Chapter 6
A Hadwiger-type theorem for general tensor valuations

Franz E. Schuster

Abstract Hadwiger's characterization of continuous rigid motion invariant real valued valuations has been the starting point for many important developments in valuation theory. In this chapter, the decomposition of the space of continuous and translation invariant valuations into a sum of $\mathrm{SO}(n)$ irreducible subspaces, derived by S. Alesker, A. Bernig and the author, is discussed. It is also explained how this result can be reformulated in terms of a Hadwiger-type theorem for translation invariant and $\mathrm{SO}(n)$ equivariant valuations with values in an arbitrary finite dimensional $\mathrm{SO}(n)$ module. In particular, this includes valuations with values in general tensor spaces. The proofs of these results will be outlined modulo a couple of basic facts from representation theory. In the final part, we survey a number of special cases and applications of the main results in different contexts of convex and integral geometry.

6.1 Statement of the principal results

Let $\mathscr{K}^n$ denote the space of convex bodies in Euclidean n-space $\mathbb{R}^n$, where $n \geq 3$, endowed with the Hausdorff metric. In this chapter we consider valuations ϕ defined on $\mathscr{K}^n$ and taking values in an Abelian semigroup $\mathscr{A}$, that is,

$$\phi(K \cup L) + \phi(K \cap L) = \phi(K) + \phi(L)$$

whenever $K \cup L$ is convex and $+$ denotes the operation of $\mathscr{A}$.

The most famous and important classical result on scalar-valued valuations (where $\mathscr{A} = \mathbb{R}$ or $\mathbb{C}$) is the characterization of continuous rigid motion invariant valuations by Hadwiger [40] (which was slightly improved later by Klain [45]).

Franz E. Schuster
Vienna University of Technology, Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria, e-mail: franz.schuster@tuwien.ac.at

Theorem ([40, 45]). *A basis for the vector space of all continuous, translation- and* $\mathrm{SO}(n)$ *invariant scalar valuations on* $\mathscr{K}^n$ *is given by the intrinsic volumes.*

The characterization theorem of Hadwiger had a transformative effect on integral geometry. It not only allows for an effortless proof of the principal kinematic formula (see, e.g., [47]) but almost all classical integral-geometric results can be derived from this landmark theorem. It also motivated subsequent characterizations of rigid motion equivariant vector-valued valuations (where $\mathscr{A} = \mathbb{R}^n$) (see [41]), valuations taking values in the set of finite Borel measures on $\mathbb{R}^n$ or $\mathbb{S}^{n-1}$ which intertwine rigid motions (see [69, 70]) and, more recently, Minkowski valuations (where $\mathscr{A} = \mathscr{K}^n$ endowed with Minkowski addition) which are translation invariant and $\mathrm{SO}(n)$ equivariant (see [44, 68, 72, 74, 76, 77]). Important parts of modern integral geometry also deal with variants of Hadwiger's characterization theorem, where either the group $\mathrm{SO}(n)$ is replaced by a subgroup acting transitively on the unit sphere (see [6, 16, 17, 19, 21, 24]) or the valuations are invariant under the larger group $\mathrm{SL}(n)$ but neither assumed to be continuous nor translation invariant (see, e.g., [36, 51, 55]).

Here we focus on continuous and translation invariant valuations which take values in a general (finite dimensional) *tensor space* Γ and are equivariant with respect to $\mathrm{SO}(n)$. The case of *symmetric* tensors, where $\Gamma = \mathrm{Sym}^k(\mathbb{R}^n)$, was first investigated by McMullen [60], who considered instead of translation invariant more general *isometry covariant* tensor valuations. Alesker [4, 3] showed that the space of all such continuous isometry covariant $\mathrm{Sym}^k(\mathbb{R}^n)$-valued valuations (of a fixed rank and given degree of homogeneity) is spanned by the Minkowski tensors. More recently, Hug, Schneider and R. Schuster [42, 43] explicitly determined the dimension of this space and obtained a full set of kinematic formulas for Minkowski tensors. Following a more algebraic approach, these kinematic formulas could be further simplified in the translation invariant case by Bernig and Hug [20]. For applications of the integral geometry of tensor valuations in different areas, see Chapters 10 to 13 and the references therein. We also mention that $\mathrm{Sym}^k(\mathbb{R}^n)$-valued valuations were also investigated in the context of affine and centro-affine geometry by Ludwig [52] and Haberl and Parapatits [37, 38].

Bernig [15] constructed an interesting translation invariant valuation with values in $\Lambda^k(\mathbb{R}^n) \otimes \Lambda^k(\mathbb{R}^n)$ which can be interpreted as a natural curvature tensor. Apart from this, not much was known for general, non-symmetric tensor valuations until recently Alesker, Bernig and the author [10] established a Hadwiger-type theorem for continuous, translation invariant and $\mathrm{SO}(n)$ equivariant valuations with values in an *arbitrary finite dimensional complex* representation space Γ of $\mathrm{SO}(n)$. In order to state this result first recall that given a Lie group G and a topological vector space Γ (finite or infinite dimensional), a (continuous) representation of G on Γ is a continuous left action $G \times \Gamma \to \Gamma$ such that for each $g \in G$ the map $v \mapsto g \cdot v$ is linear. Note that we assume throughout that all representations are continuous.

For a finite dimensional complex vector space Γ, we denote by $\Gamma\mathrm{Val}$ the vector space of all continuous and translation invariant valuations with values in Γ and write $\Gamma\mathrm{Val}_i$ for its subspace of all valuations of degree i. If $\Gamma = \mathbb{C}$, then we simply write Val and Val_i, respectively. McMullen's decomposition theorem [57] implies that

$$\Gamma\mathrm{Val} = \bigoplus_{0\le i\le n} \Gamma\mathrm{Val}_i. \tag{6.1}$$

We also recall the parametrization of the isomorphism classes of irreducible representations of $\mathrm{SO}(n)$ in terms of their highest weights. These can be identified with $\lfloor n/2 \rfloor$-tuples of integers $(\lambda_1,\dots,\lambda_{\lfloor n/2\rfloor})$ such that

$$\begin{cases} \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{\lfloor n/2\rfloor} \ge 0 & \text{for odd } n, \\ \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{n/2-1} \ge |\lambda_{n/2}| & \text{for even } n. \end{cases} \tag{6.2}$$

We write Γ_λ for any isomorphic copy of an irreducible representation of $\mathrm{SO}(n)$ with highest weight $\lambda = (\lambda_1,\dots,\lambda_{\lfloor n/2\rfloor})$. Note that since $\mathrm{SO}(n)$ is a compact Lie group, every Γ_λ is finite dimensional. Moreover, any finite dimensional representation of $\mathrm{SO}(n)$ can be decomposed into a direct sum of irreducible representations. In particular, we have a decomposition of our representation space Γ of the form

$$\Gamma = \bigoplus_{\lambda} m(\Gamma,\lambda)\,\Gamma_\lambda, \tag{6.3}$$

where the sum ranges over a finite number of highest weights $\lambda = (\lambda_1,\dots,\lambda_{\lfloor n/2\rfloor})$ satisfying (6.2). Here and in the following $m(\Theta,\lambda)$ denotes the *multiplicity* of Γ_λ in an arbitrary $\mathrm{SO}(n)$ module Θ which, by Schur's lemma, is given by

$$m(\Theta,\lambda) = \dim \mathrm{Hom}_{\mathrm{SO}(n)}(\Theta,\Gamma_\lambda),$$

where $\mathrm{Hom}_{\mathrm{SO}(n)}$ denotes as usual the space of continuous linear $\mathrm{SO}(n)$ equivariant maps. If $m(\Theta,\lambda)$ is 0 or 1 for all highest weights λ satisfying (6.2), we say that the $\mathrm{SO}(n)$ module Θ is *multiplicity free*. For explicit examples of decompositions of the form (6.3) and more background material as well as references on representation theory of compact Lie groups, see Section 6.2.

We are now ready to state the main result of [10] which is the topic of this chapter.

Theorem 6.1. *Let Γ be a finite dimensional complex $\mathrm{SO}(n)$ module and let $0 \le i \le n$. The dimension of the subspace of $\mathrm{SO}(n)$ equivariant valuations in $\Gamma\mathrm{Val}_i$ is given by*

$$\sum_{\lambda} m(\Gamma,\lambda),$$

where the sum ranges over all highest weights $\lambda = (\lambda_1,\dots,\lambda_{\lfloor n/2\rfloor})$ satisfying (6.2) and the following additional conditions:

(i) $\lambda_j = 0$ *for* $j > \min\{i, n-i\}$*;*
(ii) $|\lambda_j| \ne 1$ *for* $1 \le j \le \lfloor n/2 \rfloor$*;*
(iii) $|\lambda_2| \le 2$.

Theorem 6.1 follows from an equivalent result about the decomposition of the space Val_i into $\mathrm{SO}(n)$ irreducible subspaces.

