Log-Concavity Properties of Minkowski Valuations

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Abstract. New Orlicz Brunn–Minkowski inequalities are established for rigid motion compatible Minkowski valuations of arbitrary degree. These extend classical log-concavity properties of intrinsic volumes and generalize seminal results of Lutwak and others. Two different approaches which refine previously employed techniques are explored. It is shown that both lead to the same class of Minkowski valuations for which these inequalities hold. An appendix by Semyon Alesker contains the proof of a new classification of generalized translation invariant valuations.

1. Introduction

The fundamental log-concavity property of the volume functional is expressed by the multiplicative form of the Brunn–Minkowski inequality:

$$V_n((1-\lambda)K+\lambda L) \geq V_n(K)^{1-\lambda}V_n(L)^{\lambda}, \tag{1.1}$$

where K and L are convex bodies (non-empty compact convex sets) in $\mathbb{R}^n$ with non-empty interiors, $0<\lambda<1$, and $+$ denotes Minkowski addition. Equality holds in (1.1) if and only if K and L are translates of each other. The excellent survey of Gardner [**16**] gives a comprehensive overview of different aspects and consequences of the Brunn–Minkowski inequality. Here we just mention that it directly implies the classical *Euclidean* isoperimetric inequality.

Projection bodies of convex bodies were defined at the turn of the previous century by Minkowski. In 1984 Lutwak [**35**] discovered that an *affine* isoperimetric inequality of Petty [**50**] for *polar* projection bodies is not only significantly stronger than the Euclidean isoperimetric inequality, but in fact an optimal version of this classical inequality. For the tremendous impact of Petty's inequality and its generalizations see, e.g., [**25**, **39**, **41**, **62**, **67**]. The problem of finding sharp bounds for the volume of projection bodies, given the volume of the original body, remains a central quest in convex geometric analysis. It has led, among many other results, to the discovery of important log-concavity properties of the volume of projection bodies. In fact, Lutwak [**37**] established not only Brunn–Minkowski type inequalities for the volume of projection bodies, but for all the *intrinsic volumes* of projection bodies of arbitrary order (see Section 2).

In the present article we investigate a common generalization of Lutwak's Brunn–Minkowski inequalities for projection bodies and inequality (1.1), more specifically, its version for all the intrinsic volumes. To be more precise, we establish new log-concavity properties of intrinsic volumes of convex body valued *valuations* which intertwine rigid motions. This line of research has its origins in the discovery of the special place of projection bodies in affine geometry: Ludwig [**30**, **32**] characterized the projection body operator as the unique continuous Minkowski valuation which is translation invariant and $\mathrm{GL}(n)$ contravariant (see [**22**, **31**, **33**, **45**, **46**, **56**, **64**] for related results).

In recent years, it has become apparent that several geometric inequalities for projection bodies and, more general, valuations intertwining the group of affine transformations, in fact, hold for much larger classes of valuations intertwining merely rigid motions. First such results were obtained in [**53**], where the Brunn–Minkowski inequalities for projection bodies of Lutwak were generalized to translation invariant and $\mathrm{SO}(n)$ *equivariant* Minkowski valuations of degree $n-1$. Although considerable efforts have been invested ever since to show that these log-concavity properties extend to Minkowski valuations of arbitrary degree (see [**9**, **47**, **55**]), the conjectured complete family of inequalities has only partially been obtained (compare Section 2).

For the inequalities established so far two different approaches were used. While in [**53**] and [**55**] integral representations of (even) Minkowski valuations which are translation invariant and $\mathrm{SO}(n)$ equivariant were crucial, in [**47**] the Hard Lefschetz *derivation* operator on Minkowski valuations [**4**, **47**], together with a symmetry property of bivaluations [**9**], was the key ingredient. In this paper we show that the Hard Lefschetz *integration* operator on Minkowski valuations [**6**, **10**, **57**] on one hand and a recent representation theorem for Minkowski valuations [**57**, **58**] on the other hand lead to a natural class of Minkowski valuations which exhibit log-concavity properties. All the Brunn–Minkowski inequalities for Minkowski valuations established before turn out to be special cases of our new results. From new monotonicity properties of these Minkowski valuations, we are able to deduce a complete characterization of equality cases without any smoothness assumptions that were required before.

Moreover, all previously obtained and new Brunn–Minkowski inequalities for Minkowski valuations are shown to not only hold for Minkowski addition but for all *commutative* Orlicz Minkowski additions (introduced in [**17**]) of convex bodies. This includes, in particular, all L_p Minkowski additions.

2. Statement of principal results

Let $\mathcal{K}^n$ denote the space of convex bodies in n-dimensional Euclidean space $\mathbb{R}^n$ endowed with the Hausdorff metric. Throughout the article we assume that $n \geq 3$. A convex body K is uniquely determined by its support function $h(K,u) = \max\{u \cdot x : x \in K\}$ for $u \in S^{n-1}$. For $i \in \{0,\ldots,n\}$, let $V_i(K)$ denote the ith intrinsic volume of K (see Section 3).

A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called a *Minkowski valuation* if

$$\Phi K + \Phi L = \Phi(K \cup L) + \Phi(K \cap L)$$

whenever $K \cup L \in \mathcal{K}^n$ and addition on $\mathcal{K}^n$ is Minkowski addition.

The theory of *scalar* valued valuations has long played a prominent role in convex geometry (see, e.g., [**26**, **28**] for the history of scalar valuations and [**3**, **12**, **15**, **24**, **34**, **48**, **65**] for more recent results). Systematic investigations of Minkowski valuations have only been initiated about a decade ago by Ludwig [**30–32**]. These valuations arise naturally from data about projections and sections of convex bodies and form an integral part of geometric tomography. As first examples we mention here the projection body maps $\Pi_i : \mathcal{K}^n \to \mathcal{K}^n$ of order $i \in \{1,\ldots,n-1\}$, defined by

$$h(\Pi_i K, u) = V_i(K|u^\perp), \qquad u \in S^{n-1}.$$

While the entire family Π_i is translation invariant and $\mathrm{SO}(n)$ equivariant, the classic projection body map Π_{n-1} is the only one among them which intertwines *linear* transformations (see [**30**]). In fact, there is only a small number of Minkowski valuations which are compatible with affine transformations (see [**1**, **2**, **22**, **32**, **56**, **64**] for their classification).

