Conjectures for clones over homogenous structures

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BIRS Workshop on Homogeneous Structures

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Outline

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- Two CSP conjectures

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I Two CSP conjectures

II Two clone conjectures

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III Breakdown and decomposition of the talk

I: Two CSP conjectures

Constraint Satisfaction Problems (CSPs)

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(c): for ω-categorical cores Γ, i.e., Aut(Γ) is oligomorphic and dense in End(Γ).

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Conjecture (Bulatov + Krokhin + Jeavons '00; Feder + Vardi '93)

Let Γ be finite.

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Let Δ be finite and idempotent.
Then pos. 1-in-3-SAT is pp-interpretable in Δ, or CSP(Δ) is in P.

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Let Γ be a reduct of a finitely bounded homogeneous structure. Then:

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However: don't know if this is ideal!

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II: Clone conjectures

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Theorem (Bulatov + Jeavons + Krokhin '00; Bodirsky + Nešetřil '03)

For ω-categorical Γ, the complexity of CSP(Γ) only depends on Pol(Γ).

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If D countable: complete metric.

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- *All finite structures have a pp interpretation in* Γ*.*

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- *Both false:* we do not understand homogeneous structures.
- *Old false, new true:* method for disproving old conjecture.
- *Both true:* structural insight on clones of ω-categorical structures.

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Let Γ be an ω-categorical core such that $\mathrm{Pol}(\Gamma) \rightsquigarrow \mathbf{1}$ uniformly continuously.

Then $\mathrm{Pol}(\Gamma, \bar{c}) \to \mathbf{1}$ continuously for some $\bar{c}$?

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Proposition

There is an ω-categorical core Γ
with a uniformly continuous h1 clone homomorphism $\xi \colon \mathrm{Pol}(\Gamma) \rightsquigarrow \mathbf{1}$
such that no restriction of ξ to any $\mathrm{Pol}(\Gamma, \bar{c})$ is a clone homomorphism.

III: Breakdown and decomposition of the talk

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Are there non-trivial equations that hold in $\mathrm{Pol}(\Gamma)$, i.e., $\mathrm{Pol}(\Gamma) \not\to \mathbf{1}$?

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Then: apply Ramsey method to turn equations into an algorithm.

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Example:

$$\alpha f(\beta x, \gamma y) = \delta f(\varepsilon y, x)$$

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Let Γ be a reduct of $(\mathbb{Q}; <)$, and Δ be its core.

Then $(\Delta, \bar{c}) \to \mathbf{1}$ continuously for some $\bar{c}$,
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Binary branching homogeneous C-relation: $\alpha f(x, y) = \beta f(y, x)$.
(Bodirsky + Jonsson + Van Pham '15)

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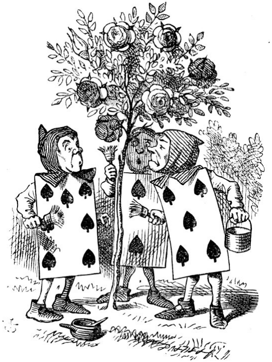
(and $\mathrm{Pol}(\Gamma) \not\to \mathbf{1}$).

Thank you!



Thanks to the organizers!

Thanks to Norbert!