Theorem 6.2. *Let $0 \leq i \leq n$. Under the action of* $\mathrm{SO}(n)$ *the space* Val_i *is multiplicity free. Moreover,* $m(\mathrm{Val}_i, \lambda) = 1$ *if and only if the highest weight* $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ *satisfies* (6.2) *and the conditions (i)–(iii) from Theorem 6.1.*

In order to outline how Theorem 6.1 can be deduced from Theorem 6.2 (for the precise argument, see [10, p. 765]), first note that we may assume that Γ is irreducible, that is, $\Gamma = \Gamma_\mu$ for some highest weight $\mu = (\mu_1, \ldots, \mu_{\lfloor n/2 \rfloor})$ satisfying (6.2). Next, observe that the linear map $\iota : \mathrm{Val}_i \otimes \Gamma \to \Gamma\mathrm{Val}_i$, induced by

$$\iota(\phi \otimes v)(K) = \phi(K)v,$$

is an isomorphism and that the subspace of $\mathrm{SO}(n)$ equivariant valuations in $\Gamma\mathrm{Val}_i$ corresponds under this isomorphism to the subspace of $\mathrm{SO}(n)$ *invariant* elements in $\mathrm{Val}_i \otimes \Gamma$. (As usual we will use the superscript $\mathrm{SO}(n)$ to denote subspaces of $\mathrm{SO}(n)$ invariant elements.) Now, if S denotes the set of highest weights of $\mathrm{SO}(n)$ satisfying conditions (i)-(iii), then, by Theorem 6.2,

$$\dim(\mathrm{Val}_i \otimes \Gamma)^{\mathrm{SO}(n)} = \sum_{\lambda \in S} \dim(\Gamma_\lambda \otimes \Gamma_\mu)^{\mathrm{SO}(n)} = \sum_{\lambda \in S} \dim \mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma_\lambda^*, \Gamma_\mu).$$

It follows from Lemma 6.3 below, that the $\mathrm{SO}(n)$ irreducible subspaces Γ_λ for $\lambda \in S$ are *not* necessarily isomorphic as $\mathrm{SO}(n)$ modules to their dual representations Γ_λ^* (see Section 6.2 for details). However, Lemma 6.3 also implies that if $\lambda \in S$, then also $\lambda' \in S$, where λ' is the highest weight of Γ_λ^*. Thus, from an application of Schur's lemma, we obtain

$$\dim(\mathrm{Val}_i \otimes \Gamma)^{\mathrm{SO}(n)} = \sum_{\lambda \in S} \dim \mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma_\lambda, \Gamma_\mu) = \begin{cases} 1 & \text{if } \mu \in S, \\ 0 & \text{otherwise} \end{cases}$$

which is precisely the statement of Theorem 6.1 in the case $\Gamma = \Gamma_\mu$. We also remark that the argument outlined here can be easily reversed to deduce Theorem 6.2 from Theorem 6.1.

A proof of Theorem 6.2 for *even* valuations was first given by Alesker and Bernstein [11] (based on the Irreducibility Theorem of Alesker [5]). They used the Klain embedding of continuous, translation invariant and even valuations and its relation to the cosine transform on Grassmannians to deduce Theorem 6.2 in this special case. We will discuss this approach in more detail in the last section of this chapter, where we also survey a number of other special cases and applications of Theorems 6.1 and 6.2.

In Section 6.2 we collect more background material about representations of $\mathrm{SO}(n)$ which is needed for the analysis of the action of $\mathrm{SO}(n)$ on the space of translation invariant differential forms on the sphere bundle. Combining this with a description of smooth translation invariant valuations via integral currents by Alesker [8] and, in a refined form, by Bernig and Bröcker [18] and Bernig [17], this will allow us to give an essentially complete proof of Theorem 6.2 in Section 6.4.

6.2 Irreducible representations of SO(n)

For an introduction to the representation theory of compact Lie groups we refer to the books by Bröcker and tom Dieck [22], Fulton and Harris [25], Goodman and Wallach [33], and Knapp [49]. These books, in particular, contain all the material on irreducible representations of SO(n) which are needed in this chapter.

In this and the next section let V be an n-dimensional Euclidean vector space and write $V_{\mathbb{C}} = V \otimes \mathbb{C}$ for its complexification. For later reference we state here a number of examples of irreducible SO(n) modules as well as reducible ones together with their direct sum decomposition into SO(n) irreducible subspaces.

Examples.

(a) Up to isomorphism, the trivial representation is the only one dimensional (complex) representation of SO(n). It corresponds to the SO(n) module $\Gamma_{(0,\dots,0)}$. The standard representation of SO(n) on $V_{\mathbb{C}}$ is isomorphic to $\Gamma_{(1,0,\dots,0)}$.

(b) The exterior power $\Lambda^i V_{\mathbb{C}}$ is SO(n) irreducible for every $0 \le i \le \lfloor n/2 \rfloor - 1$. If $n = 2i+1$ is odd, then $\Lambda^i V_{\mathbb{C}}$ is also irreducible under the action of SO(n). In these cases the highest weight tuple of $\Lambda^i V_{\mathbb{C}}$ is given by $\lambda = (1,\dots,1,0,\dots,0)$, where 1 appears i times. If $n = 2i$ is even, then $\Lambda^i V_{\mathbb{C}}$ is *not* irreducible but is a direct sum of two irreducible subspaces, namely, $\Lambda^i V_{\mathbb{C}} = \Gamma_{(1,\dots,1)} \oplus \Gamma_{(1,\dots,1,-1)}$. Moreover, for every $i \in \{0,\dots,n\}$, there is a natural isomorphism of SO(n) modules
$$\Lambda^i V_{\mathbb{C}} \cong \Lambda^{n-i} V_{\mathbb{C}}. \tag{6.4}$$
The spaces $\Lambda^i V_{\mathbb{C}}$ are called *fundamental representations* since they can be used to construct arbitrary irreducible representations of SO(n) (cf. [25]).

(c) The symmetric power $\mathrm{Sym}^k V_{\mathbb{C}}$ is *not* irreducible as SO(n) module when $k \ge 2$. Its direct sum decomposition into irreducible subspaces takes the form
$$\mathrm{Sym}^k V_{\mathbb{C}} = \bigoplus_{j=0}^{\lfloor k/2 \rfloor} \Gamma_{(k-2j,0,\dots,0)}. \tag{6.5}$$

(d) The decomposition of $L^2(\mathbb{S}^{n-1})$ into an *orthogonal* sum of SO(n) irreducible subspaces is given by
$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{k \in \mathbb{N}} \mathscr{H}_k^n, \tag{6.6}$$
where $\mathscr{H}_k^n$ is the space of spherical harmonics of dimension n and degree k. The highest weight tuple of the space $\mathscr{H}_k^n$ is given by $\lambda = (k,0,\dots,0)$.

(e) For $1 \le i \le n-1$, the space $L^2(G(n,i))$ is an orthogonal sum of SO(n) irreducible subspaces whose highest weights $(\lambda_1,\dots,\lambda_{\lfloor n/2 \rfloor})$ satisfy (6.2) and the following two additional conditions (see, e.g., [49, Theorem 8.49]):
$$\begin{cases} \lambda_j = 0 \text{ for all } j > \min\{i, n-i\}, \\ \lambda_1,\dots,\lambda_{\lfloor n/2 \rfloor} \text{ are all even.} \end{cases} \tag{6.7}$$

Let Θ be a complex finite dimensional $\mathrm{SO}(n)$ module which is not necessarily irreducible. The *dual representation* of $\mathrm{SO}(n)$ on the dual space Θ^* is defined by

$$(\vartheta\, u^*)(v) = u^*(\vartheta^{-1} v), \qquad \vartheta \in \mathrm{SO}(n),\, u^* \in \Theta^*,\, v \in \Theta.$$

We say that Θ is *self-dual* if Θ and Θ^* are isomorphic representations. The module Θ is called *real* if there exists a non-degenerate symmetric $\mathrm{SO}(n)$ invariant bilinear form on Θ. In particular, if Θ is real, then Θ is also self-dual.

Lemma 6.3 ([22]). *Let $\lambda = (\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ be a tuple of integers satisfying* (6.2).

(a) *If $n \equiv 2 \mod 4$, then the irreducible representation Γ_λ of $\mathrm{SO}(n)$ is real if and only if $\lambda_{n/2} = 0$.*
(b) *If $n \equiv 2 \mod 4$ and $\lambda_{n/2} \neq 0$, then the dual of Γ_λ is isomorphic to $\Gamma_{\lambda'}$, where $\lambda' = (\lambda_1, \dots, \lambda_{n/2-1}, -\lambda_{n/2})$.*
(c) *If $n \not\equiv 2 \mod 4$, then all representations of $\mathrm{SO}(n)$ are real.*

Now, let Γ be again a finite dimensional complex $\mathrm{SO}(n)$ module and denote by $\rho : \mathrm{SO}(n) \to \mathrm{GL}(\Gamma)$ the corresponding representation. The *character* of Γ is the function $\mathrm{char}\,\Gamma : \mathrm{SO}(n) \to \mathbb{C}$ defined by

$$(\mathrm{char}\,\Gamma)(\vartheta) = \mathrm{tr}\,\rho(\vartheta),$$

where $\mathrm{tr}\,\rho(\vartheta)$ is the trace of the linear map $\rho(\vartheta) : \Gamma \to \Gamma$.

The most important property of the character of a complex representation is that it determines the module Γ up to isomorphism. Moreover, several well known properties of the trace map immediately carry over to useful arithmetic properties of characters. For example, if Γ and Θ are finite dimensional $\mathrm{SO}(n)$ modules, then

$$\mathrm{char}(\Gamma \oplus \Theta) = \mathrm{char}\,\Gamma + \mathrm{char}\,\Theta \tag{6.8}$$

and

$$\mathrm{char}(\Gamma \otimes \Theta) = \mathrm{char}\,\Gamma \cdot \mathrm{char}\,\Theta. \tag{6.9}$$

A description of the characters of irreducible representations of compact Lie groups in terms of their highest weights is provided by Weyl's character formula. For our purposes, that is, the case of the special orthogonal group $\mathrm{SO}(n)$, a consequence of this description, known as the *second determinantal formula*, is crucial. In order to state this result let $\lambda = (\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ be a tuple of *non-negative* integers satisfying (6.2). We define the $\mathrm{SO}(n)$ module $\overline{\Gamma}_\lambda$ by

$$\overline{\Gamma}_\lambda := \begin{cases} \Gamma_\lambda \oplus \Gamma_{\lambda'} & \text{if } n \text{ is even and } \lambda_{n/2} \neq 0, \\ \Gamma_\lambda & \text{otherwise.} \end{cases}.$$

The *conjugate* of λ is the $s := \lambda_1$ tuple $\mu = (\mu_1, \dots, \mu_s)$ defined by letting μ_j be the number of terms in λ that are greater than or equal j.