In this article we establish new log-concavity properties for the class $\mathbf{MVal}_j$ of continuous, translation invariant *and* $\mathrm{SO}(n)$ equivariant Minkowski valuations of a given degree j of homogeneity (by a result of McMullen [**43**], only integer degrees $0 \leq j \leq n$ can occur; cf. Section 5). A first such result was obtained by Lutwak [**37**, Theorem 6.2] for projection bodies of arbitrary order. In an equivalent multiplicative form it states the following: If $K, L \in \mathcal{K}^n$ have non-empty interiors, $1 \leq i \leq n$, and $2 \leq j \leq n-1$, then for all $\lambda \in (0,1)$,

$$V_i(\Pi_j((1-\lambda)K + \lambda L)) \geq V_i(\Pi_j K)^{1-\lambda} V_i(\Pi_j L)^\lambda, \tag{2.1}$$

with equality if and only if K and L are translates of each other.

Inequalities (2.1) have been generalized in different directions: Abardia and Bernig [**2**] extended (2.1) to the entire class of *complex* projection bodies. Analogues of (2.1) were established in [**53**] for all valuations in $\mathbf{MVal}_{n-1}$ and then in [**55**] for *even* valuations in $\mathbf{MVal}_j$ in the case $i = j + 1$. The assumption on the parity could later be omitted in [**9**]. In the Euclidean setting, the most general result to date can be stated (in multiplicative form) as follows, where we call the Minkowski valuation which maps every convex body to the set containing only the origin *trivial*.

Theorem 1 **([47])** *Let $\Phi_j \in \mathbf{MVal}_j$, $2 \leq j \leq n-1$, be non-trivial. If $K, L \in \mathcal{K}^n$ and $1 \leq i \leq j+1$, then for all $\lambda \in (0,1)$,*

$$V_i(\Phi_j((1-\lambda)K + \lambda L)) \geq V_i(\Phi_j K)^{1-\lambda} V_i(\Phi_j L)^{\lambda}. \tag{2.2}$$

If K and L are of class C^2_+, then equality holds if and only if K and L are translates of each other.

Note that Theorem 1 establishes (2.2) only for $1 \leq i \leq j+1$, while in Lutwak's family of inequalities (2.1) the range of i does not depend on j. The proof of Theorem 1 used ideas from [**9**] and the existence of a new derivation operator Λ on Minkowski valuations established in [**47**] (see also Section 5). For $\Phi \in \mathbf{MVal}_j$, there exists $\Lambda\Phi \in \mathbf{MVal}_{j-1}$ such that

$$h((\Lambda\Phi)(K), \cdot) = \left.\frac{d}{dt}\right|_{t=0} h(\Phi(K + tB), \cdot).$$

This definition was motivated by a similar derivation operator introduced by Alesker [**4**] in the theory of *scalar valued* valuations. There it is widely used to deduce results for valuations of degree i from those for valuations of some degree $j > i$. The key to the proof of Theorem 1 was the following generalization of a symmetry property of bivaluations obtained in [**9**].

Theorem 2 **([47])** *Let $\Phi_j \in \mathbf{MVal}_j$, $2 \leq j \leq n-1$. If $1 \leq i \leq j+1$, then*

$$W_{n-i}(K, \Phi_j L) = \frac{(i-1)!}{j!} W_{n-j-1}(L, (\Lambda^{j+1-i}\Phi_j)(K)) \tag{2.3}$$

for every $K, L \in \mathcal{K}^n$.

Here $W_m(K, L)$ denotes the mixed volume $V(K[n-m-1], B[m], L)$ with $n-m-1$ copies of K and m copies of the Euclidean unit ball B. We will see in Section 6 that Theorem 2 follows directly from a recently obtained integral representation of Minkowski valuations intertwining rigid motions.

An obvious idea for a proof of (2.2) for the remaining cases $j+2 \leq i \leq n$ is to establish a counterpart of Theorem 2 for the Hard Lefschetz integration operator: For $\Phi \in \mathbf{MVal}_j$, there exists $\mathfrak{L}\Phi \in \mathbf{MVal}_{j+1}$ such that

$$h((\mathfrak{L}\Phi)(K), \cdot) = \int_{\mathrm{AGr}_{n-1,n}} h(\Phi(K \cap E), \cdot)\, dE,$$

where $\mathrm{AGr}_{n-1,n}$ denotes the affine Grassmannian of $n-1$ planes in $\mathbb{R}^n$ and where we integrate with respect to the suitably normalized invariant measure on $\mathrm{AGr}_{n-1,n}$ (see Section 5). For *scalar valued* valuations the operator $\mathfrak{L}$ was first defined in [**6**] and used to deduce results for valuations of degree i from those for valuations of some degree $j < i$. As an operator on Minkowski valuations, $\mathfrak{L}$ was first considered in [**57**].

Our first result is a version of Theorem 2 for the operator $\mathfrak{L}$. However, the situation is more delicate in this case and we will see that a full analogue of (2.3) only holds for a subclass of Minkowski valuations. To define this class let $\mathbf{MVal}_j^\infty$ denote the set of translation invariant and $\mathrm{SO}(n)$ equivariant *smooth* Minkowski valuations (cf. Section 5).

Definition *For $1 \leq i, j \leq n-1$, let $\mathbf{MVal}_{j,i}^\infty \subseteq \mathbf{MVal}_j^\infty$ be defined by*

$$\mathbf{MVal}_{j,i}^\infty = \begin{cases} \Lambda^{i-j}(\mathbf{MVal}_i^\infty) & \text{if } i > j, \\ \mathbf{MVal}_j^\infty & \text{if } i \leq j. \end{cases}$$

We write $\mathbf{MVal}_{j,i}$ for the closure of $\mathbf{MVal}_{j,i}^\infty$ in the topology of uniform convergence on compact subsets.

We will see in Section 5 that $\Lambda : \mathbf{MVal}_j^\infty \to \mathbf{MVal}_{j-1}^\infty$ is injective for $2 \leq j \leq n$. Thus, for $i > j$, the inverse map $(\Lambda^{i-j})^{-1} : \mathbf{MVal}_{j,i}^\infty \to \mathbf{MVal}_i^\infty$ is well defined and will be denoted by Λ^{j-i}.

