

Topological clones

Michael Pinsker

Technische Universität Wien / Univerzita Karlova v Praze

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Outline

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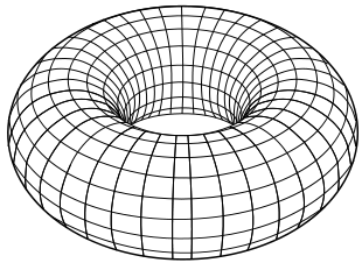
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I: Abstract clones

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$\text{Clo}(\mathfrak{A})$ is a **function clone**:

- closed under composition: $f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m))$;
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Here: algebras up to “clone equivalence”.

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Write $\mathcal{C} \rightarrow \mathcal{D}$ if there exists a clone homomorphism from \mathcal{C} into \mathcal{D} .

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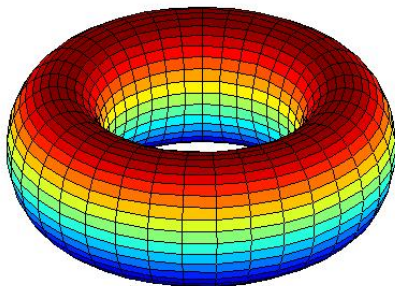
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What about HSP^{fin} of infinite function clones?

Analogy with groups and monoids

Permutation group	Abstract group
Transformation monoid	Abstract monoid
Function clone	Abstract clone



II: Topological clones

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Remark: For finite function clones: topology discrete.

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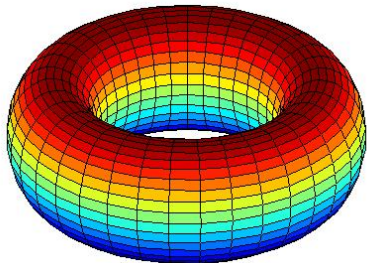
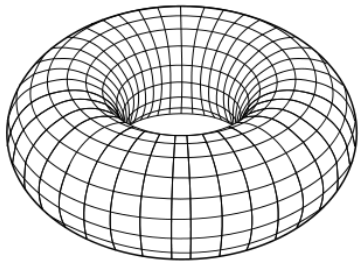
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- Edith Vargas-Garcia
- Christian Pech

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Two polymorphism clones of countable ω -categorical structures which are isomorphic, but not topologically.

(Bodirsky + Evans + Kompatscher + MP 2015)

IV: pp interpretations

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Observe: $\text{Pol}(\Gamma) \supseteq \text{End}(\Gamma) \supseteq \text{Aut}(\Gamma)$.

Closed polymorphism clones

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What does $\text{Pol}(\Delta) \in \text{HSP}^{\text{fin}}(\text{Pol}(\Gamma))$ imply for Γ, Δ ?

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- Δ has a **pp interpretation** in Γ :
*it is a pp-definable homomorphic image
of a pp-definable subuniverse
of a finite power
of a structure which is pp-definable in Γ .*

Primitive positive (pp) interpretations

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$$\exists x_1 \cdots \exists x_n \psi_1 \wedge \cdots \wedge \psi_m,$$

where ψ_i are atomic.

Theorem (Bulatov + Jeavons + Krokhin 2000; Bodirsky + MP 2011)

Let Γ, Δ be countable **ω -categorical** or finite relational structures.

TFAE:

■ $\text{Pol}(\Delta)$ contains a clone in $\text{HSP}^{\text{fin}}(\text{Pol}(\Gamma))$;

■ Δ has a **pp interpretation** in Γ :

*it is a pp-definable **homomorphic image***

*of a pp-definable **subuniverse***

*of a finite **power***

of a structure which is pp-definable in Γ .

(There is a partial mapping from some Γ^n onto Δ such that the preimage of every relation of Δ is pp-definable in Γ .)

pp interpretations and topological clones

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Let Γ be countable ω -categorical or finite, and Δ be finite. TFAE:

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Theorem (Bodirsky + MP 2011)

Let Γ, Δ be countable ω -categorical or finite. TFAE:

- $\text{Pol}(\Gamma)$ and $\text{Pol}(\Delta)$ are topologically isomorphic;
- Γ and Δ are pp biinterpretable.

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Fact: It is the polymorphism clone of the structure

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Corollary

Let Γ be countable ω -categorical or finite. TFAE:

- $\text{Pol}(\Gamma) \rightarrow \mathbf{1}$ continuously;
- Π has a pp interpretation in Γ ;
- All finite structures have a pp interpretation in Γ .

C			4		3		2	8			9				B
7						A					6			4	
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			0		7				B		D		6		E
4				9							E		1		
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			C		F		1						B		E
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V: Constraint Satisfaction Problems

Constraint Satisfaction Problems (CSPs)

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Irrelevant whether Γ is finite or infinite. But language finite.

