Topological clones

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Outline

I: Algebras, function clones, abstract clones, Birkhoff's theorem

II: Topological clones, Topological Birkhoff

Commercial break: Reconstruction

III: Logic: pp interpretations, Constraint Satisfaction Problems

IV: Topological clones revisited

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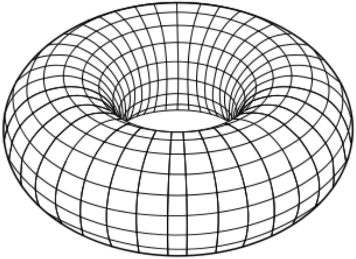


I: Abstract clones

Algebras, function clones

Let $\mathfrak{A} = (A; (f_i)_{i\in\tau})$ be an algebra with signature τ.

Every abstract τ-term t induces a term function $t^{\mathfrak{A}}$ on A.

Clo($\mathfrak{A}$) ("clone of $\mathfrak{A}$") is the set of its term functions.

Clo($\mathfrak{A}$) is a function clone:

- closed under composition: $f(g_1(x_1,\ldots,x_m),\ldots,g_n(x_1,\ldots,x_m))$;
- contains projections $\pi_i^n(x_1,\ldots,x_n) = x_i$.

Many properties of an algebra depend only on its function clone:

e.g. homomorphic images, subalgebras.

Here: algebras up to "clone equivalence".

Abstract clones, clone homomorphisms

Function clones carry algebraic structure via equations.

Can model this structure via multi-sorted algebra:

- one sort for each arity;
- composition functions;
- projections are distinguished elements (constants).

Let $\mathcal{C}, \mathcal{D}$ be function clones. $\xi \colon \mathcal{C} \to \mathcal{D}$ clone homomorphism if

- preserves arities;
- sends each projection π_i^n in $\mathcal{C}$ to same projection in $\mathcal{D}$;
- commutes with composition:
  $\xi(f(g_1, \ldots, g_n)) = \xi(f)(\xi(g_1), \ldots, \xi(g_n))$.

We write $\mathcal{C} \to \mathcal{D}$ if there exists a clone homomorphism from $\mathcal{C}$ to $\mathcal{D}$.

Birkhoff's theorem

For an algebra $\mathfrak{A}$:

- $\mathrm{H}(\mathfrak{A})\ldots$ homomorphic images / factor algebras of $\mathfrak{A}$
- $\mathrm{S}(\mathfrak{A})\ldots$ subalgebras of $\mathfrak{A}$
- $\mathrm{P}^{\mathrm{fin}}(\mathfrak{A})\ldots$ finite powers of $\mathfrak{A}$
- $\mathrm{P}(\mathfrak{A})\ldots$ powers of $\mathfrak{A}$

Similarly for function clone $\mathcal{C}$: it acts on congruence classes, invariant subsets, powers of its domain. Write $\mathrm{H}(\mathcal{C}), \mathrm{S}(\mathcal{C}), \mathrm{P}(\mathcal{C})$.

Theorem (Birkhoff 1935)

Let $\mathcal{C}, \mathcal{D}$ be function clones. TFAE:

- $\mathcal{D} \in \mathrm{HSP}(\mathcal{C})$;
- $\mathcal{D}$ *can be obtained from* $\mathcal{C}$ *applying* H, S, P;
- $\mathcal{C} \to \mathcal{D}$ *surjectively.*

Birkhoff II: Finite powers

Theorem (Birkhoff 1935)

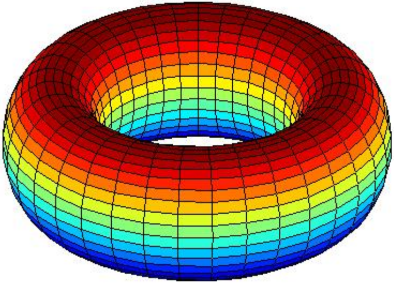
Let $\mathcal{C}, \mathcal{D}$ be function clones on a finite domain. TFAE:

- $\mathcal{D} \in \mathrm{HSP}^{\mathrm{fin}}(\mathcal{C})$;
- $\mathcal{D}$ *can be obtained from* $\mathcal{C}$ *applying* H, S, $\mathrm{P}^{\mathrm{fin}}$;
- $\mathcal{C} \to \mathcal{D}$ *surjectively.*

What about $\mathrm{HSP}^{\mathrm{fin}}$ of infinite function clones?