Our counterpart of Theorem 2 can now be stated as follows:

Theorem 3 *Let $\Phi_j \in \mathbf{MVal}_j^\infty$, $2 \leq j \leq n-1$. For $j+2 \leq i \leq n$ and every convex body $L \in \mathcal{K}^n$, there exists a* generalized valuation $\gamma_{i,j}(L, \cdot) \in \mathbf{Val}_1^{-\infty}$ *such that*

$$W_{n-i}(K, \Phi_j L) = \gamma_{i,j}(L, (\mathfrak{L}^{i-j-1}\Phi_j)(K))$$

for every $K \in \mathcal{K}^n$. Moreover, if $\Phi_j \in \mathbf{MVal}_{j,i-1}^\infty$, then

$$\gamma_{i,j}(L, (\mathfrak{L}^{i-j-1}\Phi_j)(K)) = \frac{(i-1)!}{j!} W_{n-1-j}(L, (\Lambda^{j+1-i}\Phi_j)(K)).$$

Generalized translation invariant valuations were introduced recently by Alesker and Faifman [**8**]. We recall their definition and basic properties (in particular, of the space $\mathbf{Val}_1^{-\infty}$) in Section 5. A crucial ingredient in the proof of Theorem 3 is a new classification of generalized valuations from $\mathbf{Val}_1^{-\infty}$. We are very grateful to Semyon Alesker for communicating to us a proof of this result and his permission to include it in an appendix of this article.

Using Theorem 3, we establish in Section 6 the main result of this article:

Theorem 4 *Let $1 \leq i \leq n$ and let $\Phi_j \in \mathbf{MVal}_{j,i-1}$, $2 \leq j \leq n-1$, be non-trivial. If $K, L \in \mathcal{K}^n$ have non-empty interiors, then for all $\lambda \in (0,1)$,*

$$V_i(\Phi_j((1-\lambda)K + \lambda L)) \geq V_i(\Phi_j K)^{1-\lambda} V_i(\Phi_j L)^{\lambda}, \tag{2.4}$$

with equality if and only if K and L are translates of each other.

Since $\mathbf{MVal}_{j,i-1} = \mathbf{MVal}_j$ for $i \leq j+1$, Theorem 4 includes both Lutwak's inequalities (2.1) and Theorem 1 as special cases. Also note that the smoothness assumption for the bodies K and L in the equality conditions of (2.2) is no longer required. This follows from new monotonicity properties of the Minkowski valuations in $\mathbf{MVal}_{j,i-1}$, which we prove in Section 6.

In the last part of the article we explain that our proof of Theorem 4 can be modified to yield an even stronger result. More precisely, we show that (2.4) not only holds for the usual Minkowski addition but, in fact, for all commutative Orlicz Minkowski additions introduced by Gardner, Hug, and Weil [**17**]. In particular, this includes all the L_p Minkowski additions.

Let Θ_1 denote the set of convex functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$. For $\varphi \in \Theta_1$ and $K, L \in \mathcal{K}^n$ containing the origin, we write $K +_{\varphi,\lambda} L$ for the Orlicz Minkowski convex combination of K and L (see Section 3 for the definition).

Theorem 5 *Let $\varphi \in \Theta_1$, $1 \leq i \leq n$, and let $\Phi_j \in \mathbf{MVal}_{j,i-1}$, $2 \leq j \leq n-1$, be non-trivial. If $K, L \in \mathcal{K}^n$ contain the origin, then for all $\lambda \in (0,1)$,*

$$V_i(\Phi_j(K +_{\varphi,\lambda} L)) \geq V_i(\Phi_j K)^{1-\lambda} V_i(\Phi_j L)^{\lambda}. \tag{2.5}$$

When φ is strictly convex and K and L have non-empty interiors, equality holds if and only if $K = L$.

We will explain in Section 3 that by a recent result of Gardner, Hug, and Weil [**17**] (Theorem 3.1 below), inequality (2.5) holds for all *commutative* Orlicz Minkowski additions.

3. Background material on convex bodies

For quick later reference we collect in this section some basic facts from convex geometry, in particular, on additions of convex bodies and inequalities for mixed volumes. As general reference for this material we recommend the book by Schneider [**52**] and the article [**17**].

For a convex body $K \in \mathcal{K}^n$, the definition of the support function implies $h(\vartheta K, u) = h(K, \vartheta^{-1}u)$ for every $u \in S^{n-1}$ and $\vartheta \in \mathrm{SO}(n)$. Since every twice continuously differentiable function on S^{n-1} is a difference of support functions (see, e.g., [**52**, p. 49]), the subspace spanned by support functions $\{h(K,\cdot) - h(L,\cdot) : K, L \in \mathcal{K}^n\}$ is dense in $C(S^{n-1})$. The *Steiner point* $s(K)$ of $K \in \mathcal{K}^n$ is defined by

$$s(K) = \frac{1}{\kappa_n} \int_{S^{n-1}} h(K,u)u\, du.$$

Here and in the following we use du to denote integration with respect to spherical Lebesgue measure and κ_m for the m-dimensional volume of the unit ball in $\mathbb{R}^m$. The Steiner point map is the unique vector valued, rigid motion equivariant and continuous valuation on $\mathcal{K}^n$ (see e.g., [**52**, p. 363]).

For $K, L \in \mathcal{K}^n$ and $s, t \geq 0$, the support function of the Minkowski combination $s\,K + t\,L$ is given by

$$h(s\,K + t\,L, \cdot) = s\,h(K,\cdot) + t\,h(L,\cdot).$$

On the set $\mathcal{K}^n_{\mathrm{o}}$ of convex bodies containing the origin, Firey introduced in the 1960s a more general way of combining convex sets. For $K, L \in \mathcal{K}^n_{\mathrm{o}}$, $s, t \geq 0$, and $1 \leq p < \infty$, the L_p *Minkowski combination* $s \cdot K +_p t \cdot L$ is defined by

$$h(s \cdot K +_p t \cdot L, \cdot)^p = s\,h(K,\cdot)^p + t\,h(L,\cdot)^p.$$

Initiated by Lutwak [**36**, **38**], in the last two decades an entire L_p theory of convex bodies was developed which represents a powerful extension of the classical Brunn–Minkowski theory (see, e.g., [**25**, **39**, **40**, **45**, **46**, **59**, **66**]).

A still more recent extension of the Brunn–Minkowski theory goes back to two articles of Lutwak, Yang, and Zhang [**41**, **42**] and a paper by Haberl, Lutwak, Yang, and Zhang [**23**]. While these articles form the starting point of an emerging Orlicz Brunn–Minkowski theory that generalizes the L_p theory of convex bodies in the same way that Orlicz spaces generalize L_p spaces, the fundamental notion of an Orlicz Minkowski combination of convex bodies was introduced later by Gardner, Hug, and Weil [**17**].

