VOLUME INEQUALITIES
FOR ASYMMETRIC WULFF SHAPES

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Abstract

Sharp reverse affine isoperimetric inequalities for asymmetric Wulff shapes and their polars are established, along with the characterization of all extremals. These new inequalities have as special cases previously obtained simplex inequalities by Ball, Barthe and Lutwak, Yang, and Zhang. In particular, they provide the solution to a problem by Zhang.

1. Introduction

Over the last decades considerable progress has been made in establishing reverse (affine) isoperimetric inequalities, i.e., inequalities which usually have simplices or, in the symmetric case, cubes and their polars, as extremals. By the end of the 1980s, only a very small number of significant reverse inequalities had been obtained and no systematic approach towards these inequalities seemed within reach. A breakthrough occurred when Ball [1, 2] discovered a reformulation of the Brascamp–Lieb inequality by exploiting the notion of *isotropic measure* which, in turn, is connected to a variety of extremal problems in geometric analysis (see [14, 15, 18, 30, 35]). Ball's geometric Brascamp–Lieb inequality was tailor-made to establish several important new reverse inequalities.

Settling the uniqueness of the extremals for the newly obtained reverse isoperimetric inequalities with an underlying *discrete* isotropic measure was made possible only through an optimal transport approach of Barthe [4, 6] towards establishing not only the Brascamp–Lieb inequality but also its inverse form conjectured by Ball. In this way, for example, sharp L_p volume ratio inequalities and their duals were established (see Section 6 for details and further examples). To obtain uniqueness of extremals when the isotropic measure underlying the extremal problem is not necessarily discrete, Lutwak, Yang, and Zhang [29, 31, 32] developed a new approach based on the Ball–Barthe techniques—but not on the Brascamp–Lieb inequality or its inverse—which allowed them to extend all the new reverse inequalities to the setting of general isotropic measures along with characterizations of all

extremizers. Later, Barthe [7] established a continuous version of the Brascamp–Lieb inequality and its inverse which also yields these general reverse inequalities along with their equality conditions.

The notion of Wulff shape has its origins in the classical theory of crystal growth. In more modern mathematical terms, it provides a unifying setting for several extremal problems with an underlying isotropic measure. Sharp reverse volume inequalities for *origin-symmetric* Wulff shapes and their polars were obtained by Lutwak, Yang, and Zhang [29] (see Section 3). These inequalities generalize several of the previously obtained Ball–Barthe volume ratio inequalities for unit balls of subspaces of L_p. The problem of finding similar volume estimates for not necessarily origin-symmetric Wulff shapes remained open.

In this article we establish such sharp reverse isoperimetric inequalities for asymmetric Wulff shapes and their polars including a complete description of all equality cases. Special cases of these new Wulff shape inequalities are previously obtained simplex inequalities of Ball [2], Barthe [4] and Lutwak, Yang, and Zhang [27, 31, 32].

The setting for this article is Euclidean space $\mathbb{R}^n$, $n \geq 2$. A convex body is a compact convex set and in this article will always be assumed to contain the origin in its interior. The polar body of a convex body K is given by $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}$. Throughout, all Borel measures are understood to be non-negative and finite. We write $\operatorname{supp} \nu$ for the support of a measure ν and we use $\operatorname{conv} L$ to denote the convex hull of a set L.

The main objects of this paper are Wulff shapes. This notion was introduced at the turn of the previous century by Wulff [39], who conjectured that this shape describes the minimizer of the interfacial free energy among a crystal's possible shapes of given volume (see also, e.g., [38]). Variations of Wulff's original definition (expanding the class of admitted parameters) yield versatile geometric objects that have been analyzed extensively (see e.g. [11, 12, 17] or [37, Section 6.5]).

Definition. Suppose ν is a Borel measure on S^{n-1} and f is a positive continuous function on S^{n-1}. The *Wulff shape* $W_{\nu,f}$ determined by ν and f is defined by

$$W_{\nu,f} := \{x \in \mathbb{R}^n : x \cdot u \leq f(u) \text{ for all } u \in \operatorname{supp} \nu\}.$$

Without further assumptions on the measure ν and the function f, Wulff shapes, while always convex, may be unbounded. In order to guarantee that $W_{\nu,f}$ is a convex body, we introduce the notion of *f-centered* measure and consider only Wulff shapes determined by measures ν which are *f-centered* and *isotropic*.

Let f be a positive continuous function on S^{n-1}. A Borel measure ν on S^{n-1} is called *f-centered* if

$$\int_{S^{n-1}} f(u)\, u\, d\nu(u) = o.$$

The measure ν is called *isotropic* if

$$(1) \qquad\qquad \int_{S^{n-1}} u \otimes u\, d\nu(u) = I_n,$$

where $u \otimes u$ is the orthogonal projection onto the line spanned by u and I_n denotes the identity map on $\mathbb{R}^n$.

In order to establish reverse affine isoperimetric inequalities, it is often critical to exploit special positions of convex bodies which, in turn, are characterized by isotropic measures (see, e.g., [**2, 4–6, 14, 27, 30, 32, 33**]). The notion of f-centered isotropic measure is designed to unify approaches towards reverse inequalities which are based on the isotropic-embedding technique introduced by Lutwak, Yang, Zhang [**32**] (see Section 4 for details).

Sharp volume estimates for *origin-symmetric* Wulff shapes and their polars determined by even functions and even isotropic measures (clearly, they are f-centered) can easily be deduced from previous work of Lutwak, Yang, and Zhang [**29**], see Section 3. The extremal configurations here are given by constant functions and measures for which the convex hull of their support is a cube. The natural problem to determine sharp volume bounds for not necessarily symmetric Wulff shapes was posed by Zhang [**40**].

With our first main result we establish such a sharp bound for the volume of the Wulff shape $W_{\nu,f}$ determined by an f-centered isotropic measure ν. It depends on the *displacement* of $W_{\nu,f}$ defined by

$$\operatorname{disp} W_{\nu,f} := \operatorname{cd} W_{\nu,f} \cdot \int_{S^{n-1}} \frac{u}{f(u)}\, d\nu(u),$$

where $\operatorname{cd} W_{\nu,f}$ denotes the centroid of the convex body $W_{\nu,f}$.

Theorem 1. *Suppose f is a positive continuous function on S^{n-1} and ν is an isotropic f-centered measure. If $\operatorname{disp} W_{\nu,f} = 0$, then*

$$(2) \qquad\qquad V(W_{\nu,f}) \leq \frac{(n+1)^{(n+1)/2}}{n!} \|f\|_{L^2(\nu)}^n$$

with equality if and only if $\operatorname{conv} \operatorname{supp} \nu$ is a regular simplex inscribed in S^{n-1} and f is constant on $\operatorname{supp} \nu$.