The second determinantal formula expresses the character of $\overline{\Gamma}_\lambda$ as a polynomial in the characters $E_i := \operatorname{char} \Lambda^i V_{\mathbb{C}}$, $i \in \mathbb{Z}$. (Here and in the following, we use the convention $E_i = 0$ for $i < 0$ or $i > n$.)

Theorem 6.4 ([25]). *Let $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ be a tuple of non-negative integers satisfying* (6.2) *and let $\mu = (\mu_1, \ldots, \mu_s)$ be the conjugate of λ. The character of $\overline{\Gamma}_\lambda$ is equal to the determinant of the $s \times s$-matrix whose i-th row is given by*

$$\left(E_{\mu_i - i+1} \;\; E_{\mu_i - i+2} + E_{\mu_i - i} \;\; \cdots \;\; E_{\mu_i - i+s} + E_{\mu_i - i-s+2} \right). \tag{6.10}$$

In the definition of the conjugate of λ we will later also allow $s > \lambda_1$. Note that this introduces additional zeros at the end of the conjugate tuple but does not change the determinant of the matrix defined by (6.10).

In order to analyze the action of $\mathrm{SO}(n)$ on the *infinite dimensional* space Val, we need to briefly discuss the construction of a class of such infinite dimensional representations of a Lie group G induced from closed subgroups $H \subseteq G$ (although we only need the case $G = \mathrm{SO}(n)$ and $H = \mathrm{SO}(n-1)$). To this end, we denote by $C^\infty(G;\Gamma)$ the space of all smooth functions from G to an arbitrary finite dimensional (complex) H module Γ. The *induced representation* of G by H on the space

$$\operatorname{Ind}_H^G \Gamma := \left\{ f \in C^\infty(G;\Gamma) : f(gh) = h^{-1} f(g) \text{ for all } g \in G, h \in H \right\} \subseteq C^\infty(G;\Gamma)$$

is given by left translation, that is, $(gf)(u) = f(g^{-1}u)$, $g, u \in G$. Conversely, if Θ is any representation of G, we obtain a representation $\operatorname{Res}_H^G \Theta$ of H by restriction. The fundamental *Frobenius Reciprocity Theorem* establishes a connection between induced and restricted representations.

Theorem 6.5 ([33]). *If Θ is a G module and Γ is an H module, then there is a canonical vector space isomorphism*

$$\operatorname{Hom}_G(\Theta, \operatorname{Ind}_H^G \Gamma) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G \Theta, \Gamma).$$

A for our purposes crucial consequence of the Frobenius Reciprocity Theorem (and the definition of multiplicity) is the fact that if Θ and Γ are irreducible, then the multiplicity of Θ in $\operatorname{Ind}_H^G \Gamma$ equals the multiplicity of Γ in $\operatorname{Res}_H^G \Theta$.

In order to apply Theorem 6.5 in the case $G = \mathrm{SO}(n)$ and $H = \mathrm{SO}(n-1)$, we require the following *branching formula* for decomposing $\operatorname{Res}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \Gamma$ into irreducible $\mathrm{SO}(n-1)$ modules.

Theorem 6.6 ([25]). *If Γ_λ is an irreducible representation of* $\mathrm{SO}(n)$ *with highest weight tuple $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ satisfying* (6.2)*, then*

$$\operatorname{Res}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \Gamma_\lambda = \bigoplus_\mu \Gamma_\mu, \tag{6.11}$$

where the sum ranges over all $\mu = (\mu_1, \ldots, \mu_k)$ with $k := \lfloor (n-1)/2 \rfloor$ satisfying

$$\begin{cases} \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{k-1} \geq \lambda_{\lfloor n/2 \rfloor} \geq |\mu_k| & \textit{for odd } n, \\ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_k \geq |\lambda_{n/2}| & \textit{for even } n. \end{cases}$$

6.3 Smooth valuations and the Rumin-de Rham complex

We discuss in this section a description of translation invariant *smooth* valuations via integral currents and how it relates to induced representations. This will allow us in the next section to apply the machinery from representation theory presented in Section 6.2 to prove Theorem 6.2.

First recall that by McMullen's decomposition theorem

$$\mathrm{Val} = \bigoplus_{0 \leq i \leq n} \mathrm{Val}_i^+ \oplus \mathrm{Val}_i^-, \tag{6.12}$$

where $\mathrm{Val}_i^\pm$ denote the subspaces of valuations of degree i and even or odd parity, respectively. In the cases $i = 0$ and $i = n$ a simple description of the valuations in Val_i is possible (cf. Chapter 1, Theorem 1.16 and Corollary 1.24).

Proposition 6.7 ([39]).

(a) *The space* Val_0 *is one-dimensional and spanned by the Euler characteristic.*
(b) *The space* Val_n *is one-dimensional and spanned by the volume functional.*

Note that statement (a) of Proposition 6.7 is trivial while the non-trivial statement (b) was proved by Hadwiger [39, p. 79]. We also note that Proposition 6.7 directly implies Theorem 6.2 for the cases $i = 0$ and $i = n$.

There is also an explicit description of the valuations in Val_{n-1} going back to McMullen [58] (cf. Chapter 1, Theorem 1.25). However, in this chapter we will not make use of this result and we will therefore not repeat it here. Instead we turn to the notion of smooth valuations. To this end first recall that the space Val becomes a Banach space when endowed with the norm

$$\|\phi\| = \sup\{|\phi(K)| : K \subseteq B^n\}.$$

Here, B^n denotes as usual the Euclidean unit ball. On the Banach space Val there is a natural continuous action of the group $\mathrm{GL}(n)$ given by

$$(A\phi)(K) = \phi(A^{-1}K), \qquad A \in \mathrm{GL}(n), \, \phi \in \mathrm{Val}.$$

Clearly, the subspaces $\mathrm{Val}_i^\pm \subseteq \mathrm{Val}$ are $\mathrm{GL}(n)$ invariant. In fact, they are *irreducible* as was shown by Alesker [5] (but we will not use this deep result directly).

Smooth translation invariant valuations were first introduced by Alesker [6]. By definition, they are precisely the smooth vectors (see, e.g. [79]) of the natural representation of $\mathrm{GL}(n)$ on Val.

Definition. *A valuation $\phi \in \mathrm{Val}$ is called* smooth *if the map* $\mathrm{GL}(n) \to \mathrm{Val}$*, defined by $A \mapsto A\phi$, is infinitely differentiable.*

As usual, we denote by Val^∞ the Fréchet space of smooth translation invariant valuations endowed with the Gårding topology (see, e.g., [82, Section 4.4]) and write Val_i^∞ for its subspaces of smooth valuations of degree i.

By general properties of smooth vectors (cf. [79]), the spaces Val_i^∞ are dense $\mathrm{GL}(n)$ invariant subspaces of Val_i and from (6.12) it is easy to deduce that

$$\mathrm{Val}^\infty = \bigoplus_{0\le i\le n} \mathrm{Val}_i^\infty.$$

The advantage of considering smooth translation invariant valuations instead of merely continuous ones is that the Fréchet space Val^∞ admits additional algebraic structures. Since these are precisely the topic of Chapter 4, we will discuss here only one structural property of Val^∞ which is crucial for us. To this end, first recall that McMullen's decomposition (6.12) implies that for any $\phi \in \mathrm{Val}$ and $K \in \mathscr{K}^n$, the function $t \mapsto \phi(K+tB^n)$ is a polynomial of degree at most n. This, in turn, gives rise to a derivation operator $\Lambda : \mathrm{Val} \to \mathrm{Val}$, defined by

$$(\Lambda\phi)(K) = \left.\frac{d}{dt}\right|_{t=0} \phi(K+tB^n). \tag{6.13}$$

From this definition it follows that if $\phi \in \mathrm{Val}_i$, then $\Lambda\phi \in \mathrm{Val}_{i-1}$, that Λ is continuous, $\mathrm{SO}(n)$ equivariant, and that Λ maps smooth valuations to smooth ones. Moreover, the following *Hard Lefschetz theorem* for Λ was proved by Alesker [6] for even and by Bernig and Bröcker [18] for general valuations.

Theorem 6.8 ([6, 18]). *For $\frac{n}{2} < i \le n$, the map $\Lambda^{2i-n} : \mathbf{Val}_i^\infty \to \mathbf{Val}_{n-i}^\infty$ is an $\mathrm{SO}(n)$ equivariant isomorphism of Fréchet spaces.*

The main tool used in [18] to establish Theorem 6.8 was a description of smooth valuations in terms of the normal cycle map by Alesker [8]. Since a refined version of this result by Bernig [17] (stated below as Theorem 6.9) is critical for the proof of Theorem 6.2, we discuss these results and the necessary background in the following.