As before let Θ_1 be the set of convex functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$. For $K, L \in \mathcal{K}_{\mathrm{o}}^n$, $s, t \geq 0$, and $\varphi, \psi \in \Theta_1$, the *Orlicz Minkowski combination* $+_{\varphi,\psi}(K, L, s, t)$ is defined by

$$h(+_{\varphi,\psi}(K, L, s, t), u) = \inf \left\{ \alpha > 0 : s\,\varphi\left(\frac{h(K,u)}{\alpha}\right) + t\,\psi\left(\frac{h(L,u)}{\alpha}\right) \leq 1 \right\}$$

for $u \in S^{n-1}$. The notation $+_{\varphi,\psi}(K, L, s, t)$ is necessitated by the fact that it is not possible in general to isolate an Orlicz scalar multiplication. We note that for $\varphi(t) = \psi(t) = t^p$, $p \geq 1$, the Orlicz Minkowski combination $+_{\varphi,\psi}(K, L, s, t)$ equals the L_p Minkowski combination $s \cdot K +_p t \cdot L$.

For $s = t = 1$, we write $K +_{\varphi,\psi} L$ instead of $+_{\varphi,\psi}(K, L, 1, 1)$ and call this the Orlicz Minkowski sum of K and L. In fact, Gardner, Hug, and Weil defined a more general Orlicz addition but proved (see [**17**, Theorem 5.5]) that their definition leads (essentially) to the Orlicz Minkowski addition as defined here *and* the L_∞ Minkowski addition obtained as the Hausdorff limit of the L_p Minkowski addition, that is, for $K, L \in \mathcal{K}_{\mathrm{o}}^n$,

$$K +_\infty L = \lim_{p\to\infty} K +_p L = \operatorname{conv}(K \cup L).$$

While all L_p Minkowski additions are commutative, in general, the Orlicz Minkowski addition of convex bodies is not. A classification of those Orlicz additions which are commutative was obtained by Gardner, Hug, and Weil.

Theorem 3.1 ([**17**]) *Let $\varphi, \psi \in \Theta_1$. The addition $+_{\varphi,\psi} : \mathcal{K}_{\mathrm{o}}^n \times \mathcal{K}_{\mathrm{o}}^n \to \mathcal{K}_{\mathrm{o}}^n$ is commutative if and only if there exists $\phi \in \Theta_1$ such that $+_{\varphi,\psi} = +_{\phi,\phi}$.*

In the following we will only be interested in commutative Orlicz additions. For $K, L \in \mathcal{K}_{\mathrm{o}}^n$, $\varphi \in \Theta_1$, and $\lambda \in (0,1)$ we use $K +_{\varphi,\lambda} L$ to denote the Orlicz Minkowski convex combination $+_{\varphi,\varphi}(K, L, (1-\lambda), \lambda)$. More explicitly,

$$h(K +_{\varphi,\lambda} L, u) = \inf\left\{\alpha > 0 \colon (1-\lambda)\varphi\left(\frac{h(K,u)}{\alpha}\right) + \lambda\varphi\left(\frac{h(L,u)}{\alpha}\right) \leq 1\right\}$$

for $u \in S^{n-1}$. For the proof of Theorem 5 we need the following simple fact.

Lemma 3.2 *If $\varphi \in \Theta_1$ and $K, L \in \mathcal{K}_{\mathrm{o}}^n$, then for all $\lambda \in (0,1)$,*

$$K +_{\varphi,\lambda} L \supseteq (1-\lambda)K + \lambda L. \tag{3.1}$$

Proof. For $u \in S^{n-1}$ choose $t > h(K +_{\varphi,\lambda} L, u)$. Then, by the convexity of φ and the definition of $K +_{\varphi,\lambda} L$, we have

$$\varphi\left(\frac{(1-\lambda)h(K,u)+\lambda h(L,u)}{t}\right) \leq (1-\lambda)\varphi\left(\frac{h(K,u)}{t}\right) + \lambda\varphi\left(\frac{h(L,u)}{t}\right) \leq 1.$$

Since every $\varphi \in \Theta_1$ is increasing and satisfies $\varphi(1) = 1$, we conclude that

$$(1-\lambda)h(K,u) + \lambda h(L,u) \leq t.$$

Now, letting t approach $h(K +_{\varphi,\lambda} L, u)$, we obtain the desired inclusion (3.1). ■

By a classical result of Minkowski, the volume of a Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_m K_m$, where $K_1, \ldots, K_m \in \mathcal{K}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$, can be expressed as a homogeneous polynomial of degree n,

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{j_1,\ldots,j_n=1}^{m} V(K_{j_1}, \ldots, K_{j_n})\lambda_{j_1} \cdots \lambda_{j_n}, \tag{3.2}$$

where the coefficients $V(K_{j_1}, \ldots, K_{j_n})$, called *mixed volumes* of $K_{j_1}, \ldots, K_{j_n}$, depend only on $K_{j_1}, \ldots, K_{j_n}$ and are symmetric in their arguments. For $K, L \in \mathcal{K}^n$ and $0 \leq i \leq n$, we denote the mixed volume with i copies of K and $n-i$ copies of L by $V(K[i], L[n-i])$. For $K, K_1, \ldots, K_i \in \mathcal{K}^n$ and $\mathbf{C} = (K_1, \ldots, K_i)$, we write $V_i(K, \mathbf{C})$ instead of $V(K, \ldots, K, K_1, \ldots, K_i)$.

The mixed volume $W_i(K) := W_i(K, K)$ is called the *ith quermassintegral* of K. The *ith intrinsic volume* $V_i(K)$ of K is defined by

$$\kappa_{n-i} V_i(K) = \binom{n}{i} W_{n-i}(K).$$

A special case of (3.2) is the classical *Steiner formula* for the volume of the parallel set of K at distance $r > 0$,

$$V(K + rB) = \sum_{i=0}^{n} r^i \binom{n}{i} W_i(K) = \sum_{i=0}^{n} r^{n-i} \kappa_{n-i} V_i(K).$$

A fundamental inequality for mixed volumes is the general Minkowski inequality (see [**52**, p. 427]): If $2 \leq i \leq n$ and $K, L \in \mathcal{K}^n$ have dimension at least i, then

$$W_{n-i}(K, L)^i \geq W_{n-i}(K)^{i-1} W_{n-i}(L), \tag{3.3}$$

with equality if and only if K and L are homothetic.