Analogy with groups and monoids

Permutation group	Abstract group
Transformation monoid	Abstract monoid
Function clone	Abstract clone



II: Topological clones

Pointwise convergence

Functions clones carry also topological structure:

Pointwise convergence on functions $f\colon D^n \to D$.

$(f_i)_{i\in\omega}$ converges to f iff $f(\overline{c}) = f_i(\overline{c})$ eventually, for every $\overline{c} \in D^n$.

Equivalently: $D\ldots$ discrete; D^{D^n} product topology.

Set of all finitary functions $\bigcup_n D^{D^n}\ldots$ sum space.

Function clones subspace.

If D countable: $\bigcup_n D^{D^n}$ is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$.

Complete metric separable (=Polish) space.

For finite function clones: topology discrete.

Topological clones

Definition

A topological clone is an abstract clone + topology such that composition is continuous.

Structure preserving mappings: continuous clone homomorphisms.

Permutation group	Topological group	Abstract group
Transformation monoid	Topological monoid	Abstract monoid
Function clone	Topological clone	Abstract clone

Theorem (Variant of "Topological Birkhoff", Bodirsky + MP 2011)

Let $\mathcal{C}, \mathcal{D}$ be function clones on an at most countable domain, where $\mathcal{D}$ is finitely generated. TFAE:

- $\mathcal{D} \in \mathsf{HSP}^{\mathsf{fin}}(\mathcal{C})$;
- $\mathcal{C} \to \mathcal{D}$ *surjectively + uniformly continuously.*

Commercial break: Reconstruction

For many closed function clones $\mathcal{C}$,
the algebraic structure determines the topological structure!

Every isomorphism between $\mathcal{C}$ and another closed function clone $\mathcal{D}$ is a homeomorphism ("reconstruction").

Example: automorphism group, polymorphism clone of the random graph.

Impossible to construct non-continuous homomorphism between closed permutation groups on ω in ZF+DC (need full AC).

Michael Kompatscher, Wednesday 10:30, in ZFC:

Two closed function clones which are isomorphic, but not topologically. (Bodirsky + Evans + Kompatscher + MP 2015)

C			4		3		2	8			9				B
7						A				6			4		
	E		8	D				F		5	2		C	7	
			0		7				B		D		6		E
4				9							E		1		
	6		2							0		5			3
	0	B	1	4		2			9				E		
	9	5			A	B	C	6			7				
	C		B		6		F	A	2		5			0	4
A		2			5	D	0			C	8	3	B		1
		0	F	B								D		2	
5			3		8				1		0	9	F		
3	8			5		6	E	0		F				9	
		C		F		1						B		E	
0							8				6	7			D
		4		A	D		7		E		C	2			5

III: pp interpretations, Constraint Satisfaction Problems

Polymorphism clones

Let $\Gamma = (A; (R_i)_{i \in \tau})$ be a relational structure.

Pol(Γ)...set of all homomorphisms $f\colon \Gamma^n \to \Gamma$, where $1 \leq n < \omega$.

So $f(x_1, \ldots, x_n) \in \text{Pol}(\Gamma)$ iff $f(r_1, \ldots, r_n) \in R$
for all $r_1, \ldots, r_n \in R$ and all relations R of Γ.

Elements of Pol(Γ) are called polymorphisms of Γ.

Pol(Γ) is a function clone:

- closed under composition
- contains projections.

Observe: $\text{Pol}(\Gamma) \supseteq \text{End}(\Gamma) \supseteq \text{Aut}(\Gamma)$.

Closed polymorphism clones

Automorphism group	Perm. group	Top. gr.	Abstr. gr.
Endomorphism monoid	Transf. monoid	Top. mon.	Abstr. mon.
Polymorphism clone	Function clone	Top. clone	Abstr. clone

The closed function clones are precisely the polymorphism clones of structures.

Let Γ, Δ be relational structures.

$$\text{What does } \mathrm{Pol}(\Delta) \in \mathrm{HSP}^{\mathrm{fin}}(\mathrm{Pol}(\Gamma)) \text{ imply for } \Gamma, \Delta?$$

Primitive positive (pp) interpretations

A formula is primitive positive (pp) iff it is of the form

$$\exists x_1 \cdots \exists x_n \, \psi_1 \wedge \cdots \wedge \psi_m,$$

where ψ_i are atomic.

