure $\Delta$: the set of homomorphisms $\Delta_n \to \Delta$, for all $n \geq 1$. 

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Closed clones

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A first-order formula is called primitive-positive iff it is of the form

\[ \exists x_1, \ldots, x_n \psi_1 \land \cdots \land \psi_m, \]

where each \( \psi_i \) is an atomic formula.

Theorem (Bodirsky and Nešetřil, 2006)

If \( \Delta \) is countable and \( \omega \)-categorical, then

\[ \{ \text{reducts of } \Delta \} / \sim \mapsto \{ \text{closed clones containing } \text{Aut}(\Delta) \} \]

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Constraint satisfaction problems

Definition

Let $\Gamma$ be a structure in a finite relational language $\tau$. $\text{CSP}(\Gamma)$ is the computational problem of deciding whether a given primitive-positive $\tau$-sentence holds in $\Gamma$.

Theorem (Bulatov, Krokhin, Jeavons, 2000)

Let $\Gamma = (D; R_1, \ldots, R_n)$ be a structure and let $R$ be a relation with a primitive-positive definition in $\Gamma$. Then $\text{CSP}(D; R_1, \ldots, R_n)$ and $\text{CSP}(D; R_1, \ldots, R_n, R)$ are polynomial-time equivalent. Therefore, the complexity of $\text{CSP}(\Gamma)$ depends only on $\text{Pol}(\Gamma)$.

Problem

Classify the computational complexity of $\text{CSP}(\Gamma)$ for all reducts $\Gamma$ of $\Pi$.
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Classify the computational complexity of CSP($\Gamma$) for all reducts $\Gamma$ of $\Pi$. 
## Related problems

- Can these results be extended to structures with $n$ linear orders, for $n \geq 3$?
- Does Thomas's conjecture hold for Ramsey structures?
- Does every structure which is homogeneous in a finite relational language have a homogeneous Ramsey expansion?
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Thank you!