A consequence of (3.3) and the homogeneity of quermassintegrals is the (multiplicative) Brunn–Minkowski inequality: If $2 \le i \le n$ and $K, L \in \mathcal{K}^n$ have dimension at least i, then for all $\lambda \in (0,1)$,

$$W_{n-i}((1-\lambda)K + \lambda L) \ge W_{n-i}(K)^{1-\lambda} W_{n-i}(L)^{\lambda}, \tag{3.4}$$

with equality if and only if K and L are translates of each other.

A further generalization of inequality (3.4) (where the equality conditions are not yet known) is the following (see [**52**, p. 406]): If $0 \le i \le n-2$, $K, L, K_1, \dots, K_i \in \mathcal{K}^n$ and $\mathbf{C} = (K_1, ..., K_i)$, then for all $\lambda \in (0,1)$,

$$V_i((1-\lambda)K + \lambda L, \mathbf{C}) \ge V_i(K, \mathbf{C})^{1-\lambda} V_i(L, \mathbf{C})^{\lambda}. \tag{3.5}$$

Associated with a convex body $K \in \mathcal{K}^n$ is a family of Borel measures $S_i(K, \cdot)$, $0 \le i \le n-1$, on S^{n-1}, called the *area measures of order* i of K. They are uniquely determined by the property that

$$W_{n-1-i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)\, dS_i(K, u) \tag{3.6}$$

for all $L \in \mathcal{K}^n$. If $K \in \mathcal{K}^n$ has non-empty interior, then, by a theorem of Aleksandrov–Fenchel–Jessen (see, e.g., [**52**, p. 449]), each of the measures $S_i(K, \cdot)$, $1 \le i \le n-1$, determines K up to translations.

For $1 \le j \le n-1$ and $r > 0$, we have the Steiner type formula

$$S_j(K + rB, \cdot) = \sum_{i=0}^{j} r^{j-i} \binom{j}{i} S_i(K, \cdot).$$

A body $K \in \mathcal{K}^n$ is of class C^2_+ if the boundary of K is a C^2 submanifold of $\mathbb{R}^n$ with everywhere positive curvature. In this case, each measure $S_i(K, \cdot)$, $0 \le i \le n-1$, is absolutely continuous with respect to spherical Lebesgue measure and its density is (up to a constant) given by the ith elementary symmetric function of the principal radii of curvature of K.

The center of mass (centroid) of every area measure of a convex body is at the origin, that is, for every $K \in \mathcal{K}^n$ and all $i \in \{0, \dots, n-1\}$, we have

$$\int_{S^{n-1}} u\, dS_i(K, u) = o.$$

The set $\mathcal{S}_i$ of all area measures of order i of convex bodies in $\mathcal{K}^n$ is dense in the set of all non-negative finite Borel measures on S^{n-1} with centroid at the origin, endowed with the weak topology, if and only if $i = n-1$. However, $\mathcal{S}_i - \mathcal{S}_i$, $1 \le i \le n-1$, is dense in the set $\mathcal{M}_o(S^{n-1})$ of all *signed* finite Borel measures on S^{n-1} with centroid at the origin (see, e.g., [**52**, p. 477]).

4. Spherical harmonics and distributions

In this section we collect facts about spherical harmonics, in particular, on the series expansion of distributions on the sphere. We also recall C. Berg's functions used in his solution of the Christoffel problem, since they are closely related to the action of the Hard Lefschetz integration operator on Minkowski valuations (see Section 5). In the final part of this section we give a new proof of the bijectivity of integral transforms involving C. Berg's functions. For the background material we refer the reader to [**52**, Chapter 8.3], [**20**], and [**44**].

We write Δ_S for the Laplacian (or Laplace–Beltrami operator) on S^{n-1}. If $f, g \in C^2(S^{n-1})$, then we have

$$\int_{S^{n-1}} f(u)\, \Delta_S g(u)\, du = \int_{S^{n-1}} g(u)\, \Delta_S f(u)\, du.$$

The finite dimensional vector space of spherical harmonics of dimension n and degree k will be denoted by $\mathcal{H}_k^n$ and we write $N(n,k)$ for its dimension. Spherical harmonics are eigenfunctions of Δ_S, more precisely, for $Y_k \in \mathcal{H}_k^n$,

$$\Delta_S Y_k = -k(k+n-2)\, Y_k. \tag{4.1}$$

Let $L^2(S^{n-1})$ denote the Hilbert space of square-integrable functions on S^{n-1} with the usual inner product $(\,\cdot\,,\,\cdot\,)$. The spaces $\mathcal{H}_k^n$ are pairwise orthogonal with respect to this inner product. If $\{Y_{k,1}, \ldots, Y_{k,N(n,k)}\}$ is an orthonormal basis of $\mathcal{H}_k^n$, then the collection $\{Y_{k,1}, \ldots, Y_{k,N(n,k)} : k \in \mathbb{N}\}$ is a complete orthogonal system in $L^2(S^{n-1})$, that is, the Fourier series

$$f \sim \sum_{k=0}^{\infty} \pi_k f \tag{4.2}$$

converges to f in the L^2 norm for every $f \in L^2(S^{n-1})$. Here, we used $\pi_k : L^2(S^{n-1}) \to \mathcal{H}_k^n$ to denote the orthogonal projection. Since the *Legendre polynomial* $P_k^n \in C([-1,1])$ of dimension n and degree k satisfies

$$\sum_{i=1}^{N(n,k)} Y_{k,i}(u)\, Y_{k,i}(v) = \frac{N(n,k)}{\omega_n}\, P_k^n(u \cdot v),$$

where ω_m denotes the surface area of the m-dimensional unit ball, we have

$$(\pi_k f)(v) = \sum_{i=1}^{N(n,k)} (f, Y_{k,i}) Y_{k,i}(v) = \frac{N(n,k)}{\omega_n} \int_{S^{n-1}} f(u)\, P_k^n(u \cdot v)\, du. \tag{4.3}$$

Throughout the article we use $\bar{e} \in S^{n-1}$ to denote a fixed but arbitrarily chosen pole of the sphere and we write $\mathrm{SO}(n-1)$ for the stabilizer in $\mathrm{SO}(n)$ of $\bar{e}$. A function or measure on S^{n-1} is called *zonal* if it is $\mathrm{SO}(n-1)$ invariant. Clearly, zonal functions depend only on the value of $u \cdot \bar{e}$.