Theorem (Bulatov + Jeavons + Krokhin 2000; Bodirsky + MP 2011)

Let Γ, Δ be countable ω-categorical or finite relational structures. TFAE:

- $\mathrm{Pol}(\Delta)$ *contains a clone in* $\mathrm{HSP}^{\mathrm{fin}}(\mathrm{Pol}(\Gamma))$*;*
- Δ *has a pp interpretation in* Γ*:*

  it is a pp-definable homomorphic image
  of a pp-definable subuniverse
  of a finite power
  of a structure which is pp-definable in Γ*.*

pp interpretations and topological clones

Theorem (Bodirsky + MP '11)

Let Γ be countable ω-categorical or finite, and Δ be finite. TFAE:

- $\mathrm{Pol}(\Gamma) \to \mathrm{Pol}(\Delta)$ *continuously;*
- Δ *has a pp interpretation in* Γ*.*

Remark: Continuity $\implies$ uniform continuity since Γ is ω-categorical.

For ω-categorical Δ: have to require that $\xi[\mathrm{Pol}(\Gamma)]$ is dense in the polymorphism clone of an ω-categorical structure.

Let $\mathbf{1}$ be the clone of projections on a 2-element set.

Corollary (Bodirsky + MP '11)

Let Γ be countable ω-categorical or finite. TFAE:

- $\mathrm{Pol}(\Gamma) \to \mathbf{1}$ *continuously;*
- *All finite structures have a pp interpretation in* Γ*.*

Constraint Satisfaction Problems (CSPs)

Let Γ be a structure in a finite relational language.

Definition

CSP(Γ) is the decision problem:

INPUT: variables $x_1, \ldots, x_n$ and atomic statements about them.

QUESTION: is there a satisfying assignment $h \colon \{x_1, \ldots, x_n\} \to \Gamma$?

Γ is called the template of the CSP.

Can see input as conjunction of atomic formulas.

Or can see it as pp sentence (existentially quantified conjunction).

Irrelevant whether Γ is finite or infinite. But language finite.

Examples

Diophantine

Input: A finite system of equations using $=, +, \cdot, 1$

Question: Is there a solution in $\mathbb{Z}$?

Is CSP: template $(\mathbb{Z}; 1, +, \cdot, =)$

n-colorability

Input: A finite undirected graph

Question: Is it *n*-colorable?

Is a CSP: template clique of size *n*

Examples

Positive 1-in-3-SAT

Input: A finite set of triples of variables

Question: Can one assign Boolean values to the variables so that every triple contains exactly one 1?

Is CSP: template $(\{0,1\};\{(0,0,1),(0,1,0),(1,0,0)\})$

Betweenness

Input: A finite set of triples of variables

Question: Is there a linear order on the variables such that for each triple (x,y,z) either $x<y<z$ or $z<y<x$?

Is CSP: template $(\mathbb{Q};\{(x,y,z) \mid (x<y<z) \vee (z<y<x)\})$

CSPs and pp interpretations

Observation

If Δ has a pp interpretation in Γ,
then CSP(Δ) is polynomial-time reducible to CSP(Γ).

Structure Π with polymorphism clone $\mathbf{1}$:

$$\Pi := (\{0,1\}; \{(0,0,1),(0,1,0),(1,0,0)\})$$

CSP(Π) is positive 1-in-3-SAT. NP-complete.

Corollary

Let Γ be finite or countable ω-categorical.
If Pol(Γ) $\to$ $\mathbf{1}$ continuously, then CSP(Γ) is NP-hard.

Finite tractability conjecture

Observation (Bulatov + Krokhin + Jeavons 2000)

For every finite structure Γ there is a finite structure $\mathfrak{C}(\Gamma)$ such that

- $f(x, \ldots, x) = x$ for all polymorphisms of $\mathfrak{C}(\Gamma)$
- CSP($\mathfrak{C}(\Gamma)$) is polynomial-time equivalent to CSP(Γ).

$\mathfrak{C}(\Gamma)$ is called the idempotent core of Γ.

In a sense unique.