Let $SV = V \times S^{n-1}$ denote the unit sphere bundle on the n-dimensional Euclidean vector space V. The natural (smooth) action of $\mathrm{SO}(n)$ on SV is given by

$$l_\vartheta(x,u) := (\vartheta x, \vartheta u), \qquad \vartheta \in \mathrm{SO}(n), (x,u) \in SV. \tag{6.14}$$

Similarly, each $y \in V$ determines a smooth map $t_y : SV \to SV$ by

$$t_y(x,u) = (x+y,u), \qquad (x,u) \in SV. \tag{6.15}$$

The canonical *contact form* α on SV is the one form given by

$$\alpha|_{(x,u)}(w) = \langle u, d_{(x,u)}\pi(w)\rangle, \qquad w \in T_{(x,u)}SV,$$

where $\pi : SV \to V$ denotes the canonical projection and $d_{(x,u)}\pi$ is its differential at $(x,u) \in SV$. Endowed with the contact form α the manifold SV becomes a $2n-1$ dimensional *contact manifold*. The kernel of α defines the so-called *contact distribution* $Q := \ker \alpha$. Note that the restriction of $d\alpha$ to Q is a non-degenerate two form and, therefore, each space $Q_{(x,u)}$ becomes a symplectic vector space.

Since SV is a product manifold, the vector space $\Omega^*(SV)$ of complex valued smooth differential forms on SV admits a bigrading given by

$$\Omega^*(SV) = \bigoplus_{i,j} \Omega^{i,j}(SV),$$

where $\Omega^{i,j}(SV)$ are the subspaces of $\Omega^*(SV)$ of forms of bidegree (i,j). We denote by $\Omega^{i,j}$ the subspace of translation invariant forms in $\Omega^{i,j}(SV)$, that is,

$$\Omega^{i,j} = \{\omega \in \Omega^{i,j}(SV) : t_y^*\omega = \omega \text{ for all } y \in V\}.$$

The natural (continuous) action of $\mathrm{SO}(n)$ on the vector space $\Omega^{i,j}$ is given by

$$\vartheta\omega = l_{\vartheta^{-1}}^*\omega, \qquad \vartheta \in \mathrm{SO}(n), \omega \in \Omega^{i,j}.$$

Here, t_y^* and $l_{\vartheta^{-1}}^*$ are the pullbacks of the maps defined in (6.14) and (6.15). We also note that the restriction of the exterior derivative d to $\Omega^{i,j}$ has bidegree $(0,1)$.

For $K \in \mathscr{K}^n$ and $x \in \partial K$, let $N(K,x)$ denote the normal cone of K at x. The *normal cycle* of K is the Lipschitz submanifold of SV defined by

$$\mathbf{nc}(K) = \{(x,u) \in SV : x \in \partial K, u \in N(K,x)\}.$$

For $0 \leq i \leq n-1$, Alesker [8, Theorem 5.2.1] proved that the $\mathrm{SO}(n)$ equivariant map $\nu : \Omega^{i,n-i-1} \to \mathrm{Val}_i^\infty$, defined by

$$\nu(\omega)(K) = \int_{\mathbf{nc}(K)} \omega, \tag{6.16}$$

is *surjective*. However, for our purposes we need a more precise version of this statement which includes, in particular, a description of the kernel of ν first obtained by Bernig and Bröcker [18]. In order to state this refinement, we first have to recall the notion of primitive forms.

Let $\mathfrak{I}^{i,j}$ denote the $\mathrm{SO}(n)$ submodule of $\Omega^{i,j}$ defined by

$$\mathfrak{I}^{i,j} := \{\omega \in \Omega^{i,j} : \omega = \alpha \wedge \xi + d\alpha \wedge \psi,\, \xi \in \Omega^{i-1,j}, \psi \in \Omega^{i-1,j-1}\}.$$

The $\mathrm{SO}(n)$ module $\Omega_p^{i,j}$ of *primitive* forms is defined as the quotient

$$\Omega_p^{i,j} := \Omega^{i,j}/\mathfrak{I}^{i,j}. \tag{6.17}$$

Primitive forms are very important for the study of translation invariant valuations since, by a theorem of Bernig [17], the space Val_i^∞, $0 \leq i \leq n$, fits into an exact sequence of the spaces $\Omega_p^{i,j}$. In order to state Bernig's result precisely, first note that $d\mathfrak{I}^{i,j} \subseteq \mathfrak{I}^{i,j+1}$. Thus, by (6.17), on one hand the exterior derivative induces a linear $\mathrm{SO}(n)$ equivariant operator $d_Q : \Omega_p^{i,j} \to \Omega_p^{i,j+1}$. On the other hand, integration over the normal cycle (6.16) induces a linear map $\nu : \Omega_p^{i,n-i-1} \to \mathrm{Val}_i^\infty$ which, clearly, is also $\mathrm{SO}(n)$ equivariant.

Theorem 6.9 ([17]). *For every $0 \leq i \leq n$, the $\mathrm{SO}(n)$ equivariant sequence of $\mathrm{SO}(n)$ modules*

$$0 \longrightarrow \Lambda^i V_{\mathbb{C}} \longrightarrow \Omega_p^{i,0} \xrightarrow{d_Q} \Omega_p^{i,1} \xrightarrow{d_Q} \cdots \xrightarrow{d_Q} \Omega_p^{i,n-i-1} \xrightarrow{\nu} \mathrm{Val}_i^\infty \longrightarrow 0$$

is exact.

In order to apply Theorem 6.9 in the proof of Theorem 6.2, we require an equivalent description of primitive forms involving horizontal forms. To this end, let R denote the *Reeb vector field* on SV defined by $R_{(x,u)} = (u,0)$. Note that $\alpha(R) = 1$ and that $i_R d\alpha = 0$, where i_R denotes the interior product with the vector field R. The $\mathrm{SO}(n)$ submodule $\Omega_h^{i,j} \subseteq \Omega^{i,j}$ of *horizontal* forms is defined by

$$\Omega_h^{i,j} := \{\omega \in \Omega^{i,j} : i_R\omega = 0\}.$$

From this definition it is not difficult to see that the multiplication by the symplectic form $-d\alpha$ induces an $\mathrm{SO}(n)$ equivariant linear operator $L : \Omega_h^{i,j} \to \Omega_h^{i+1,j+1}$ which is injective if $i + j \leq n-2$. Moreover, it follows from (6.17) that in this case

$$\Omega_p^{i,j} = \Omega_h^{i,j}/L\Omega_h^{i-1,j-1}. \tag{6.18}$$

Let us now fix an arbitrary point $u_0 \in S^{n-1}$ and let $\mathrm{SO}(n-1)$ denote the stabilizer of $\mathrm{SO}(n)$ at u_0. For $u \in S^{n-1}$, we denote by $W_u := T_u S^{n-1} \otimes \mathbb{C}$ the complexification of the tangent space $T_u S^{n-1}$ and we write W_0 to denote W_{u_0}. The advantage of using description (6.18) instead of definition (6.17) of primitive forms becomes clear from the next lemma which relates horizontal and primitive forms to certain $\mathrm{SO}(n)$ representations induced from $\mathrm{SO}(n-1)$.

Lemma 6.10 ([10]). *For $i, j \in \mathbb{N}$, there is an isomorphism of $\mathrm{SO}(n)$ modules*

$$\Omega_h^{i,j} \cong \mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)}(\Lambda^i W_0^* \otimes \Lambda^j W_0^*). \tag{6.19}$$

Moreover, if $i + j \leq n-1$, then there is an isomorphism of $\mathrm{SO}(n)$ modules

$$\Omega_p^{i,j} \oplus \mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)}(\Lambda^{i-1} W_0^* \otimes \Lambda^{j-1} W_0^*) \cong \mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)}(\Lambda^i W_0^* \otimes \Lambda^j W_0^*). \tag{6.20}$$

Note that (6.20) is an immediate consequence of (6.19) and (6.18).

6.4 Proof of the main result

With the auxiliary results from the last two sections at hand, we are now in a position to complete the proof of Theorem 6.2. To this end, first recall that the cases $i = 0$ and $i = n$ are immediate consequences of Proposition 6.7. Hence, by Theorem 6.8, we may assume that $n/2 \leq i < n$.

Let $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ be a highest weight tuple of $\mathrm{SO}(n)$. As a consequence of (6.20), the multiplicity $m(\Omega_p^{i,j}, \lambda)$ is finite for all $i, j \in \mathbb{N}$ such that $i + j \leq n-1$. Since, by Theorem 6.9, the spaces Val_i^∞ are quotients of $\Omega_p^{i,n-i-1}$, it follows that also $m(\mathrm{Val}_i^\infty, \lambda)$ is finite. Moreover, since $m(\mathrm{Val}_i, \lambda) = m(\mathrm{Val}_i^\infty, \lambda)$, we deduce from Theorem 6.9 that

$$m(\mathrm{Val}_i, \lambda) = (-1)^{n-i} m(\Lambda^i V_{\mathbb{C}}, \lambda) + \sum_{j=0}^{n-i-1} (-1)^{n-1-i-j} m(\Omega_p^{i,j}, \lambda) \tag{6.21}$$

and another application of (6.20) yields

$$\begin{aligned} m(\Omega_p^{i,j}, \lambda) \\ &= m\Big(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)}(\Lambda^i W \otimes \Lambda^j W), \lambda\Big) - m\Big(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)}(\Lambda^{i-1} W \otimes \Lambda^{j-1} W), \lambda\Big), \end{aligned}$$

where $W \cong W^*$ denotes the complex standard representation of $\mathrm{SO}(n-1)$. In order to further simplify the last expression, we require a consequence of the second determinantal formula, Theorem 6.4. In order to state this simple auxiliary result, let $\#(\lambda, j)$ denote the number of integers in λ which are equal to j.