The subspace of zonal functions in $\mathcal{H}_k^n$ is 1-dimensional for every $k \in \mathbb{N}$ and spanned by the function $u \mapsto P_k^n(u \cdot \bar{e})$. Since the spaces $\mathcal{H}_k^n$ are invariant under the natural action of $\mathrm{SO}(n)$, the functions $u \mapsto P_k^n(u \cdot v)$, for fixed $v \in S^{n-1}$, are elements of $\mathcal{H}_k^n$. The orthogonality of the spaces $\mathcal{H}_k^n$ is reflected by the fact that the Legendre polynomials P_k^n form a complete orthogonal system with respect to the inner product on $C([-1,1])$ defined by

$$[p,q]_n = \int_{-1}^{1} p(t)\, q(t)\, (1-t^2)^{\frac{n-3}{2}}\, dt.$$

From the orthogonality property of the Legendre polynomials and (4.3), it is not difficult to show that any function $\phi \in L^2([-1,1])$ (or, equivalently, any zonal $g \in L^2(S^{n-1})$) admits a series expansion

$$\phi \sim \sum_{k=0}^{\infty} \frac{N(n,k)}{\omega_n}\, a_k^n[\phi]\, P_k^n, \tag{4.4}$$

where

$$a_k^n[\phi] = \omega_{n-1} \int_{-1}^{1} \phi(t)\, P_k^n(t)\, (1-t^2)^{\frac{n-3}{2}}\, dt = \omega_{n-1}\, [P_k^n, \phi]_n\,. \tag{4.5}$$

For the explicit calculation of integrals of the form (4.5) the following formula of Rodrigues for the Legendre polynomials is often very useful:

$$P_k^n(t) = \frac{(-1)^k}{2^k \left(\frac{n-1}{2}\right)_k} (1-t^2)^{-\frac{n-3}{2}} \frac{d^k}{dt^k} (1-t^2)^{\frac{n-3}{2}+k}, \tag{4.6}$$

where, for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$, we have used $(\alpha)_k$ to abbreviate the product $\alpha(\alpha+1)\cdots(\alpha+k-1)$. Using (4.6) one can show that the derivatives of Legendre polynomials are again Legendre polynomials. For $l \geq k$, we have

$$\frac{d^k}{dt^k} P_l^n(t) = 2^k \left(\frac{n}{2}\right)_k \frac{N(n+2k, l-k)}{N(n,l)}\, P_{l-k}^{n+2k}. \tag{4.7}$$

Next we recall the *Gegenbauer polynomials* which can be defined for $\alpha > 0$ by means of the generating function

$$\frac{1}{(1+r^2-2rt)^{\alpha}} = \sum_{k=0}^{\infty} C_k^{\alpha}(t)\, r^n.$$

For $n \geq 3$, their relation to Legendre polynomials can be expressed by

$$C_k^{(n-2)/2} = \binom{n+k-3}{n-3} P_k^n. \tag{4.8}$$

For the following well known auxiliary result about the spherical harmonic expansion of smooth functions, see, e.g., [**44**, p. 36].

Lemma 4.1 *If $f \in C^\infty(S^{n-1})$, then the sequence $\|\pi_k f\|_\infty$, $k \in \mathbb{N}$, is* rapidly decreasing*, that is, for any $m \in \mathbb{N}$, we have* $\sup\{k^m \|\pi_k f\|_\infty : k \in \mathbb{N}\} < \infty$. *Conversely, if $Y_k \in \mathcal{H}_k^n$, $k \in \mathbb{N}$, is a sequence of spherical harmonics such that $\|Y_k\|_\infty$ is rapidly decreasing, then the function*

$$f(u) = \sum_{k=0}^\infty Y_k(u), \qquad u \in S^{n-1},$$

is C^∞ and $\pi_k f = Y_k$ for every $k \in \mathbb{N}$.

For $f \in C^\infty(S^{n-1})$ and $m \in \mathbb{N}$, define

$$(-\Delta_S)^{\frac{m}{2}} f = \sum_{k=0}^\infty (k(k+n-2))^{\frac{m}{2}} \pi_k f.$$

Note that, by Lemma 4.1, $(-\Delta_S)^{\frac{m}{2}} f \in C^\infty(S^{n-1})$.

If we endow the vector space $C^\infty(S^{n-1})$ with the topology defined by the family of semi norms $\|(-\Delta_S)^{\frac{m}{2}} f\|_\infty$, $m \in \mathbb{N}$, then $C^\infty(S^{n-1})$ becomes a Fréchet space. Moreover, the spherical harmonic expansion (4.2) of any $f \in C^\infty(S^{n-1})$ converges to f in this topology.

A *distribution* on S^{n-1} is a continuous linear functional on $C^\infty(S^{n-1})$. We write $C^{-\infty}(S^{n-1})$ for the space of distributions on S^{n-1} equipped with the topology of weak convergence and use $\langle\,\cdot\,,\,\cdot\,\rangle$ to denote the canonical bilinear pairing on $C^\infty(S^{n-1}) \times C^{-\infty}(S^{n-1})$.

A (signed) measure σ on S^{n-1} defines a distribution T_σ by

$$\langle f, T_\sigma \rangle = \int_{S^{n-1}} f(u)\, d\sigma(u), \qquad f \in C^\infty(S^{n-1}).$$

Using the continuous linear injection $\sigma \mapsto T_\sigma$, we can regard $\mathcal{M}(S^{n-1})$ as a subspace of $C^{-\infty}(S^{n-1})$. In the same way, the spaces $C^\infty(S^{n-1})$, $C(S^{n-1})$, and $L^2(S^{n-1})$ can be viewed as subspaces of $C^{-\infty}(S^{n-1})$ and we have

$$C^\infty(S^{n-1}) \subseteq C(S^{n-1}) \subseteq L^2(S^{n-1}) \subseteq \mathcal{M}(S^{n-1}) \subseteq C^{-\infty}(S^{n-1}). \tag{4.9}$$

Since $\pi_k : L^2(S^{n-1}) \to \mathcal{H}_k^n$ is self-adjoint, that is, $(\pi_k f, g) = (f, \pi_k g)$ for all $f, g \in L^2(S^{n-1})$ and $k \in \mathbb{N}$, it is consistent to define the k-spherical harmonic component $\pi_k T$ of $T \in C^{-\infty}(S^{n-1})$ as the distribution given by

$$\langle f, \pi_k T\rangle = \langle \pi_k f, T\rangle, \qquad f \in C^\infty(S^{n-1}).$$