While restrictions on the location of the Wulff shape are necessary for inequality (2) to hold (see, e.g., Corollary 6.2), the assumption $\operatorname{disp} W_{\nu,f} = 0$ is not necessary. Our proof of Theorem 1 yields an explicit description of how the displacement enters the sharp upper bound for the volume of $W_{\nu,f}$ (see Theorem 5.1).

Our second main result, a natural dual to Theorem 1, provides a sharp lower bound for the volume of the polar of the Wulff shape $W_{\nu,f}$. Note that it is independent of the displacement of the Wulff shape.

Theorem 2. *Suppose f is a positive continuous function on S^{n-1} and ν is an isotropic f-centered measure. Then*

$$V\left(W_{\nu,f}^*\right) \geq \frac{(n+1)^{(n+1)/2}}{n!} \|f\|_{L^2(\nu)}^{-n}$$

with equality if and only if $\operatorname{conv} \operatorname{supp} \nu$ is a regular simplex inscribed in S^{n-1} and f is constant on $\operatorname{supp} \nu$.

Our proofs of Theorems 1 and 2 are based on a refinement of the approach towards recently established simplex inequalities by Lutwak, Yang, and Zhang [31, 32], which, in turn, uses many ideas of Ball and Barthe. We remark, however, that our results can also be obtained by applications of Barthe's continuous Brascamp–Lieb inequality and its inverse. The more direct approach we have chosen has the advantage to be at the same time reasonably self contained and elementary.

In Section 6 we show how Theorems 1 and 2 directly imply a number of reverse isoperimetric inequalities (including all equality conditions) obtained by Ball [1,2], Barthe [4] and Lutwak, Yang, Zhang [29,31,32]. As an example, we state here one corollary of Theorem 2. It was first established in [32] and provides a lower bound for the volume of the polar of a convex body K in terms of the volume of its dual Legendre ellipsoid $\Gamma_{-2}K$ introduced in [27] (see Section 2 for precise definitions).

Corollary 3. *If K is a convex body in $\mathbb{R}^n$, then*

$$V(K^*)V(\Gamma_{-2}K) \geq \frac{\kappa_n(n+1)^{(n+1)/2}}{n!n^{n/2}}$$

with equality if and only if K is a simplex whose centroid is at the origin.

Here and in the following, κ_n denotes the volume of the Euclidean unit ball in $\mathbb{R}^n$.

2. Background material

For quick later reference, we collect in this section the necessary background material. In particular, we list basic auxiliary facts from the L_p Brunn–Minkowski theory and recall a number of special positions of convex bodies (needed in the last section). As a general reference, the reader may wish to consult the books [13, 37] and the articles [25, 26].

Throughout we denote by $e_1, \ldots, e_n$ the standard Euclidean basis of $\mathbb{R}^n$ and we use $\|\cdot\|$ to denote the standard Euclidean norm on $\mathbb{R}^n$. We emphasize that in this article a convex body in $\mathbb{R}^n$ is a compact convex

set that contains the origin in its interior. A convex body K is uniquely determined by its (positive, sublinear) support function defined by

$$h(K, x) := \max\{x \cdot y : y \in K\}, \qquad x \in \mathbb{R}^n.$$

A convex body K in $\mathbb{R}^n$ is also determined up to translation by its surface area measure $S(K, \cdot)$. Recall that for a Borel set $\omega \subseteq S^{n-1}$, $S(K, \omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K at which there exists a normal vector of K belonging to ω. It is well known that the surface area measure of a convex body K is 1-centered, that is,

$$(3) \qquad \int_{S^{n-1}} u \, dS(K, u) = o.$$

If K and L are convex bodies and $\alpha, \beta \geq 0$ (not both zero), then their L_p Minkowski combination $\alpha \cdot K +_p \beta \cdot L$ is the convex body whose support function is given by

$$h(\alpha \cdot K +_p \beta \cdot L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p.$$

In [**25, 26**], Lutwak showed that merging the notion of volume with these L_p Minkowski combinations of convex bodies, introduced by Firey, leads to a Brunn–Minkowski theory for each $p \geq 1$. In particular, the L_p mixed volume $V_p(K, L)$ was defined in [**25**] by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Clearly, for $K = L$, we have

$$(4) \qquad V_p(K, K) = V(K).$$

It was shown in [**25**] that for each convex body K, there exists a positive Borel measure on S^{n-1}, the L_p *surface area measure* $S_p(K, \cdot)$ of K, such that for every convex body L,

$$(5) \qquad V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_p(K, u).$$

The measure $S_1(K, \cdot)$ is just the surface area measure of K.

The L_p surface area measure is absolutely continuous with respect to $S(K, \cdot)$, more precisely,

$$(6) \qquad dS_p(K, u) = h(K, u)^{1-p} \, dS(K, u), \qquad u \in S^{n-1}.$$

From (6) and the definition of surface area measures, it follows easily that, for a given convex body K, the Wulff shape determined by the L_p surface area measure $S_p(K, \cdot)$ and the support function $h(K, \cdot)$ of K is precisely the body K, i.e.,

$$(7) \qquad W_{S_p(K, \cdot), h(K, \cdot)} = K.$$

A $\mathrm{GL}(n)$ image of a convex body is often called a *position* of the body. Special positions have been the focus of intensive investigations,

in particular, in relation with a variety of extremal problems for geometric invariants of the bodies in special position (see e.g. [2, 5, 14, 15, 30]).

A classical example of an important special position of a convex body K is the *John position*: Let JK denote the unique ellipsoid of maximal volume contained in K. The body K is said to be in John position, if JK coincides with the Euclidean unit ball B. The following well known characterization of this position goes back to John [18]:

Proposition 2.1. *A convex body K which contains the unit ball B is in John position if and only if there exists a 1-centered isotropic measure on S^{n-1} supported on contact points of K and B.*

A natural dual to the John position of a convex body K is the *Loewner position*, here the ellipsoid of minimal volume containing K is the unit ball. It was also characterized in [18] by the existence of a 1-centered isotropic measure supported on the contact points of K and B.

Another classical position is closely related to the problem of finding a reverse form of the Euclidean isoperimetric inequality. Since convex bodies of a given volume may have arbitrarily large surface area, it is natural to consider convex bodies in *minimal surface area position*, that is, the surface area of these bodies is minimal among all their affine images of the same volume. Petty [35] showed that a convex body K is in minimal surface area position if and only if its surface area measure $S(K, \cdot)$ is isotropic (up to scaling).

In a more recent article, Lutwak, Yang, Zhang [30] have shown that the John position and the minimal surface area position are in fact special cases ($p = \infty$ and $p = 1$) of a family of L_p John positions of a given convex body.