Lemma 6.11. *If $i, j \in \mathbb{N}$ are such that $n/2 \leq i \leq n$ and $i + j \leq n$, then*

$$E_i E_j - E_{i-1} E_{j-1} = \sum_{\lambda} \mathrm{char}\,\overline{\Gamma}_\lambda, \tag{6.22}$$

where the sum ranges over all tuples of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ satisfying (6.2) *and*

$$\lambda_1 \leq 2, \qquad \#(\lambda, 1) = n - i - j, \qquad \#(\lambda, 2) \leq j. \tag{6.23}$$

Proof. The conjugate of an $\lfloor n/2 \rfloor$-tuple of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ satisfying (6.2) and (6.23) is given by $\mu = (\mu_1, \mu_2)$, where $\mu_2 = \#(\lambda, 2) \leq j$ and $\mu_1 - \mu_2 = \#(\lambda, 1) = n - i - j$. Thus, by Theorem 6.4,

$$\mathrm{char}\,\overline{\Gamma}_\lambda = \det \begin{pmatrix} E_{\mu_2+k} & E_{\mu_2+k+1} + E_{\mu_2+k-1} \\ E_{\mu_2-1} & E_{\mu_2} + E_{\mu_2-2} \end{pmatrix},$$

where $k = n - i - j$. Since, by (6.4), $E_{n-i} = E_i$, we therefore obtain for the right hand side of (6.22),

$$\begin{aligned}\sum_{\lambda} \operatorname{char} \overline{\Gamma}_\lambda &= \sum_{\mu_2=0}^{j} \big(E_{\mu_2+k}(E_{\mu_2}+E_{\mu_2-2}) - E_{\mu_2-1}(E_{\mu_2+k+1}+E_{\mu_2+k-1})\big) \\ &= E_{n-i}E_j - E_{n-(i-1)}E_{j-1} = E_iE_j - E_{i-1}E_{j-1}. \qquad \square\end{aligned}$$

An application of Lemma 6.11 with n replaced by $n-1$ and $0 \le j \le n-1-i$ now yields

$$m(\Omega_p^{i,j},\lambda) = \sum_{\sigma} m\Big(\operatorname{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \overline{\Gamma}_\sigma, \lambda\Big), \tag{6.24}$$

where the sum ranges over all $k := \lfloor (n-1)/2 \rfloor$-tuples of non-negative highest weights $\sigma = (\sigma_1, \dots, \sigma_k)$ of $\mathrm{SO}(n-1)$ such that

$$\sigma_1 \le 2, \qquad \#(\sigma,1) = n-1-i-j, \qquad \#(\sigma,2) \le j.$$

Let $\mathscr{W}_i$ denote the union of these k-tuples of non-negative highest weights of $\mathrm{SO}(n-1)$. By (6.21) and (6.24), we now have

$$m(\mathrm{Val}_i,\lambda) = (-1)^{n-i} m(\Lambda^i V_{\mathbb{C}},\lambda) + \sum_{\sigma \in \mathscr{W}_i} (-1)^{|\sigma|} m\Big(\operatorname{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \overline{\Gamma}_\sigma, \lambda\Big). \tag{6.25}$$

The Frobenius Reciprocity Theorem (Theorem 6.5), the branching formula from Theorem 6.6, and the definition of $\overline{\Gamma}_\sigma$ yield

$$\sum_{\sigma \in \mathscr{W}_i} (-1)^{|\sigma|} m\Big(\operatorname{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \overline{\Gamma}_\sigma, \lambda\Big) = \sum_{\mu} (-1)^{|\mu|},$$

where the sum on the right ranges over all tuples $\mu = (\mu_1, \dots, \mu_k)$ with $\mu_{n-i} = 0$ and

$$\begin{cases} \lambda_1^* \ge \mu_1 \ge \lambda_2^* \ge \mu_2 \ge \ldots \ge \mu_{k-1} \ge \lambda^*_{\lfloor n/2 \rfloor} \ge |\mu_k| & \text{for odd } n, \\ \lambda_1^* \ge \mu_1 \ge \lambda_2^* \ge \mu_2 \ge \ldots \ge \mu_k \ge \lambda^*_{n/2} & \text{for even } n. \end{cases}$$

Here, $\lambda_1^* := \min\{\lambda_1, 2\}$ and $\lambda_j^* := |\lambda_j|$ for every $1 < j \le \lfloor n/2 \rfloor$. Thus, if $\lambda^*_{n-i+1} > 0$, then there is no such tuple μ. However, if $\lambda^*_{n-i+1} = 0$, then

$$\sum_{\sigma \in \mathscr{W}_i} (-1)^{|\sigma|} m\Big(\operatorname{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \overline{\Gamma}_\sigma, \lambda\Big) = \prod_{j=1}^{n-i-1} \sum_{\mu_j = \lambda^*_{j+1}}^{\lambda_j^*} (-1)^{\mu_j}.$$

This product vanishes if the λ_j^*, $j = 1, \dots, n-i$, do not all have the same parity. If the λ_j^* all do have the same parity, then the product above equals $(-1)^{(n-i-1)\lambda_1^*}$. Hence, we obtain for $i > n/2$,

$$\sum_{\sigma \in \mathscr{W}_i} (-1)^{|\sigma|} m\Big(\operatorname{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \overline{\Gamma}_\sigma, \lambda\Big) = \begin{cases} (-1)^{n-i-1} & \text{if } \Gamma_\lambda \cong \Lambda^{n-i} V_{\mathbb{C}}, \\ 1 & \text{if } \lambda \text{ satisfies (i), (ii), (iii)}, \\ 0 & \text{otherwise.} \end{cases}$$

If $i = n/2$, in which case n must be even, then

$$\sum_{\sigma \in \mathscr{W}_i} (-1)^{|\sigma|} m\Big(\operatorname{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \overline{\Gamma}_\sigma, \lambda\Big) = \begin{cases} (-1)^{i-1} & \text{if } \lambda = (1,\dots,1,\pm 1), \\ 1 & \text{if } \lambda \text{ satisfies (i), (ii) and (iii)}, \\ 0 & \text{otherwise}. \end{cases}$$

Plugging these expressions into (6.25) and using that $\Lambda^{n/2} V_{\mathbb{C}} = \Gamma_{(1,\dots,1)} \oplus \Gamma_{(1,\dots,1,-1)}$ if n is even and $\Lambda^{n-i} V_{\mathbb{C}} \cong \Lambda^i V_{\mathbb{C}}$ for every $i \in \{0,\dots,n\}$, completes the proof of Theorem 6.2.

6.5 Special cases and applications

In this final section we discuss numerous special cases and recent applications of Theorems 6.1 and 6.2. In particular, these results should illustrate the variety of implications that the study of valuations has for different areas.

6.5.1 Special cases

The following is a list of special cases and immediate consequences of Theorem 6.1.

- If $\Gamma = \Gamma_{(0,\dots,0)} \cong \mathbb{C}$ is the trivial representation, then the subspace of $\mathrm{SO}(n)$ equivariant valuations in $\Gamma\mathrm{Val}$ coincides with the vector space $\mathrm{Val}^{\mathrm{SO}(n)}$ of all continuous and rigid motion invariant scalar valuations on $\mathscr{K}^n$. By (6.1) and Theorem 6.1, we have
$$\dim \mathrm{Val}^{\mathrm{SO}(n)} = \sum_{i=0}^{n} \dim \mathrm{Val}_i^{\mathrm{SO}(n)} = n+1.$$
Together with the fact that intrinsic volumes of different degrees of homogeneity are linearly independent, this yields Hadwiger's characterization theorem.
- Let $\Gamma = \Gamma_{(1,0,\dots,0)} \cong V_{\mathbb{C}}$ be the complex standard representation of $\mathrm{SO}(n)$. By Theorem 6.1, there is *no* non-trivial continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant vector valued valuation. While this result seems of no particular interest at first, it directly implies a classical characterization of the Steiner point map by Schneider [67]. Recall that the Steiner point $s(K)$ of a convex body $K \in \mathscr{K}^n$ is defined by
$$s(K) = \frac{1}{n} \int_{S^{n-1}} u h(K,u)\, du,$$
where $h(K,\cdot)$ is the support function of K and du denotes integration with respect to the rotation invariant probability measure on the unit sphere.

  Theorem ([67]). *A map $\phi : \mathscr{K}^n \to \mathbb{R}^n$ is a continuous, rigid motion equivariant valuation if and only if ϕ is the Steiner point map.*

Proof. It is well known that $s : \mathscr{K}^n \to \mathbb{R}^n$ has the asserted properties (cf. [71]). Assume that ϕ is another such valuation. Then $\phi - s$ is a continuous, translation invariant, and $\mathrm{SO}(n)$ *equivariant* valuation and, hence, $\phi - s = 0$. □

- Next, let $\Gamma = \mathrm{Sym}^k V_{\mathbb{C}}$ be the space of symmetric tensors of rank $k \geq 2$ over $V_{\mathbb{C}}$. The subspace of $\mathrm{SO}(n)$ equivariant valuations in $\Gamma\mathrm{Val}$ is then just the vector space $\mathrm{TVal}^{k,\mathrm{SO}(n)}$ of all continuous, translation invariant and $\mathrm{SO}(n)$ equivariant valuations on $\mathscr{K}^n$ with values in $\mathrm{Sym}^k V_{\mathbb{C}}$.