Conjecture (Feder + Vardi 1993; Bulatov + Jeavons + Krokhin 2000)

Let Γ be finite. Then:

- Pol($\mathfrak{C}(\Gamma)$) $\to$ **1** (and CSP(Γ) is NP-complete), or
- CSP(Γ) is polynomial-time solvable.

What does this mean for Pol(Γ)?

Infinite tractability conjecture

For every ω-categorical structure Γ there is
an ω-categorical structure $\mathfrak{C}(\Gamma)$ ("model-complete core of Γ") such that

- the automorphisms of $\mathfrak{C}(\Gamma)$ are dense in its endomorphisms
- $\mathrm{CSP}(\mathfrak{C}(\Gamma))$ is polynomial-time equivalent to $\mathrm{CSP}(\Gamma)$.

If $\bar{c}$ is a finite tuple of elements of $\mathfrak{C}(\Gamma)$, then the CSP of the expansion $(\mathfrak{C}(\Gamma), \bar{c})$ is polynomial-time equivalent to the CSP of $\mathfrak{C}(\Gamma)$ (and of Γ).

Hence $f(x, \ldots, x) = x$ in the finite case.

Conjecture (Bodirsky + MP)

Let Γ be first-order definable in a countable finitely bounded homogeneous structure (implies ω-categorical). Then:

- there exists a finite tuple $\bar{c}$ such that $\mathrm{Pol}(\mathfrak{C}(\Gamma), \bar{c}) \to \mathbf{1}$ continuously (and $\mathrm{CSP}(\Gamma)$ is NP-complete), or
- $\mathrm{CSP}(\Gamma)$ is polynomial-time solvable.

IV: Topological clones revisited

Homomorphic equivalence

How does one obtain $\mathfrak{C}(\Gamma)$ from Γ?

Let Γ, Δ be structures, same signature.
Γ, Δ homomorphically equivalent if $\Gamma \to \Delta$ and $\Delta \to \Gamma$.

Observation. In that case, $\mathrm{CSP}(\Gamma) = \mathrm{CSP}(\Delta)$.

Theorem (Bodirsky 2006)

Every finite or ω-categorical structure Γ is homomorphically equivalent to a unique ω-categorical model-complete core $\mathfrak{C}(\Gamma)$.

This reduction is not covered by pp interpretations.

How does $\mathrm{Pol}(\mathfrak{C}(\Gamma))$ relate to $\mathrm{Pol}(\Gamma)$?

Double shrinks

Let $\mathfrak{A} = (A; (f_i^{\mathfrak{A}})_{i \in \tau})$ be an algebra.
Let B be a set, and let $h_1 \colon B \to A$ and $h_2 \colon A \to B$ be functions.

Define an algebra $\mathfrak{B}$ on B with signature τ by setting

$$f_i^{\mathfrak{B}}(\bar{x}) := h_2(f_i^{\mathfrak{A}}(h_1(\bar{x}))) \; .$$

$\mathfrak{B}$ is called a double shrink of $\mathfrak{A}$.

Problem: the double shrink of a finite algebra can be infinite.

WANTED: New name!

Proposition

Let Γ, Δ be structures, where Γ is ω-categorical. TFAE:

- Δ is homomorphically equivalent to a pp definable structure of Γ
- $\mathrm{Pol}(\Delta)$ contains a double shrink of $\mathrm{Pol}(\Gamma)$.

D, HSP, and weak clone homomorphisms

D$(\mathfrak{A})$... all double shrinks of $\mathfrak{A}$.

Note: Double shrink does not preserve equations. Nor projections.

Let $\mathcal{C}, \mathcal{D}$ be function clones.
Function $\xi \colon \mathcal{C} \to \mathcal{D}$ called weak homomorphism iff

- it preserves arities
- it preserves linear equations:
  $\xi(f(\pi^m_{i_1}, \ldots, \pi^m_{i_n})) = \xi(f)(\xi(\pi^m_{i_1}), \ldots, \xi(\pi^m_{i_n}))$

If there exists such a function, we write $\mathcal{C} \rightsquigarrow \mathcal{D}$.