Definition. ([30]) Suppose K is a convex body and $1 \leq p < \infty$. Amongst all origin-symmetric ellipsoids E, the unique ellipsoid that solves the constrained extremal problem

$$\max_E V(E) \quad \text{subject to} \quad V_p(K, E) \leq V(K)$$

will be called the L_p *John ellipsoid* $\mathrm{E}_p K$ of K. We say K is in L_p *John position* if $\mathrm{E}_p K$ coincides with the Euclidean unit ball B.

The L_1 John ellipsoid of a convex body K is also called the Petty ellipsoid. It is not difficult to show (cf. [30]) that (up to scaling) K is in L_1 John position if and only if K is in minimal surface area position. The L_∞ John ellipsoid is the origin-centered ellipsoid of maximal volume contained in K. Hence, $\mathrm{E}_\infty K = JK$ if the John ellipsoid of K is centered at the origin.

Of particular importance among the family of L_p John ellipsoids of a given body K is also the L_2 John ellipsoid. This ellipsoid was previously discovered by Lutwak, Yang, and Zhang (see [27, 28]) and denoted

by $\Gamma_{-2}K$. This notation should indicate a duality with the classical Legendre ellipsoid $\Gamma_2 K$. In fact, Ludwig [21] showed that the operators Γ_{-2} and Γ_2 are the only linearly intertwining matrix-valued maps on convex bodies that satisfy the inclusion-exclusion principle (see also [16, 22–24] for related results).

The following characterization of the L_p John position of a convex body in terms of isotropic measures was also established in [30]:

Proposition 2.2. *Suppose that $p \geq 1$. A convex body K is in L_p John position if and only if its L_p surface area measure $S_p(K, \cdot)$ is isotropic up to volume normalization.*

We conclude this section with another auxiliary result [30, Theorem 5.1] concerning monotonicity properties of L_p John ellipsoids: If K is a convex body and $1 \leq p \leq q \leq \infty$, then

$$(8) \qquad V(\mathrm{E}_q K) \leq V(\mathrm{E}_p K).$$

3. Volume inequalities for symmetric Wulff shapes

In this section we state the volume inequalities corresponding to our main results when the considered Wulff shapes are all origin-symmetric, that is, they are determined by even isotropic measures and even functions. In a slightly different formulation these inequalities were established by Lutwak, Yang, and Zhang [29] and generalize previous results by Ball and Barthe. We also sketch a proof of one of these inequalities using Ball's geometric Brascamp–Lieb inequality, in order to emphasize the close connection between this analytic inequality and volume inequalities for (symmetric as well as asymmetric) Wulff shapes.

The Brascamp–Lieb Inequality. *Suppose that $u_1, \ldots, u_m \in S^{n-1}$ and $c_1, \ldots, c_m > 0$ such that*

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n.$$

If $g_i : \mathbb{R} \to [0, \infty)$, $1 \leq i \leq m$, are integrable functions, then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{m} g_i(u_i \cdot x)^{c_i} \, dx \leq \prod_{i=1}^{m} \left(\int_{\mathbb{R}} g_i \right)^{c_i}.$$

The Brascamp–Lieb inequality [9] was established to prove the sharp form of Young's convolution inequality. Around 1990 Ball [1] discovered the geometric reformulation stated above (later generalized by Barthe [7], after characterizing its equality cases in [6]) which allowed a simple computation of the optimal constant. It directly yields the following sharp volume bound for $W_{\nu,f}$, when the underlying isotropic measure is even and discrete.

Theorem 3.1. ([**29**]) *Suppose f is an even positive continuous function on S^{n-1} and ν is an even isotropic measure. Then*

$$
(9) \qquad V(W_{\nu,f}) \leq \left(\frac{2}{\sqrt{n}}\right)^n \|f\|_{L^2(\nu)}^n
$$

with equality if and only if conv supp ν *is a cube inscribed in S^{n-1} and f is constant on* supp ν.

Sketch of the proof. We prove inequality (9) without its equality conditions for the case of a discrete measure ν supported, say, on $\pm u_1, \ldots, \pm u_m \in S^{n-1}$. The Wulff shape $W_{\nu,f}$ is thus given by

$$
(10) \qquad W_{\nu,f} = \bigcap_{i=1}^{m} \{x \in \mathbb{R}^n : \ |x \cdot u_i| \leq f(u_i)\} .
$$

For $1 \leq i \leq m$, let $c_i > 0$ be the total mass of ν at the points $\pm u_i$ and define the function $g_i : \mathbb{R} \to [0, \infty)$ by

$$
(11) \qquad g_i(t) = \mathbb{I}_{[-f(u_i), f(u_i)]}(t).
$$

By (10), (11) and the Brascamp–Lieb inequality, we now obtain

$$
V(W_{\nu,f}) = \int_{\mathbb{R}^n} \mathbb{I}_{W_{\nu,f}}(x)\, dx = \int_{\mathbb{R}^n} \prod_{i=1}^{m} g_i(x \cdot u_i)^{c_i}\, dx \leq 2^{\sum_{i=1}^{m} c_i} \prod_{i=1}^{m} f(u_i)^{c_i}.
$$

Since the measure ν is isotropic, taking traces in (1) shows that $\sum_{i=1}^{m} c_i = n$. Consequently, an application of the arithmetic-geometric mean inequality yields

$$
V(W_{\nu,f}) \leq 2^n \left(\frac{1}{n} \sum_{i=1}^{m} c_i f(u_i)^2\right)^{n/2} = \left(\frac{2}{\sqrt{n}}\right)^n \|f\|_{L^2(\nu)}^n.
$$

<div align="right">q.e.d.</div>

The following result is dual to Theorem 3.1; for the case of even and discrete isotropic measures, it follows from Barthe's inverse Brascamp–Lieb inequality [**6**] (by arguments similar to the ones sketched above).

Theorem 3.2. ([**29**]) *Suppose f is an even positive continuous function on S^{n-1} and ν is an even isotropic measure. Then*

$$
V(W_{\nu,f}^*) \geq \frac{(2\sqrt{n})^n}{n!} \|f\|_{L^2(\nu)}^{-n}
$$

with equality if and only if conv supp ν *is a cube inscribed in S^{n-1} and f is constant on* supp ν.

The case $f \equiv 1$ of Theorem 3.1 was proved by Ball [**2**], the equality conditions for discrete measures were obtained by Barthe [**6**]. Theorem 3.2 for $f \equiv 1$ and discrete measures was proved by Barthe [**6**].