  By Theorem 6.1 and (6.5), we have, for $1 \leq i \leq n-1$,

$$\dim \mathrm{TVal}_i^{k,\mathrm{SO}(n)} = \begin{cases} k/2+1 & \text{if } k \text{ is even,} \\ (k-1)/2 & \text{if } k \text{ is odd.} \end{cases}$$

  In order to recall a basis of the space $\mathrm{TVal}_i^{k,\mathrm{SO}(n)}$, let $e_1, \dots, e_n$ be an orthonormal basis of $V_{\mathbb{C}}$ and denote by $Q = \sum_{l=1}^n e_l^2 \in \mathrm{Sym}^2 V_{\mathbb{C}}$ the metric tensor. Moreover, for $s \in \mathbb{N}$, $1 \leq i \leq n-1$ and $K \in \mathscr{K}^n$, let

$$\Psi_{i,s}(K) = \int_{S^{n-1}} u^s \, dS_i(K,u),$$

  where, as usual, u^s denotes the s-fold symmetric tensor product of $u \in S^{n-1}$ and $S_i(K,\cdot)$ denotes the ith area measure of the body K. Then the valuations $Q^r\Psi_{i,s}$, where $r,s \geq 0$, $s \neq 1$, and $2r+s=k$, form a basis of the space $\mathrm{TVal}_i^{k,\mathrm{SO}(n)}$. The dimensions and bases of the spaces $\mathrm{TVal}_i^{k,\mathrm{SO}(n)}$ and more general spaces of *isometry covariant* tensor valuations were first determined in the articles [3, 42]. For more information, we refer to Chapters 2, 3, and 5 of this volume.
- In [83], Yang posed the problem to classify valuations compatible with some subgroup of affine transformations with values in *skew-symmetric* tensors of rank two. Using Theorem 6.1, we can give a partial solution to Yang's problem. Taking $\Gamma = \Gamma_{(1,1,0\dots,0)} = \Lambda^2 V_{\mathbb{C}}$, it follows that there is *no* non-trivial continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant valuation with values in Γ.

  In contrast to this negative result, we note that Bernig [15] constructed for each $0 \leq k \leq i \leq n-1$ a family of continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant valuations of degree i with values in $\Lambda^k V_{\mathbb{C}} \otimes \Lambda^k V_{\mathbb{C}} = \Gamma_{(2,\dots,2,0,\dots,0)}$. By Theorem 6.1, Bernig's curvature tensor valuations are (up to scalar multiples) the uniquely determined $\mathrm{SO}(n)$ equivariant valuations in $\Gamma\mathrm{Val}_i$.

6.5.2 *Even valuations and the cosine transform*

As already mentioned in the introduction, Theorem 6.2 was first proved for *even* valuations by Alesker and Bernstein [11] using a fundamental relation between even translation invariant valuations and the cosine transform on Grassmannians. This relation will be the topic of this subsection.

First recall that the cosine of the angle between $E,F \in G(n,i)$, $1 \le i \le n-1$, is given by $|\cos(E,F)| = \mathrm{vol}_i(M|E)$, where M is an arbitrary subset of F with $\mathrm{vol}_i(M) = 1$. The *cosine transform* on smooth functions is the $\mathrm{SO}(n)$ equivariant linear operator $\mathrm{C}_i : C^\infty(G(n,i)) \to C^\infty(G(n,i))$ defined by

$$(\mathrm{C}_i f)(F) = \int_{G(n,i)} |\cos(E,F)| f(E)\, dE,$$

where integration is with respect to the Haar probability measure on $G(n,i)$.

Next, we also briefly recall the *Klain map* (for more information, see Chapter 1). For $1 \le i \le n-1$, Klain defined a map $\mathrm{Kl}_i : \mathrm{Val}_i^+ \to C(G(n,i))$, $\phi \mapsto \mathrm{Kl}_i\phi$, as follows: For $\phi \in \mathrm{Val}_i^+$ and every $E \in G(n,i)$, consider the restriction ϕ_E of ϕ to convex bodies in E. This is a continuous translation invariant valuation of degree i in E and, thus, a constant multiple of i-dimensional volume, that is, $\phi_E = (\mathrm{Kl}_i\phi)(E)\,\mathrm{vol}_i$. This gives rise to a function $\mathrm{Kl}_i\phi \in C(G(n,i))$, called the *Klain function* of the valuation ϕ. It is not difficult to see that Kl_i is $\mathrm{SO}(n)$ equivariant and maps smooth valuations to smooth ones. Moreover, by an important result of Klain [46], the Klain map Kl_i is *injective* for every $i \in \{1,\ldots,n-1\}$ (see also [64]).

Now, for $1 \le i \le n-1$, consider the map $\mathrm{Cr}_i : C^\infty(G(n,i)) \to \mathrm{Val}_i^{+,\infty}$, defined by

$$(\mathrm{Cr}_i f)(K) = \int_{G(n,i)} \mathrm{vol}_i(K|E) f(E)\, dE.$$

Clearly, Cr_i is an $\mathrm{SO}(n)$ equivariant linear operator. Moreover, if $F \in G(n,i)$, then, for any $f \in C^\infty(G(n,i))$ and convex body $K \subseteq F$,

$$(\mathrm{Cr}_i f)(K) = \mathrm{vol}_i(K) \int_{G(n,i)} |\cos(E,F)| f(E)\, dE.$$

In other words, the Klain function of the valuation $\mathrm{Cr}_i f$ is the cosine transform $\mathrm{C}_i f$ of f. Hence, the image of the cosine transform is contained in the image of the Klain map. From the main result of [11] and an application of the Casselman-Wallach Theorem [23], Alesker [6] proved that, in fact, these images coincide.

Theorem 6.12 ([11, 6]). *Let* $1 \le i \le n-1$. *The image of the restriction of the Klain map to smooth valuations* $\mathrm{Kl}_i : \mathrm{Val}_i^{+,\infty} \to C^\infty(G(n,i))$ *coincides with the image of the cosine transform* $\mathrm{C}_i : C^\infty(G(n,i)) \to C^\infty(G(n,i))$.

Theorem 6.12 was essential in the discovery of algebraic structures on the space of continuous translation invariant even valuations (see [6] and Chapter 4 for more detailed information). Using a variant of Theorem 6.12 combined with certain computations from the proof of Alesker's Irreducibility Theorem [5], Alesker and Bernstein [11] gave the following precise description of the range of the cosine transform in terms of the decomposition under the action of $\mathrm{SO}(n)$. This description is equivalent to Theorem 6.2 for even valuations.

Theorem 6.13 ([11]). *Let $1 \leq i \leq n-1$. The image of the cosine transform consists of irreducible representations of* $\mathrm{SO}(n)$ *with highest weights $\lambda = (\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ satisfying* (6.2), (6.7)*, and $|\lambda_2| \leq 2$.*

As a concluding remark for this subsection we note that the structural analysis of intertwining transforms on Grassmannians, such as Radon- and cosine transforms, has a long tradition in integral geometry and is still to this day a focus of research (see, e.g., [13, 26, 34, 61, 62, 65, 84]).

6.5.3 Unitary vector valued valuations

We have seen in Section 6.5.1 that there exists no non-trivial continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant valuation from $\mathscr{K}^n$ to $\mathbb{R}^n$. As Wannerer [81] discovered, the situation changes when the translation invariant vector valued valuations are no longer required to be equivariant with respect to $\mathrm{SO}(n)$ but merely with respect to the smaller group $\mathrm{U}(n)$ (for classifications of vector valued valuations in the non-translation invariant case, see [41, 37, 50]).

Since the natural domain of the unitary group $\mathrm{U}(n)$ is $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we consider in this (and only in this) subsection valuations defined on the space $\mathscr{K}^{2n}$ of convex bodies in $\mathbb{R}^{2n}$. In particular, in the following also the spaces Val, Val_i, ... will refer to translation invariant continuous valuations on $\mathscr{K}^{2n}$.

We denote by Vec the (real) vector space of continuous and translation invariant valuations $\phi : \mathscr{K}^{2n} \to \mathbb{C}^n$ and we write $\mathrm{Vec}^{\mathrm{U}(n)}$ for its subspace of $\mathrm{U}(n)$ equivariant valuations. It follows from McMullen's decomposition (6.1) that

$$\mathrm{Vec} = \bigoplus_{0 \leq i \leq 2n} \mathrm{Vec}_i,$$

where as usual Vec_i denotes the subspace of valuations of degree i.

Theorem 6.14 ([81]). *Suppose that $0 \leq i \leq 2n$. Then*

$$\dim_{\mathbb{R}} \mathrm{Vec}_i^{\mathrm{U}(n)} = 2 \min \left\{ \left\lfloor \frac{i}{2} \right\rfloor, \left\lfloor \frac{2n-i}{2} \right\rfloor \right\}. \tag{6.26}$$

Proof. We put $V = \mathbb{R}^{2n}$ and write again $V_{\mathbb{C}}$ for the complexification of V. Since Vec_i is isomorphic as vector space to $\mathrm{Val}_i \otimes V$, we have, by Theorem 6.2,

$$\dim_{\mathbb{C}} (\mathrm{Vec}_i \otimes \mathbb{C})^{\mathrm{U}(n)} = \dim_{\mathbb{C}} (\mathrm{Val}_i \otimes V_{\mathbb{C}})^{\mathrm{U}(n)} = \sum_{\lambda} \dim_{\mathbb{C}} (\Gamma_\lambda \otimes V_{\mathbb{C}})^{\mathrm{U}(n)}, \tag{6.27}$$

where the sum ranges over all highest weights $\lambda = (\lambda_1, \dots, \lambda_n)$ of $\mathrm{SO}(2n)$ satisfying

(i) $\lambda_j = 0$ for $j > \min\{i, 2n-i\}$; (ii) $|\lambda_j| \neq 1$ for $1 \leq j \leq n$; (iii) $|\lambda_2| \leq 2$.