Lemma 4.2 ([**44**, p. 38]) *If $T \in C^{-\infty}(S^{n-1})$, then $\pi_k T \in \mathcal{H}_k^n$ for every $k \in \mathbb{N}$ and the sequence $\|\pi_k T\|_\infty$, $k \in \mathbb{N}$, is* slowly increasing, *that is, there exist $C > 0$ and $j \in \mathbb{N}$ such that $\|\pi_k T\|_\infty \le C(1 + k^j)$ for every $k \in \mathbb{N}$. Conversely, if $Y_k \in \mathcal{H}_k^n$, $k \in \mathbb{N}$, is a sequence of spherical harmonics such that $\|Y_k\|_\infty$ is slowly increasing, then*

$$\langle g, T\rangle = \sum_{k=0}^{\infty} \int_{S^{n-1}} g(u) Y_k(u)\, du, \qquad g \in C^\infty(S^{n-1}),$$

defines a distribution $T \in C^{-\infty}(S^{n-1})$ for which $\pi_k T = Y_k$ for every $k \in \mathbb{N}$.

We can also extend the Laplacian to distributions $T \in C^{-\infty}(S^{n-1})$, by defining $\Delta_S T$ as the distribution given by

$$\langle f, \Delta_S T\rangle = \langle \Delta_S f, T\rangle, \qquad f \in C^\infty(S^{n-1}).$$

Note that, by (4.9), Δ_S can now also act on continuous functions on S^{n-1}. This is of particular importance for us, since the support function $h(K, \cdot)$ and the first-order area measure $S_1(K, \cdot)$ of a convex body $K \in \mathcal{K}^n$ are related by

$$\Box_n h(K, \cdot) = S_1(K, \cdot), \tag{4.10}$$

where $\Box_n$ is the differential operator given by

$$\Box_n h = h + \frac{1}{n-1} \Delta_S h.$$

From the definition of $\Box_n$ and (4.1), we see that for $f \in C^\infty(S^{n-1})$ the spherical harmonic expansion of $\Box_n f$ is given by

$$\Box_n f \sim \sum_{k=0}^{\infty} \frac{(1-k)(k+n-1)}{n-1} \pi_k f. \tag{4.11}$$

Thus, the kernel of the linear operator $\Box_n : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$ is given by $\mathcal{H}_1^n$ and consists precisely of the restrictions of linear functions on $\mathbb{R}^n$ to S^{n-1}. Let $C_{\mathrm{o}}^\infty(S^{n-1})$ denote the Fréchet subspace of $C^\infty(S^{n-1})$ given by

$$C_{\mathrm{o}}^\infty(S^{n-1}) = \{f \in C^\infty(S^{n-1}) : \pi_1 f = 0\}$$

and define $C_{\mathrm{o}}^{-\infty}(S^{n-1})$ analogously.

Since the linear operator $\Box_n : C^\infty_\mathrm{o}(S^{n-1}) \to C^\infty_\mathrm{o}(S^{n-1})$ is an isomorphism, it is a natural problem to find an (explicit) inversion formula. This was accomplished by C. Berg [**13**] in the late 1960s and, due to (4.10), is closely related to his solution of the classical Christoffel problem which consists in finding necessary and sufficient conditions for a Borel measure on S^{n-1} to be the first-order area measure of a convex body.

In order to describe C. Berg's inversion formula for $\Box_n$, let us recall the *Funk–Hecke Theorem*: If $\phi \in C([-1,1])$ and F_ϕ is the integral transform on $\mathcal{M}(S^{n-1})$ defined by

$$(\mathrm{F}_\phi\sigma)(u) = \int_{S^{n-1}} \phi(u \cdot v)\, d\sigma(v), \qquad u \in S^{n-1},$$

then the spherical harmonic expansion of $\mathrm{F}_\phi\sigma \in C(S^{n-1})$ is given by

$$\mathrm{F}_\phi\sigma \sim \sum_{k=0}^{\infty} a_k^n[\phi]\, \pi_k\sigma, \tag{4.12}$$

where the numbers $a_k^n[\phi]$ are given by (4.5) and called the *multipliers* of F_ϕ.

Using the theory of subharmonic functions on S^{n-1}, C. Berg proved that for every $n \geq 2$ there exists a uniquely determined C^∞ function g_n on $(-1,1)$ such that the zonal function $u \mapsto g_n(u \cdot \bar{e})$ is in $L^1(S^{n-1})$ and

$$a_1^n[g_n] = 0, \qquad\qquad a_k^n[g_n] = \frac{n-1}{(1-k)(k+n-1)}, \quad k \neq 1. \tag{4.13}$$

For later reference, we just state here

$$g_2(t) = \frac{1}{2\pi}\left((\pi - \arccos t)(1-t^2)^{\frac{1}{2}} - \frac{t}{2}\right) \tag{4.14}$$

and

$$g_3(t) = \frac{1}{2\pi}\left(1 + t\ln(1-t) + \left(\frac{4}{3} - \ln 2\right) t\right). \tag{4.15}$$

We note that, by (4.13), our normalization of the g_n differs from C. Berg's original one. It follows from (4.11), (4.12), and (4.13) that

$$f(u) = \int_{S^{n-1}} g_n(u \cdot v)(\Box_n f)(v)\, dv, \qquad u \in S^{n-1},$$

for every $f \in C^\infty_\mathrm{o}(S^{n-1})$, which is the desired inversion formula. However, for our purposes we need the following more general fact.

Theorem 4.3 *For every $n \geq 2$ and $2 \leq j \leq n$, the integral transform $\mathrm{F}_{g_j} : C^\infty_{\mathrm{o}}(S^{n-1}) \to C^\infty_{\mathrm{o}}(S^{n-1})$, given by*

$$(\mathrm{F}_{g_j} f)(u) = \int_{S^{n-1}} g_j(u \cdot v) f(v)\, dv, \qquad u \in S^{n-1},$$

is an isomorphism.

Theorem 4.3 follows for example from a recent result of Goodey and Weil [**19**, Theorem 4.3]. However, we give a different and more elementary proof below that also yields additional information required after the proof of Theorem 6.1. For this, note that, by Lemma 4.1, it is sufficient to show that the multipliers $a^n_k[g_j]$ are non-zero for $k \neq 1$ and that they are slowly increasing. Therefore, Theorem 4.3 is a direct consequence of the following.