In order to establish Theorems 3.1 and 3.2 for general even isotropic measures, Lutwak, Yang, and Zhang [**29**] used a direct approach based

on optimal mass transport and a determinant inequality, called the Ball–Barthe Lemma (see the next section), that has easily stated equality cases obtained in [**29**]. Another possibility towards proving Theorems 3.1 and 3.2 is to employ Barthe's continuous versions of the Brascamp–Lieb inequality and its inverse, the equality conditions of which are also based on the Ball–Barthe Lemma. It is therefore no surprise that this basic inequality is also critical in the proofs of our main results. Moreover, to demonstrate the extremal property of the regular simplex in our inequalities, we also need an important embedding of f-centered isotropic measures introduced by Lutwak, Yang, and Zhang [**31, 32**], which we review in the next section.

4. Isotropic embeddings and the Ball–Barthe Lemma

In the following we recall the concept of isotropic embeddings which was introduced by Lutwak, Yang, Zhang [**32**]. These embeddings lift f-centered isotropic measures on S^{n-1} to isotropic measures on S^n and at the same time map the vertices of the regular n-simplex inscribed in S^{n-1} to an orthonormal basis in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. This latter property ensures that we can apply the Ball–Barthe Lemma to obtain sharp bounds in our main results.

Definition. If ν is a Borel measure on S^{n-1}, then a continuous function $g : S^{n-1} \to \mathbb{R}^{n+1} \setminus \{o\}$ is called an *isotropic embedding* of ν if the Borel measure $\overline{\nu}$ on S^n, defined by

$$(12) \qquad \int_{S^n} t(w)\, d\overline{\nu}(w) = \int_{S^{n-1}} t\left(\frac{g(u)}{\|g(u)\|} \right) \|g(u)\|^2 d\nu(u)$$

for every continuous $t : S^n \to \mathbb{R}$, is isotropic.

Of particular interest for us are isotropic embeddings of already isotropic measures. A natural class of such embeddings can be characterized in the following way.

Lemma 4.1. *Suppose f is a positive continuous function on S^{n-1} and ν is an isotropic measure on S^{n-1}. Then the functions*

$$g_\pm : S^{n-1} \to \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R},$$

defined by

$$(13) \qquad\qquad g_\pm(u) = (\pm u, f(u)),$$

are isotropic embeddings of ν if and only if ν is f-centered and $\|f\|_{L_2(\nu)} = 1$.

Proof. If $\overline{\nu}$ is defined as in (12), where g is replaced by $g_\pm$, then we have

$$\int_{S^n} w \otimes w\, d\overline{\nu}(w) = \int_{S^{n-1}} \begin{pmatrix} u \otimes u & \pm f(u)u \\ \pm f(u)u^{\mathrm{T}} & f^2(u) \end{pmatrix} d\nu(u).$$

Consequently, since ν is isotropic,

$$\int_{S^n} w \otimes w \, d\overline{\nu}(w) = I_{n+1}$$

if and only if ν is f-centered and $\|f\|_{L_2(\nu)} = 1$.   q.e.d.

Note that for isotropic embeddings of the form (13) the last coordinate (with respect to the decomposition $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$) of all the points in the support of the measure $\overline{\nu}$, defined by (12), is positive.

Examples. The following two special cases of isotropic embeddings of the form (13) have played a critical role in the proofs of a number of reverse isoperimetric inequalities having simplices as extremals (see [**2, 5, 20, 31, 32**]):

(a) If ν is a 1-centered isotropic measure (e.g., the normalized surface area measure of a convex body in minimal surface area position), then $g_\pm : S^{n-1} \to \mathbb{R}^{n+1}$, defined by

(14) $$g_\pm(u) = \left(\pm u, n^{-1/2} \right),$$

are isotropic embeddings of ν.

(b) Suppose that K is a convex body in L_2 John position. Since, by (3) and (6),

$$\int_{S^{n-1}} u h(K, u) \, dS_2(K, u) = o$$

and, by (4) and (5),

$$\frac{1}{V(K)} \int_{S^{n-1}} h(K, u)^2 \, dS_2(K, u) = n,$$

it follows from Proposition 2.2 that $g_\pm : S^{n-1} \to \mathbb{R}^{n+1}$, defined by

$$g_\pm(u) = \left(\pm u, n^{-1/2} h(K, u) \right)$$

are isotropic embeddings of $S_2(K, \cdot)/V(K)$.

The approaches towards sharp reverse isoperimetric inequalities of both Ball, Barthe and Lutwak, Yang, Zhang make critical use of the following basic estimate for the determinant of a weighted sum of rank-one projections:

The Ball–Barthe Lemma. *If $\overline{\nu}$ is an isotropic measure on S^n and t is a positive continuous function on $\operatorname{supp} \overline{\nu}$, then*

(15) $$\det \int_{S^n} t(w) w \otimes w \, d\overline{\nu}(w) \geq \exp \left(\int_{S^n} \log t(w) \, d\overline{\nu}(w) \right),$$

with equality if and only if $t(v_1) \cdots t(v_{n+1})$ is constant for linearly independent $v_1, \ldots, v_{n+1} \in \operatorname{supp} \overline{\nu}$.

For discrete measures, inequality (15) goes back to Ball. In [6] Barthe provides a simple proof. The equality conditions for (15) were obtained using mixed discriminants and Hölder's inequality by Lutwak, Yang, Zhang [29].

The Ball–Barthe Lemma also plays a crucial role in the proof of our main results, in particular, for establishing the equality cases. Our next lemma goes back to arguments employed by Lutwak, Yang, Zhang [31]. It uses the equality conditions for (15) to characterize the support of 1-centered isotropic measures which are embedded by the functions given in (14).

Lemma 4.2. *Let ν be a 1-centered isotropic measure on S^{n-1}, let $\overline{\nu}_{\pm}$ denote the isotropic measures on S^n defined by (12), isotropically embedded by $g_{\pm}$ defined in (14), and let $D \subseteq \mathbb{R}^{n+1}$ be an open cone with apex at the origin containing e_{n+1} such that $w \cdot z > 0$ for every $w \in \operatorname{supp} \overline{\nu}_{\pm}$ and $z \in D$.*

For every $z \in D$, define $t_z : \operatorname{supp} \overline{\nu}_{\pm} \to (0, \infty)$ by

$$t_z(w) = \phi_w(w \cdot z),$$

where $\phi_w : (0, \infty) \to (0, \infty)$ is smooth nonconstant such that $t_z(w)$, for every fixed $z \in D$, depends continuously on $w \in \operatorname{supp} \overline{\nu}_{\pm}$. If there is equality in (15) for $\overline{\nu}_+$, or $\overline{\nu}_-$ respectively, and every t_z, $z \in D$, then $\operatorname{conv} \operatorname{supp} \nu$ is a regular simplex inscribed in S^{n-1}.