In order to determine the sum on the right hand side of (6.27), we first apply a formula of Klimyk [48] to $\Gamma_\lambda \otimes V_{\mathbb{C}}$ to obtain the decomposition of this tensor product into $\mathrm{SO}(2n)$ irreducible subspaces:

$$\Gamma_\lambda \otimes V_{\mathbb{C}} = \bigoplus_{\nu} \Gamma_\nu, \tag{6.28}$$

where the sum ranges over all $\nu = \lambda \pm e_k$ for some n-tuple $e_k = (0,\ldots,0,1,0,\ldots,0)$.

Next, a theorem of Helgason (see, e.g., [78, p. 151]) applied to the symmetric space $\mathrm{SO}(2n)/\mathrm{U}(n)$ implies that the highest weights $\nu = (\nu_1,\ldots,\nu_n)$ we need to consider have to satisfy the following additional condition

$$\begin{cases} \nu_1 = \nu_2 \geq \nu_3 = \nu_4 \geq \cdots \geq \nu_{n-1} = \nu_n & \text{if } n \text{ is even,} \\ \nu_1 = \nu_2 \geq \nu_3 = \nu_4 \geq \cdots \geq \nu_{n-2} = \nu_{n-1} \geq \nu_n = 0 & \text{if } n \text{ is odd.} \end{cases} \tag{6.29}$$

since

$$\dim_{\mathbb{C}} \Gamma_\nu^{\mathrm{U}(n)} = \begin{cases} 1 & \text{if } \nu \text{ satisfies (6.29),} \\ 0 & \text{otherwise.} \end{cases}$$

From this, conditions (i), (ii), (iii), and (6.28), it follows now that $(\Gamma_\lambda \otimes V_{\mathbb{C}})^{\mathrm{U}(n)}$ is non-trivial if and only if λ if of the form

$$\lambda_1 = 3, \qquad \lambda_2 = \cdots = \lambda_{2m} = 2, \qquad \lambda_j = 0 \text{ for } j > 2m$$

for some integer $1 \leq m \leq \min\{\lfloor \frac{i}{2} \rfloor, \lfloor \frac{2n-i}{2} \rfloor\}$ and that in this case

$$\dim_{\mathbb{C}} (\Gamma_\lambda \otimes V_{\mathbb{C}})^{\mathrm{U}(n)} = 2.$$

To see this, fix some λ satisfying (i), (ii), and (iii) and suppose that $\nu = \lambda + e_k$ for some k. If ν satisfies in addition (6.29), then necessarily $k = 2$, $\nu_1 = \nu_2 = 3$, and, thus, $\lambda_1 = 3$, $\lambda_2 = \cdots = \lambda_{2m} = 2$, and $\lambda_j = 0$ for $j > 2m$. If $\nu = \lambda - e_k$, then (6.29) forces $k = 1$, $\nu_1 = \nu_2 = 2$, and, again, $\lambda_1 = 3$, $\lambda_2 = \cdots = \lambda_{2m} = 2$, and $\lambda_j = 0$ for $j > 2m$.

Finally, since $\dim_{\mathbb{R}} \mathrm{Vec}_i^{\mathrm{U}(n)} = \dim_{\mathbb{C}} (\mathrm{Vec}_i \otimes \mathbb{C})^{\mathrm{U}(n)}$, we obtain now from (6.27) the desired dimension formula. □

As an application of Theorem 6.14, Wannerer [81] obtained the following new characterization of the Steiner point map in hermitian vector spaces.

Corollary 6.15 ([81]). *A map* $\phi : \mathscr{K}^{2n} \to \mathbb{C}^n$ *is a continuous, translation and* $\mathrm{U}(n)$ *equivariant map satisfying* $\phi(K+L) = \phi(K) + \phi(L)$ *for all* $K, L \in \mathscr{K}^{2n}$ *if and only if* ϕ *is the Steiner point map.*

Corollary 6.15 is a generalization of a similar result by Schneider [66], where the unitary group $\mathrm{U}(n)$ is replaced by the lager group $\mathrm{SO}(2n)$.

6.5.4 The symmetry of bivaluations

In this subsection we outline how Theorem 6.2 can be used to prove a remarkable symmetry property of rigid motion invariant continuous bivaluations which in turn has important consequences in geometric tomography and the study of geometric inequalities for Minkowski valuations.

Definition. *A map $\varphi : \mathcal{K}^n \times \mathcal{K}^n \to \mathbb{C}$ is called a* bivaluation *if φ is a valuation in both arguments. We call φ* translation biinvariant *if φ is invariant under independent translations of its arguments and say that φ has* bidegree (i, j) *if $\varphi(aK, bL) = a^i b^j \varphi(K, L)$ for all $K, L \in \mathcal{K}^n$ and $a, b > 0$. If G is some group of linear transformations of $\mathbb{R}^n$, we say φ is G* invariant *provided $\varphi(gK, gL) = \varphi(K, L)$ for all $K, L \in \mathcal{K}^n$ and $g \in G$.*

The problem to classify invariant bivaluations was already posed in the book by Klain and Rota [47] on geometric probability. A first such classification was obtained by Ludwig [54] in connection with notions of surface area in normed spaces. However, here we want to discuss a structural property of rigid motion invariant bivaluations. To this end we denote by BVal the vector space of all continuous translation biinvariant complex valued bivaluations. An immediate consequence of McMullen's decomposition (6.1) of the space Val is an analogous result for the space BVal:

$$\mathrm{BVal} = \bigoplus_{i,j=0}^{n} \mathrm{BVal}_{i,j}, \tag{6.30}$$

where $\mathrm{BVal}_{i,j}$ denotes the subspace of all bivaluations of bidegree (i, j). In turn, (6.30) can be used to show that BVal becomes a Banach space when endowed with the norm

$$\|\varphi\| = \sup\{|\varphi(K, L)| : K, L \subseteq B^n\}.$$

The following symmetry property of rigid motion invariant bivaluations was established in [10].

Theorem 6.16. *Let $0 \le i \le n$. Then*

$$\varphi(K, L) = \varphi(L, K) \qquad \text{for all } K, L \in \mathcal{K}^n \tag{6.31}$$

holds for all $\mathrm{SO}(n)$ *invariant* $\varphi \in \mathrm{BVal}_{i,i}$ *if and only if* $(i, n) \neq (2k+1, 4k+2)$*,* $k \in \mathbb{N}$*. Moreover,* (6.31) *holds for all* $\mathrm{O}(n)$ *invariant* $\varphi \in \mathrm{BVal}_{i,i}$*.*

In the following we outline the proof of the 'if' part of the first statement of Theorem 6.16 using Theorem 6.2. We refer to [10, p. 768], for the construction of an $\mathrm{SO}(n)$ invariant bivaluation $\zeta \in \mathrm{BVal}_{i,i}$, where $(i, n) = (2k+1, 4k+2)$ for some $k \in \mathbb{N}$, such that $\zeta(K, L) \neq \zeta(L, K)$ for some pair of convex bodies. Similarly, we will not treat $\mathrm{O}(n)$ invariant bivaluations here. For the proof of (6.31) in this case, a description of the irreducible representations of $\mathrm{O}(n)$ in terms of the irreducible representations of $\mathrm{SO}(n)$ is needed and we also refer to [10] for that.

Now, assume that $\varphi \in \mathrm{BVal}_{i,i}$ is $\mathrm{SO}(n)$ invariant and that $(i,n) \neq (2k+1,4k+2)$. Moreover, since for $i=0$ or $i=n$, (6.31) follows easily from Proposition 6.7, we may assume that $0<i<n$. Now, by Theorem 6.2,

$$\mathrm{BVal}_{i,i}^{\mathrm{SO}(n)} \cong (\mathrm{Val}_i \otimes \mathrm{Val}_i)^{\mathrm{SO}(n)} \cong \bigoplus_{\gamma,\lambda} (\Gamma_\gamma \otimes \Gamma_\lambda)^{\mathrm{SO}(n)}, \tag{6.32}$$

where the sum ranges of all highest weights γ and λ of $\mathrm{SO}(n)$ satisfying conditions (i), (ii), and (iii) from Theorem 6.1. (In fact, in order to make the isomorphisms in (6.32) precise, we have to consider the dense subset of all bivaluations with finite $\mathrm{SO}(n) \times \mathrm{SO}(n)$ orbit, compare [10, p. 766]).