Theorem (Barto + MP 2015)

Let $\mathcal{C}, \mathcal{D}$ be function clones. TFAE:

- $\mathcal{D} \in \mathrm{D}\,\mathrm{P}(\mathcal{C})$;
- $\mathcal{D}$ *can be obtained from* $\mathcal{C}$ *by* $\mathrm{D}, \mathrm{H}, \mathrm{S}, \mathrm{P}$.
- $\mathcal{C} \rightsquigarrow \mathcal{D}$ *surjectively.*

D and $\mathrm{HSP}^{\mathrm{fin}}$

Theorem (Barto + MP 2015)

Let $\mathcal{C}, \mathcal{D}$ be function clones, $\mathcal{D}$ finite. TFAE:

- $\mathcal{D} \in \mathrm{D}\,\mathrm{P}^{\mathrm{fin}}(\mathcal{C})$;
- $\mathcal{D}$ *can be obtained from* $\mathcal{C}$ *by* $\mathrm{D}, \mathrm{H}, \mathrm{S}, \mathrm{P}^{\mathrm{fin}}$;
- $\mathcal{C} \rightsquigarrow \mathcal{D}$ *surjectively + uniformly continuously.*

Meditation: What happened to $\mathcal{D}$ which is finitely generated?

Theorem (Barto + MP 2015)

Let Γ be finite or ω-categorical, let Δ be finite. TFAE:

- Δ *can be obtained from* Γ *by homomorphic equivalence, adding of constants to model-complete cores, and pp interpretations.*
- $\mathrm{Pol}(\Gamma) \rightsquigarrow \mathrm{Pol}(\Delta)$ *uniformly contiuously.*

Infinite tractability conjecture, revisited

Old Conjecture (Bodirsky + MP)

Let Γ be definable in a countable finitely bounded homogeneous structure (implies ω-categorical). Then:

- there exists a finite tuple $\bar{c}$ such that $\mathrm{Pol}(\mathfrak{C}(\Gamma), \bar{c}) \to \mathbf{1}$ continuously (and CSP(Γ) is NP-complete), or
- CSP(Γ) is polynomial-time solvable.

New Conjecture

Let Γ be as above or finite. Then:

- $\mathrm{Pol}(\Gamma) \rightsquigarrow \mathbf{1}$ uniformly continuously (and CSP(Γ) is NP-complete), or
- CSP(Γ) is polynomial-time solvable.

Observation: Old $\implies$ New.

Weak topological clones

- "Right" (for the moment) abstraction of function clones for CSP are weak clone homomorphisms.

Autom. group	Perm. gr.	Top. gr.	Abstr. gr.	-
Endom. monoid	Transf. mon.	Top. mon.	Abstr. mon.	-
Polym. clone	Function clone	Top. clone	Abstr. clone	Weak abstr. clone

- Any mapping between transformation monoids is a weak homomorphism. Any better name?
- Cannot expect weak homomorphism theorem with Δ infinite.
- Variant: preservation of equation of the form
  $g = \alpha(f(\beta_1(\pi^m_{i_1}), \ldots, \beta_n(\pi^m_{i_n}))$,
  where $\alpha, \beta_1, \ldots, \beta_n$ are (unary) permutations.

What have we gained?

- Avoids talking about (and proving) model-complete core $\mathfrak{C}(\Gamma)$.
- No loss of nice properties of Γ when going to $\mathfrak{C}(\Gamma)$ (e.g., finitely bounded, Ramsey property).
- Explains importance of pseudolinear equations $g = \alpha(f(\beta_1(\pi^m_{i_1}), \ldots, \beta_n(\pi^m_{i_n}))$.
- Conjecture nicer.
- Conjecture weaker (for infinite Γ)?
- Useful?

Open problems

- Is there a countable Γ such that $\mathrm{Pol}(\Gamma) \to \mathbf{1}$, but not continuously?
- Is there a closed clone $\mathcal{C}$ such that $\mathbf{1} \in \mathrm{HSP}(\mathcal{C})$, but $\mathbf{1} \notin \mathrm{HSP}^{\mathrm{fin}}(\mathcal{C})$?
- Is there a countable Γ such that $\mathrm{Pol}(\Gamma) \rightsquigarrow \mathbf{1}$, but not continuously?
- If so, is AC needed?
- Is there a better name than “double shrink”?

Reference

L. Barto, J. Opršal, and M. Pinsker

The wonderland of the double shrink

In preparation.

Wayne Ferrebee, *Torus with Spearman, Bagpipes and Barnacle*