Proof. We prove the statement for $\overline{\nu}_+$. The argument for $\overline{\nu}_-$ is almost verbatim the same. Since $\overline{\nu}_+$ is isotropic, we can find $n + 1$ linearly independent vectors in its support, say $\{w_1, \ldots, w_{n+1}\}$. If $w_0 = \sum_{i=1}^{n+1} c_i w_i$ is an arbitrary vector in $\operatorname{supp} \overline{\nu}_+$ such that, without loss of generality, $c_1 \neq 0$, then by the equality conditions of the Ball–Barthe Lemma,

$$\phi_{w_0}(w_0 \cdot z)\phi_{w_2}(w_2 \cdot z) \cdots \phi_{w_{n+1}}(w_{n+1} \cdot z)$$
$$= \phi_{w_1}(w_1 \cdot z)\phi_{w_2}(w_2 \cdot z) \cdots \phi_{w_{n+1}}(w_{n+1} \cdot z)$$

for every $z \in D$. Since ϕ_w is positive for every $w \in \operatorname{supp} \overline{\nu}_+$, evaluating partial derivatives with respect to z at λe_{n+1} yields

$$\phi'_{w_0}(w_0 \cdot \lambda e_{n+1})w_0 = \phi'_{w_1}(w_1 \cdot \lambda e_{n+1})w_1$$

for every $\lambda > 0$. By the remark after Lemma 4.1, the support of $\overline{\nu}_+$ cannot contain two antipodal points. Therefore, we either have that $w_0 = w_1$ or $\phi'_{w_0}(w_0 \cdot \lambda e_{n+1}) = 0$ for all $\lambda > 0$. Since $w_0 \cdot e_{n+1} > 0$, the latter implies that ϕ_{w_0} is constant, a contradiction. Consequently, $\operatorname{supp} \overline{\nu}_+ = \{w_1, \ldots, w_{n+1}\}$. Since $\overline{\nu}_+$ is isotropic, it is easy to see (cf. [29]) that $w_1, \ldots, w_{n+1}$ must be an orthonormal basis of $\mathbb{R}^{n+1}$.

From the definition of $\overline{\nu}_+$, it follows that $\operatorname{supp} \nu = \{u_1, \ldots, u_{n+1}\}$, where $g_+(u_i)/\|g_+(u_i)\| = w_i$ for $1 \leq i \leq n + 1$. Using definition (14) of

g_+, we obtain

$$0 = w_i \cdot w_j = \frac{(u_i, \frac{1}{\sqrt{n}}) \cdot (u_j, \frac{1}{\sqrt{n}})}{\sqrt{1 + \frac{1}{n}}\sqrt{1 + \frac{1}{n}}}, \quad 1 \leq i \neq j \leq n+1.$$

In other words, $u_i \cdot u_j = -\frac{1}{n}$ for all $i \neq j$. Hence, $\operatorname{conv} \operatorname{supp} \nu$ must be a regular simplex.   q.e.d.

5. Proof of the main results

After these preparations, we are now in a position to give the proofs of Theorems 1 and 2. We start with the following refinement of Theorem 1:

Theorem 5.1. *Suppose f is a positive continuous function on S^{n-1} and ν is an isotropic f-centered measure. Then*

$$(16) \qquad V(W_{\nu,f}) \leq \frac{(n+1 - \operatorname{disp} W_{\nu,f})^{n+1}}{n!(n+1)^{(n+1)/2}} \|f\|_{L^2(\nu)}^n$$

with equality if and only if $\operatorname{conv} \operatorname{supp} \nu$ is a regular simplex inscribed in S^{n-1} and f is constant on $\operatorname{supp} \nu$.

Proof. By the definition of $W_{\nu,f}$ and $\operatorname{disp} W_{\nu,f}$, we may assume $\|f\|_{L^2(\nu)} = 1$. Let $\bar{\nu}$ denote the measure on S^n defined by (12), isotropically embedded by $g_-(u) = (-u, f(u))$, $u \in S^{n-1}$ (here we use Lemma 4.1).

Next, let $C \subseteq \mathbb{R}^{n+1}$ denote the cone with apex at the origin defined by

$$C = \bigcup_{r>0} rW_{\nu,f} \times \{r\} \subseteq \mathbb{R}^{n+1}.$$

Clearly, $e_{n+1} \in C$. Moreover, since $w \in \operatorname{supp} \bar{\nu} \subseteq \mathbb{R}^n \times \mathbb{R}$ if and only if

$$(17) \qquad w = \frac{(-u, f(u))}{\sqrt{1 + f(u)^2}}$$

for some $u \in \operatorname{supp} \nu$, we have that, for every $w \in \operatorname{supp} \bar{\nu}$ and $z = (rx, r) \in C$,

$$w \cdot z = \frac{-u \cdot rx + rf(u)}{\sqrt{1 + f(u)^2}} \geq 0.$$

For $w \in \operatorname{supp} \bar{\nu}$, define the smooth and strictly increasing function $T_w : (0, \infty) \to \mathbb{R}$ by

$$\int_{-\infty}^{T_w(t)} e^{-\pi s^2} \, ds = \frac{1}{e_{n+1} \cdot w} \int_0^t \exp\left(-\frac{s}{e_{n+1} \cdot w}\right) ds.$$

Differentiating both sides with respect to t yields

$$T_w'(t) \, e^{-\pi T_w(t)^2} = \frac{1}{e_{n+1} \cdot w} \exp\left(-\frac{t}{e_{n+1} \cdot w}\right).$$

Taking the log of both sides and substituting $t = w \cdot z$ for $w \in \operatorname{supp} \overline{\nu}$ and $z \in \operatorname{int} C$, we obtain

$$(18) \qquad \log T'_w(w \cdot z) - \pi\, T_w(w \cdot z)^2 = -\log(e_{n+1} \cdot w) - \frac{w \cdot z}{e_{n+1} \cdot w}.$$

Furthermore define $T : \operatorname{int} C \to \mathbb{R}^{n+1}$ by

$$T(z) = \int_{S^n} T_w(w \cdot z)\, w\, d\overline{\nu}(w).$$

Let dT denote the Jacobian of T. A straightforward computation yields that, for every $z \in \operatorname{int} C$,

$$(19) \qquad dT(z) = \int_{S^n} T'_w(w \cdot z)\, w \otimes w\, d\overline{\nu}(w).$$