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Input: A finite system of equations using $=, +, \cdot, 1$

Question: Is there a solution in \mathbb{Z} ?

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Question: Is there a linear order on the variables such that for each triple (x, y, z) either $x < y < z$ or $z < y < x$?

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Corollary

Let Γ be finite or countable ω -categorical.

If $\text{Pol}(\Gamma) \rightarrow \mathbf{1}$ continuously, then $\text{CSP}(\Gamma)$ is NP-hard.

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What does this mean for $\text{Pol}(\Gamma)$?

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Topological clones

Part II

Michael Pinsker

Technische Universität Wien / Univerzita Karlova v Praze

Funded by Austrian Science Fund (FWF) grant P27600

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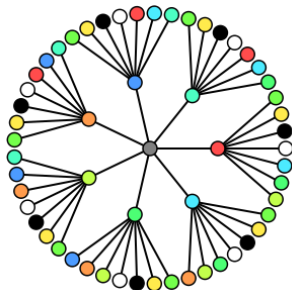
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V: Projective clone homomorphisms

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Open problem

Is there a function clone with a projective clone homomorphism, but not a continuous one?

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So the Betweenness problem is NP-hard.

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Continuous iff U is principal.

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$\text{Pol}(\Gamma) \not\rightarrow \mathbf{1}$.

But $\text{Pol}(\Gamma, 0) \rightarrow \mathbf{1}$ continuously.



VII: Topological clones revisited

Homomorphic equivalence

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How does $\text{Pol}(\mathfrak{C}(\Gamma))$ relate to $\text{Pol}(\Gamma)$?

Double shrinks

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Proposition

Let Γ, Δ be structures, where Γ is ω -categorical. TFAE:

- Δ is homomorphically equivalent to a pp definable structure of Γ
- $\text{Pol}(\Delta)$ contains a double shrink of $\text{Pol}(\Gamma)$.

D, H, S, and weak homomorphisms

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If there exists such a function, we write $\mathcal{C} \rightsquigarrow \mathcal{D}$.

Birkhoff's theorem for D and P

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Theorem (Barto + MP 2015)

Let \mathcal{C}, \mathcal{D} be function clones. TFAE:

- $\mathcal{D} \in \text{DP}(\mathcal{C})$;
- \mathcal{D} can be obtained from \mathcal{C} by $\mathcal{D}, \text{H}, \text{S}, \text{P}$.
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Meditation: What happened to \mathcal{D} which is finitely generated?

The relational side

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Theorem (Barto + MP 2015)

Let Γ be finite or ω -categorical, let Δ be finite. TFAE:

- *Δ can be obtained from Γ by homomorphic equivalence, adding of constants to model-complete cores, and pp interpretations.*

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Infinite tractability conjecture, revisited

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Old Conjecture

Let Γ be definable in a countable finitely bounded homogeneous structure (implies ω -categorical). Then:

- there exists a finite tuple \bar{c} such that $\text{Pol}(\mathfrak{C}(\Gamma), \bar{c}) \rightarrow \mathbf{1}$ continuously (and $\text{CSP}(\Gamma)$ is NP-complete), or
- $\text{CSP}(\Gamma)$ is polynomial-time solvable.

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- there exists a finite tuple \bar{c} such that $\text{Pol}(\mathfrak{C}(\Gamma), \bar{c}) \rightarrow \mathbf{1}$ continuously (and $\text{CSP}(\Gamma)$ is NP-complete), or
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Let Γ be as above or finite. Then:

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Infinite tractability conjecture, revisited

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Observation: Old \implies New.

C			4		3		2	8			9				B
7						A				6			4		
	E		8	D				F		5	2		C	7	
			0		7				B		D		6		E
4				9							E		1		
	6		2							0		5			3
	0	B	1	4		2				9				E	
	9	5			A	B	C	6			7				
	C		B		6		F	A	2		5			0	4
A		2			5	D	0				C	8	3	B	1
		0	F	B								D			2
5			3		8				1		0	9	F		
3	8			5		6	E	0		F					9
		C		F		1						B			E
0						8					6	7			D
		4		A	D		7		E		C	2			5



VIII: Discussion & Open Problems

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- Cannot expect weak homomorphism theorem with Δ infinite.

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- If a closed function clone satisfies a linear equation, does it satisfy a special equation?

Reference

L. Barto, J. Opršal, and M. Pinsker

The wonderland of the double shrink

In preparation.



Wayne Ferrebee, *Torus with Spearman, Bagpipes and Barnacle*