Since $(i,n) \neq (2k+1,4k+2)$, it follows from condition (i) and Lemma 6.3, that all irreducible representations of $\mathrm{SO}(n)$ which appear in (6.32) are real and, thus, self-dual. Hence, we have

$$(\Gamma_\gamma \otimes \Gamma_\lambda)^{\mathrm{SO}(n)} \cong \mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma_\gamma, \Gamma_\lambda) \cong \mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma_\gamma \otimes \Gamma_\lambda, \mathbb{C}).$$

Since Γ_γ and Γ_λ are irreducible, Schur's lemma implies that

$$\dim \mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma_\gamma, \Gamma_\lambda) = \begin{cases} 1 & \text{if } \gamma = \lambda, \\ 0 & \text{if } \gamma \neq \lambda. \end{cases}$$

Using again that the $\mathrm{SO}(n)$ irreducible representations which we consider are real, the space

$$\mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma_\lambda \otimes \Gamma_\lambda, \mathbb{C}) = (\mathrm{Sym}^2 \Gamma_\lambda)^{\mathrm{SO}(n)} \oplus (\Lambda^2 \Gamma_\lambda)^{\mathrm{SO}(n)}$$

of $\mathrm{SO}(n)$ invariant bilinear forms on Γ_λ must coincide with $(\mathrm{Sym}^2 \Gamma_\lambda)^{\mathrm{SO}(n)}$. Hence,

$$\mathrm{BVal}_{i,i}^{\mathrm{SO}(n)} \cong \bigoplus_{\lambda} (\mathrm{Sym}^2 \Gamma_\lambda)^{\mathrm{SO}(n)}$$

which implies (6.31).

Using partial derivation operators on bivaluations (the definition of which is motivated by the operator $\Lambda : \mathrm{Val} \to \mathrm{Val}$ defined in (6.13)), one can easily obtain a corollary of Theorem 6.16 which is particularly useful for applications. To state this result, define for $m = 1,2$ the operators $\Lambda_m : \mathrm{BVal} \to \mathrm{BVal}$ by

$$(\Lambda_1 \phi)(K,L) = \left.\frac{d}{dt}\right|_{t=0} \phi(K + tB^n, L)$$

and

$$(\Lambda_2 \phi)(K,L) = \left.\frac{d}{dt}\right|_{t=0} \phi(K, L + tB^n).$$

Clearly, if $\phi \in \mathrm{BVal}_{i,j}$, then $\Lambda_1 \phi \in \mathrm{BVal}_{i-1,j}$ and $\Lambda_2 \phi \in \mathrm{BVal}_{i,j-1}$.

Also define an operator $\mathrm{T} : \mathrm{BVal} \to \mathrm{BVal}$ by

$$(\mathrm{T}\phi)(K,L) = \phi(L,K).$$

Note that, by Theorem 6.16, the restriction of T to $\mathrm{BVal}_{i,i}^{\mathrm{O}(n)}$ acts as the identity.

Corollary 6.17. *Suppose that* $0 \le j \le n$ *and* $0 \le i \le j$*. Then the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{BVal}_{j,j}^{\mathrm{O}(n)} & \xrightarrow{\mathrm{T}=\mathrm{Id}} & \mathrm{BVal}_{j,j}^{\mathrm{O}(n)} \\ \Big\downarrow \Lambda_2^{j-i} & & \Big\downarrow \Lambda_1^{j-i} \\ \mathrm{BVal}_{j,i}^{\mathrm{O}(n)} & \xrightarrow{\mathrm{T}} & \mathrm{BVal}_{i,j}^{\mathrm{O}(n)}. \end{array}$$

Corollary 6.17 has found several interesting applications in connection with *Minkowski valuations*, that is, maps $\Phi : \mathscr{K}^n \to \mathscr{K}^n$ such that

$$\Phi(K) + \Phi(L) = \Phi(K \cup L) + \Phi(K \cap L),$$

whenever $K \cup L$ is convex and addition here is the usual Minkowski addition. Recently, Ludwig [53] started an important line of research concerned with the classification of Minkowski valuations intertwining *linear* transformations, see [1, 2, 35, 51, 54, 75, 80] and Chapter 7. However, first investigations of Minkowski valuations by Schneider [68] in 1974 were concentrating on rigid motion compatible valuations which are still a focus of intensive research, see [44, 72, 74, 76, 77].

In order to explain how Corollary 6.17 can be used in the theory of Minkowski valuations, let MVal denote the set of all continuous and translation invariant Minkowski valuations. Parapatits and the author [63] proved that for any $\Phi \in \mathrm{MVal}$, there exist $\Phi^{(j)} \in \mathrm{MVal}$, where $0 \le j \le n$, such that for every $K \in \mathscr{K}^n$ and $t \ge 0$,

$$\Phi(K + tB^n) = \sum_{j=0}^{n} t^{n-j} \Phi^{(j)}(K)$$

This Steiner type formula shows that an analogue of the operator Λ from (6.13) can be defined for Minkowski valuations $\Lambda :\ \mathrm{MVal} \ \to\ \mathrm{MVal}$ by

$$h((\Lambda\Phi)(K),u) = \frac{d}{dt}\bigg|_{t=0} h(\Phi(K + tB^n,u), \qquad u \in S^{n-1}.$$

For $K, L \in \mathscr{K}^n$, we use $W_i(K,L)$ to denote the mixed volume $V(K[n-i-1], B^n[i], L)$.

Corollary 6.18 ([63]). *Suppose that* $\Phi_j \in \mathbf{MVal}_j$*,* $2 \le j \le n-1$*, is* $\mathrm{O}(n)$ *equivariant. If* $1 \le i \le j+1$*, then*

$$W_{n-i}(K, \Phi_j(L)) = \frac{(i-1)!}{j!} W_{n-1-j}(L, (\Lambda^{j+1-i}\Phi_j)(K))$$

for every $K, L \in \mathscr{K}^n$*.*

Proof. For $K,L \in \mathscr{K}^n$, define $\phi \in \mathbf{BVal}_{j,j}^{\mathrm{O}(n)}$ by $\phi(K,L) = W_{n-1-j}(K,\Phi_j(L))$. Then it is not difficult to show that

$$W_{n-i}(K,\Phi_j(L)) = \frac{(i-1)!}{j!}(\Lambda_1^{j+1-i}\phi)(K,L).$$

Thus, an application of Corollary 6.17 completes the proof. □

Corollary 6.18 as well as variants and generalizations of this result have been a critical tool in the proof of log-concavity properties of rigid motion compatible Minkowski valuations (see [2, 10, 14, 56, 63, 73, 74]). Corollary 6.18 was also important in the solution of injectivity questions for certain Minkowski valuations arising naturally from tomographic data, more precisely, the so-called mean section operators, defined and investigated by Goodey and Weil [30, 31, 32].

6.5.5 *Miscellaneous applications*

In this short final subsection we collect three more applications of Theorem 6.2 in various contexts. We will not outline proofs here but rather refer to the original source material. We begin with the following fact about Minkowski valuations.

Proposition 6.19 ([10]). *If $\Phi \in \mathrm{MVal}$ is $\mathrm{SO}(n)$ equivariant, then Φ is also $\mathrm{O}(n)$ equivariant.*

Note that, by Proposition 6.19, Corollary 6.18 in fact holds for $\mathrm{SO}(n)$ equivariant Minkowski valuations.

The proof of Proposition 6.19 is based on the simple fact that any continuous Minkowski valuation which is translation invariant and $\mathrm{SO}(n)$ equivariant is uniquely determined by a *spherical* valuation.

Definition. *For $0 \le i \le n$, the subspaces $\mathrm{Val}_i^{\mathrm{sph}}$ and $\mathrm{Val}_i^{\infty,\mathrm{sph}}$ of translation invariant continuous and smooth* spherical *valuations of degree i are defined as the closure (w.r.t. the respective topologies) of the direct sum of all $\mathrm{SO}(n)$ irreducible subspaces in Val_i and Val_i^∞ which contain a non-zero $\mathrm{SO}(n-1)$ invariant valuation.*

Since, by Theorem 6.2, the space Val_i is multiplicity free under the action of $\mathrm{SO}(n)$, it follows from basic facts about spherical representations (see [77]) that

$$\mathrm{Val}_i^{\infty,\mathrm{sph}} = \mathrm{cl}_\infty \bigoplus_{k\in\mathbb{N}} \Gamma_{(k,0,\dots,0)}. \tag{6.33}$$

Spherical valuations also play an important role in the recent article [20] by Bernig and Hug, where they compute kinematic formulas for translation invariant and $\mathrm{SO}(n)$ equivariant tensor valuations. It follows from Theorem 6.2 and (6.5) that tensor valuations from $\mathrm{TVal}_i^{\mathrm{SO}(n)}$ are also determined by spherical valuations. In order to bring the algebraic machinery from modern integral geometry into play in the computation of the kinematic formulas in [20], the main step was to determine the Alesker–Fourier transform $\mathbb{F}$ (see [9]) of spherical valuations. Note that since $\mathbb{F}$ is a linear and $\mathrm{SO}(n)$ equivariant map, (6.33) and Schur's lemma imply that the restriction of $\mathbb{F}$ to $\mathrm{Val}_i^{\infty,\mathrm{sph}}$ is determined by a sequence of *multipliers* which was computed in [20].

As a final application of Theorem 6.2, we mention that it was used by Bernig and Solanes [21] to give a complete classification of valuations on the quaternionic plane which are invariant under the action of the group $\mathrm{Sp}(2)\mathrm{Sp}(1)$. Note that since $\mathrm{Sp}(2)\mathrm{Sp}(1)$ contains $-\mathrm{Id}$ all such valuations are even and, thus, determined by their Klain functions. For the proof of their classification theorem, Bernig and Solanes now identify certain $\mathrm{Sp}(2)\mathrm{Sp}(1)$ invariant functions on the Grassmannian as eigenfunctions of the Laplace-Beltrami operator on $G(n,i)$ and determine the $\mathrm{SO}(n)$ irreducible subspaces that they are contained in. Then Theorem 6.2 is applied to show that these subspaces also appear in Val_i. Finally, the computation of $\dim \mathrm{Val}_i^{\mathrm{Sp}(2)\mathrm{Sp}(1)}$ from [17] is used to show that the so-constructed valuations form a basis.

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