Since T'_w is a positive function, it follows that the matrix $dT(z)$ is positive definite for $z \in \operatorname{int} C$. Consequently, T is a diffeomorphism onto its image. Moreover, since

$$\|T(z)\|^2 = \int_{S^n} T_w(w \cdot z)(T(z) \cdot w)\, d\overline{\nu}(w)$$

and $\overline{\nu}$ is isotropic, we obtain from an application of the Cauchy–Schwarz inequality that

$$(20) \qquad \|T(z)\|^2 \le \int_{S^n} T_w(w \cdot z)^2\, d\overline{\nu}(w).$$

Now, by (18), followed by an application of the Ball–Barthe Lemma with $t(w) = T'_w(w \cdot z)$, (19), (20), and a change of variables it follows that

$$\exp\left(-\int_{S^n} \log(e_{n+1} \cdot w)\, d\overline{\nu}(w)\right) \int_{\operatorname{int} C} \exp\left(\int_{S^n} -\frac{w \cdot z}{e_{n+1} \cdot w}\, d\overline{\nu}(w)\right)\, dz$$

$$= \int_{\operatorname{int} C} \exp\left(\int_{S^n} \log T'_w(w \cdot z)\, d\overline{\nu}(w) - \pi \int_{S^n} T_w(w \cdot z)^2\, d\overline{\nu}(w)\right)\, dz$$

$$\le \int_{\operatorname{int} C} \det dT(z) \exp\left(-\pi \|T(z)\|^2\right)\, dz \le \int_{\mathbb{R}^{n+1}} \exp\left(-\pi \|z\|^2\right)\, dz = 1.$$

Equivalently, by the definition of the cone C,

$$(21) \qquad \begin{aligned} \int_0^\infty \int_{rW_{\nu,f}} &\exp\left(\int_{S^n} -\frac{(x,r) \cdot w}{e_{n+1} \cdot w}\, d\overline{\nu}(w)\right)\, dx\, dr \\ &\le \exp\left(\frac{1}{n+1} \int_{S^n} \log\left((e_{n+1} \cdot w)^2\right)\, d\overline{\nu}(w)\right)^{(n+1)/2}. \end{aligned}$$

Since $\overline{\nu}$ is isotropic, an application of Jensen's inequality to the right-hand side of (21) yields

$$(22)$$
$$\int_0^\infty \int_{rW_{\nu,f}} \exp\left(\int_{S^n} -\frac{(x,r) \cdot w}{e_{n+1} \cdot w}\, d\overline{\nu}(w)\right)\, dx\, dr \le \left(\frac{1}{n+1}\right)^{(n+1)/2}.$$

In order to obtain the desired inequality (16), it remains to show that the left-hand side of (22) dominates

$$V(W_{\nu,f})\, n!\, (n+1 - \operatorname{disp} W_{\nu,f})^{-(n+1)}.$$

To see this, first note that, by definition (12) of $\bar{\nu}$, the fact that we use the embedding $g(u) = (-u, f(u))$ and since ν is f-centered, the left-hand side of (22) is equal to

$$(23) \qquad \int_0^\infty e^{-(n+1)r} \int_{rW_{\nu,f}} \exp\left(\int_{S^{n-1}} \frac{x \cdot u}{f(u)}\, d\nu(u) \right)\, dx\, dr.$$

Since

$$(24) \qquad r \operatorname{disp} W_{\nu,f} = \frac{1}{V(rW_{\nu,f})} \int_{rW_{\nu,f}} \int_{S^{n-1}} \frac{x \cdot u}{f(u)}\, d\nu(u)\, dx,$$

another application of Jensen's inequality yields

$$\int_{rW_{\nu,f}} \exp\left(\int_{S^{n-1}} \frac{x \cdot u}{f(u)}\, d\nu(u) \right)\, dx \geq V(rW_{\nu,f})\, e^{r \operatorname{disp} W_{\nu,f}}.$$

Consequently, (23) is larger than

$$V(W_{\nu,f}) \int_0^\infty r^n e^{-(n+1 - \operatorname{disp} W_{\nu,f})r}\, dr$$
$$= V(W_{\nu,f})\, n!\, (n+1 - \operatorname{disp} W_{\nu,f})^{-(n+1)},$$

where we have used that $\operatorname{disp} W_{\nu,f} \leq n$, which follows easily from (24) and the definition of $W_{\nu,f}$. This completes the proof of inequality (16).

Assume now that there is equality in inequality (16). By the equality conditions of Jensen's inequality, equality in (22) can hold only if $e_{n+1} \cdot w$ is constant for every $w \in \operatorname{supp} \bar{\nu}$. Since any $w \in \operatorname{supp} \bar{\nu}$ is of the form (17), this implies that f is constant on the support of ν. By the normalization $\|f\|_{L^2(\nu)} = 1$, we must have $f \equiv \frac{1}{\sqrt{n}}$ on $\operatorname{supp} \nu$. Now it is easy to check that the assumptions of Lemma 4.2, where $D = \operatorname{int} C$ and $\phi_w = T'_w$, are satisfied. Hence, an application of Lemma 4.2 concludes the proof.   q.e.d.

In order to establish Theorem 2, we will use a transport map $\widehat{T}_w$ that is in some sense dual to the function T_w used in the proof above.

Proof of Theorem 2. As before we may assume that $\|f\|_{L^2(\nu)} = 1$. In the following we denote by $\bar{\nu}$ the measure on S^n defined by (12), isotropically embedded by $g_+(u) = (u, f(u))$, $u \in S^{n-1}$ (where we again use Lemma 4.1).

For $w \in \operatorname{supp} \overline{\nu}$, define the smooth and strictly increasing function $\widehat{T}_w : \mathbb{R} \to (0, \infty)$ by

$$e_{n+1} \cdot w \int_0^{\widehat{T}_w(t)} e^{-s(e_{n+1} \cdot w)} \, ds = \int_{-\infty}^t e^{-\pi s^2} \, ds.$$

Differentiating both sides with respect to t yields

$$(e_{n+1} \cdot w) \, \widehat{T}_w'(t) \, e^{-\widehat{T}_w(t)(e_{n+1} \cdot w)} = e^{-\pi t^2}.$$

Taking the log of both sides and substituting $t = w \cdot z$ for $w \in \operatorname{supp} \overline{\nu}$ and $z \in \mathbb{R}^{n+1}$, we obtain

$$(25) \quad \log \widehat{T}_w'(w \cdot z) = \widehat{T}_w(w \cdot z)(e_{n+1} \cdot w) - \pi(w \cdot z)^2 - \log(e_{n+1} \cdot w).$$

Define the map $\widehat{T} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$\widehat{T}(z) := \int_{S^n} \widehat{T}_w(w \cdot z) \, w \, d\overline{\nu}(w).$$

The Jacobian of $\widehat{T}$ is given by

$$d\widehat{T}(z) = \int_{S^n} \widehat{T}_w'(w \cdot z) \, w \otimes w \, d\overline{\nu}(w),$$

which shows that $d\widehat{T}(z)$ is a positive definite matrix for every $z \in \mathbb{R}^{n+1}$. Consequently, $\widehat{T}$ is a diffeomorphism onto its image. In fact, we claim that its image is contained in the cone

$$\widehat{C} := \bigcup_{r>0} r W_{\nu,f}^* \times \{r\}.$$

To prove this, we have to show that if $\widehat{T}(z) = (x, r) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and $y \in W_{\nu,f}$, then $x \cdot y \leq r$. By definition (12) of $\overline{\nu}$, the definition of $\widehat{T}$ and the fact that $u \cdot y \leq f(u)$ for every $u \in \operatorname{supp} \nu$, we obtain

$$x \cdot y = \int_{S^{n-1}} \widehat{T}_{\frac{(u,f(u))}{\sqrt{1+f(u)^2}}} \left(\frac{(u, f(u))}{\sqrt{1+f(u)^2}} \cdot z \right) (u \cdot y) \sqrt{1 + f(u)^2} \, d\nu(u)$$

$$\leq \int_{S^{n-1}} \widehat{T}_{\frac{(u,f(u))}{\sqrt{1+f(u)^2}}} \left(\frac{(u, f(u))}{\sqrt{1+f(u)^2}} \cdot z \right) f(u) \sqrt{1 + f(u)^2} \, d\nu(u)$$

$$= \int_{S^n} \widehat{T}_w(w \cdot z)(e_{n+1} \cdot w) \, d\overline{\nu}(w) = r.$$

Since

$$n! \, V(W_{\nu,f}^*) = \int_0^\infty \int_{r W_{\nu,f}^*} e^{-r} \, dx \, dr = \int_{\widehat{C}} e^{-e_{n+1} \cdot z} \, dz,$$

a change of variables, followed by an application of the Ball–Barthe Lemma, (25) and the fact that $\overline{\nu}$ is isotropic, yields

$$
\begin{aligned}
n!\, V(W_{\nu,f}^*) &\geq \int_{\mathbb{R}^{n+1}} e^{-e_{n+1}\cdot\widehat{T}(z)} \det d\widehat{T}(z)\, dz \\
&\geq \int_{\mathbb{R}^{n+1}} e^{-e_{n+1}\cdot\widehat{T}(z)} \exp\left(\int_{S^n} \log \widehat{T}_w'(w\cdot z)\, d\overline{\nu}(w)\right) dz \\
&= \exp\left(-\int_{S^n} \log(e_{n+1}\cdot w)\, d\overline{\nu}(w)\right) \int_{\mathbb{R}^{n+1}} \exp\left(-\pi\|z\|^2\right) dz \\
&= \exp\left(\frac{1}{n+1}\int_{S^n} \log\left((e_{n+1}\cdot w)^2\right) d\overline{\nu}(w)\right)^{-(n+1)/2}.
\end{aligned}
$$

Thus, using Jensen's inequality and again the fact that $\overline{\nu}$ is isotropic, we obtain the desired inequality

$$
(26) \qquad\qquad n!\, V(W_{\nu,f}^*) \geq (n+1)^{(n+1)/2}.
$$

Assume now that there is equality in inequality (26). As in the proof of Theorem 5.1 we conclude that this is possible only if f is constant on the support of ν. Thus, by the normalization $\|f\|_{L^2(\nu)} = 1$, we must have $f \equiv \frac{1}{\sqrt{n}}$ on $\operatorname{supp}\nu$. In order to apply Lemma 4.2, define the open cone $D \subseteq \mathbb{R}^{n+1}$ by

$$
D = \{z \in \mathbb{R}^{n+1} : w\cdot z > 0 \text{ for every } w \in \operatorname{supp}\overline{\nu}\}.
$$

Clearly, the assumptions of Lemma 4.2, where $\phi_w = \widehat{T}_w'$, are satisfied. Hence an application of Lemma 4.2 concludes the proof.   q.e.d.

6. Applications

In this final section, we show how Theorem 1 and 2 directly imply previously established reverse isoperimetric inequalities which have simplices as extremals. We begin with consequences of Theorem 1, such as Ball's volume ratio inequality and its L_2 analog by Lutwak, Yang, and Zhang, and conclude this section with dual results (including Corollary 3), which can be deduced from Theorem 2.

First suppose that ν is a 1-centered isotropic measure on S^{n-1}. Then

$$
W_{\nu,1} = (\operatorname{conv}\operatorname{supp}\nu)^*
$$

and $\operatorname{disp} W_{\nu,1} = 0$. Consequently, Theorem 1 reduces to the following result of Lutwak, Yang and Zhang [31].

Corollary 6.1. ([31]) *If ν is a 1-centered isotropic measure on S^{n-1}, then*

$$
(27) \qquad\qquad V((\operatorname{conv}\operatorname{supp}\nu)^*) \leq \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!},
$$

with equality if and only if $\operatorname{conv}\operatorname{supp}\nu$ is a regular simplex inscribed in S^{n-1}.

Ball [2] had first established inequality (27), but without the equality conditions. For discrete measures, these were obtained by Barthe [6].

Corollary 6.1 allows for a geometric interpretation, known as Ball's volume ratio inequality, which gives an upper bound for the ratio between the volume of a convex body and its John ellipsoid:

Ball's Volume Ratio Inequality. ([2, 6]) *If $K \subseteq \mathbb{R}^n$ is a convex body, then*

$$(28) \qquad \frac{V(K)}{V(\mathrm{J}K)} \leq \frac{n^{n/2}(n+1)^{(n+1)/2}}{\kappa_n n!},$$

with equality if and only if K is a simplex.

Proof. Without loss of generality we may assume that K is in John position, that is $\mathrm{J}K = B$. By Proposition 2.1, there exists a 1-centered isotropic measure ν on S^{n-1} supported by contact points of K and B.

Clearly, $K \subseteq W_{\nu,1} = (\mathrm{conv}\, \mathrm{supp}\, \nu)^*$. Thus, Corollary 6.1 implies (28) along with its equality conditions.   q.e.d.

If $K \subseteq \mathbb{R}^n$ is a convex body such that $\mathrm{J}K$ is centered at the origin, then we can replace $\mathrm{J}K$ in inequality (28) by the L_∞ John ellipsoid $\mathrm{E}_\infty K$. A combination of (28) and (8) thus yields the following L_p *volume ratio inequality* for the entire family of L_p John ellipsoids $\mathrm{E}_p K$, where $1 \leq p \leq \infty$:

$$(29) \qquad \frac{V(K)}{V(\mathrm{E}_p K)} \leq \frac{n^{n/2}(n+1)^{(n+1)/2}}{\kappa_n n!},$$

with equality if and only if K is a simplex with centroid at the origin.

Using a similar approach, Lutwak, Yang and Zhang established the case $p = 2$ of inequality (29) in [27]. This L_2 volume ratio inequality is also a *direct* consequence of Theorem 1, where in addition we can replace the assumption that $\mathrm{J}K$ is centered at the origin by the more natural assumption that K has centroid at the origin. Note that, in contrast, the L_2 volume ratio is unbounded on the set of *all* convex bodies.

Corollary 6.2. *If $K \subseteq \mathbb{R}^n$ is a convex body with centroid at the origin, then*

$$(30) \qquad \frac{V(K)}{V(\mathrm{E}_2 K)} \leq \frac{n^{n/2}(n+1)^{(n+1)/2}}{\kappa_n n!},$$

with equality if and only if K is a simplex with centroid at the origin.

Proof. Without loss of generality we may assume that K is in L_2 John position, that is $E_2 K = B$. By Proposition 2.2, this implies that the measure $\nu := \frac{1}{V(K)} S_2(K, \cdot)$ is isotropic. Moreover, by (3) and (6), the measure ν is $h(K, \cdot)$-centered. By (7), (4) and (5), we have

$$W_{\nu, h(K, \cdot)} = K \qquad \text{and} \qquad \|h(K, \cdot)\|_{L^2(\nu)} = \sqrt{n}.$$

Thus, $\operatorname{disp} W_{\nu, h(K, \cdot)} = 0$ and Theorem 1 implies inequality (30) along with its equality conditions.   q.e.d.

A combination of (30) and (8) shows that inequality (30) (under the assumption that $\operatorname{cd} K = o$) holds true if $E_2 K$ is replaced by $E_p K$, $1 \leq p \leq 2$.

We now turn to special cases of Theorem 2. To this end, it is useful to keep the following (easily verified) alternative representation of the polar Wulff shape $W_{\nu, f}^*$ of a given f-centered isotropic measure ν in mind:

$$(31) \qquad W_{\nu, f}^* = \operatorname{conv} \left\{ \frac{u}{f(u)} : \ u \in \operatorname{supp} \nu \right\}.$$

If ν is now a 1-centered isotropic measure on S^{n-1}, then, by (31), Theorem 2 reduces to the following result of Lutwak, Yang, Zhang [31].

Corollary 6.3. ([31]) *If ν is a 1-centered isotropic measure on S^{n-1}, then*

$$(32) \qquad V(\operatorname{conv} \operatorname{supp} \nu) \geq \frac{(n+1)^{(n+1)/2}}{n^{n/2} n!},$$

with equality if and only if $\operatorname{conv} \operatorname{supp} \nu$ *is a regular simplex inscribed in* S^{n-1}.

For discrete measures, Corollary 6.3 was first established by Barthe [3]. A more geometric reformulation of inequality (32) is a dual result to Ball's volume ratio inequality:

Barthe's Dual Volume Ratio Inequality. ([3]) *If $K \subseteq \mathbb{R}^n$ is a convex body, then*

$$(33) \qquad V(K^*) V(\mathrm{J}K) \geq \frac{(n+1)^{(n+1)/2} \kappa_n}{n^{n/2} n!},$$

with equality if and only if K is a simplex with centroid at the origin.

Proof. First, we use Corollary 6.3 to deduce Barthe's [3] outer volume ratio inequality

$$(34) \qquad \frac{V(K)}{V(\mathrm{L}K)} \geq \frac{(n+1)^{(n+1)/2}}{n^{n/2} n! \kappa_n},$$

where $\mathrm{L}K$ denotes the Loewner ellipsoid of K, that is the ellipsoid of minimal volume containing K. To this end, we may assume without loss

of generality that $\mathrm{L}K = B$. Then, there exists a 1-centered isotropic measure ν on S^{n-1} supported by contact points of K and B (compare the remark after Proposition 2.1). Clearly, $\mathrm{conv\,supp}\,\nu \subseteq K$. Thus, using Corollary 6.3, we obtain inequality (34) with equality if and only if K is a simplex.

Using the definitions of $\mathrm{J}K$ and $\mathrm{L}K$ and the fact that for every ellipsoid E containing the origin in its interior, $V(E)V(E^*) \geq \kappa_n^2$ with equality precisely for origin-symmetric ellipsoids, we obtain

$$V(\mathrm{J}K)V(\mathrm{L}K^*) \geq V((\mathrm{L}K^*)^*)V(\mathrm{L}K^*) \geq \kappa_n^2.$$

Combining this with inequality (34), where K is replaced by K^*, yields the desired inequality (33) along with its equality conditions.   q.e.d.

If $K \subseteq \mathbb{R}^n$ is a convex body such that $\mathrm{J}K$ is centered at the origin, that is $\mathrm{J}K = \mathrm{E}_\infty K$, then a combination of (33) and (8) yields the following dual to inequality (29) for $1 \leq p \leq \infty$:

$$(35) \qquad V(K^*)V(\mathrm{E}_p K) \geq \frac{(n+1)^{(n+1)/2}\kappa_n}{n^{n/2}n!},$$

with equality if and only if K is a simplex with centroid at the origin.

It is an important task in convex geometry to find sharp lower bounds for the volume of K^* in terms of other natural geometric quantities of K (see e.g. [8, 10, 19, 34, 36] for results in this direction). Unfortunately, for inequality (35), the requirement that the John ellipsoid of K is centered at the origin cannot, in general, be omitted for all $p > 2$. For the L_2 case, Lutwak, Yang, and Zhang [32] have discovered that this additional assumption is in fact unnecessary. Their result was stated in the introduction as Corollary 3 and is (in the following equivalent formulation) also a direct consequence of Theorem 2:

Corollary 6.4. ([32]) *If $K \subseteq \mathbb{R}^n$ is a convex body, then*

$$(36) \qquad V(K^*)V(\mathrm{E}_2 K) \geq \frac{(n+1)^{(n+1)/2}\kappa_n}{n^{n/2}n!},$$

with equality if and only if K is a simplex with centroid at the origin.

Proof. Without loss of generality we may assume that K is in L_2 John position. By Proposition 2.2, this implies that the $h(K,\cdot)$-centered measure $\nu := \frac{1}{V(K)}S_2(K,\cdot)$ is isotropic. By (7), (4) and (5), we have

$$W_{\nu,h(K,\cdot)} = K \qquad \text{and} \qquad \|h(K,\cdot)\|_{L^2(\nu)} = \sqrt{n}.$$

Thus, Theorem 2 reduces to the desired statement.   q.e.d.

A combination of (36) and (8) again shows that inequality (36) remains true if $\mathrm{E}_2 K$ is replaced by $\mathrm{E}_p K$, $1 \leq p \leq 2$.

We finally remark that all the special cases of Theorem 1 and 2 presented in this section have natural analogues for origin-symmetric

convex bodies, where the cube instead of the simplex plays the extremal role. These are of course special cases of the volume inequalities for symmetric Wulff shapes stated in Section 3.

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