Theorem 4.4 *For $n \geq 2$, $2 \leq j \leq n$, and $k \neq 1$, we have*

$$a^n_k[g_j] = -\frac{\pi^{\frac{n-j}{2}}(j-1)}{4} \frac{\Gamma\left(\frac{n-j+2}{2}\right)\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{j+k-1}{2}\right)}{\Gamma\left(\frac{n-j+k+1}{2}\right)\Gamma\left(\frac{n+k+1}{2}\right)}.$$

Proof. For $n \geq 2$, $d \geq 0$, and $k \neq 1$, by (4.5), we have to determine

$$a^{n,d}_k := a^{n+d}_k[g_n] = \omega_{n+d-1}\left[P^{n+d}_k, g_n\right]_{n+d}, \tag{4.16}$$

where we know from (4.13) that

$$a^{n,0}_k = \frac{n-1}{(1-k)(n-1+k)}. \tag{4.17}$$

We start with the case $d = 1$. By (4.4) and (4.13), we have

$$g_n \sim \sum_{l=0}^{\infty} \frac{N(n,l)}{\omega_n}\, a^{n,0}_l P^n_l,$$

where the sum converges in the topology induced by $[\,\cdot\,,\cdot\,]_n$, which implies the convergence in the topology induced by $[\,\cdot\,,\cdot\,]_{n+1}$. Consequently,

$$a^{n,1}_k = \omega_n\left[P^{n+1}_k, g_n\right]_{n+1} = \sum_{l=0}^{\infty} N(n,l)\, a^{n,0}_l \left[P^{n+1}_k, P^n_l\right]_{n+1}. \tag{4.18}$$

Since Legendre polynomials of degree k are even if k is even, and odd otherwise, we may assume that k and l have the same parity. Since $\left[P^{n+1}_k, P^n_l\right]_{n+1}$ vanishes for $l < k$ (see the next calculation), let $l \geq k$ and put

$$\beta := \frac{n-2}{2}.$$

If $\beta + k \geq \frac{1}{2}$, that is, $(n,k) \neq (2,0)$, then it follows from (4.6), integration by parts, (4.7), and (4.8) that

$$\begin{aligned}
\left[P_k^{n+1}, P_l^n\right]_{n+1} &= \frac{(-1)^k}{2^k(\beta+1)_k} \int_{-1}^{1} \left(\frac{d^k}{dt^k}(1-t^2)^{\beta+k}\right) P_l^n(t)\,dt \\
&= \frac{1}{2^k(\beta+1)_k} \int_{-1}^{1} (1-t^2)^{\beta+k} \left(\frac{d^k}{dt^k} P_l^n(t)\right) dt \\
&= \frac{N(n+2k,l-k)}{N(n,l)} \int_{-1}^{1} (1-t^2)^{\beta+k}\, P_{l-k}^{n+2k}(t)\,dt \\
&= \frac{\beta+l}{N(n,l)(\beta+k)} \int_{-1}^{1} (1-t^2)^{\beta+k}\, C_{l-k}^{\beta+k}(t)\,dt.
\end{aligned}$$

For $\alpha \in \frac{1}{2}\mathbb{N}$ and even m, we have (cf. [**18**, p. 424])

$$c_m^\alpha = \int_{-1}^{1} (1-t^2)^\alpha\, C_m^\alpha(t)\,dt = -\frac{\alpha\, 4^{\alpha+\frac{1}{2}}\, m!\,\Gamma\left(\frac{m}{2}+\alpha+1\right)^2}{(m-1)\left(\frac{m}{2}+\alpha\right)(m+2\alpha+1)!\,\Gamma\left(\frac{m}{2}+1\right)^2}.$$

Plugging this into (4.18) and changing the summation index, yields

$$a_k^{n,1} = \sum_{l=0}^{\infty} \frac{\beta+k+2l}{\beta+k}\, a_{k+2l}^{n,0}\, c_{2l}^{\beta+k} = \left(\beta+\frac{1}{2}\right) 4^{\beta+k+1} \sum_{l=0}^{\infty} q(\beta,k,l),$$

where

$$q(\beta,k,l) = \frac{2l\,(2l-2)!\,(\beta+k+2l)\,\Gamma(\beta+k+l+1)^2}{(k+2l-1)\,(2\beta+k+2l+1)\,(\beta+k+l)\,(2\beta+2k+2l+1)!\,\Gamma(l+1)^2}.$$

Using Zeilberger's algorithm (see, e.g., [**49**]), we find that q satisfies the following recurrence relation

$$A(\beta,k)q(\beta+1,k,l)+B(\beta,k)q(\beta,k,l) = q(\beta,k,l+1)C(\beta,k,l+1)-q(\beta,k,l)C(\beta,k,l),$$

where

$$A(\beta,k) = 4(2\beta+k+4), \quad B(\beta,k) = -(2\beta+k+1), \quad C(\beta,k,l) = -\frac{l(k+2l-1)}{\beta+k+2l}.$$

If we let $Q(\beta,k) = \sum_{l=0}^{\infty} q(\beta,k,l)$, then we obtain

$$Q(\beta+1,k) = \frac{2\beta+k+1}{4(2\beta+k+4)}\, Q(\beta,k)$$

or, in terms of the multipliers,

$$a_k^{n+2,1} = \frac{(n+1)\,(n+k-1)}{(n-1)\,(n+k+2)}\, a_k^{n,1}. \tag{4.19}$$

The function q also satisfies the recurrence relation

$$D(\beta,k)q(\beta,k{+}2,l){+}E(\beta,k)q(\beta,k,l) = q(\beta,k,l{+}1)F(\beta,k,l{+}1){-}q(\beta,k,l)F(\beta,k,l),$$

where

$$D(\beta,k) = 16(k+2)(2\beta+k+4), \quad E(\beta,k) = -(k-1)(2\beta+k+1)$$

and

$$F(\beta,k,l) = -\frac{1}{(\beta+k+2l)(2\beta+2k+2l+3)} \sum_{i=1}^{3} l^i \, p_i(\beta,k),$$

with polynomials p_1, p_2, p_3 given by

$$p_1(\beta,k) = 8\beta^3 + 16k\beta^2 + 20\beta^2 + 12k^2\beta + 26k\beta + 10\beta + 4k^3 + 9k^2 + 4k + 3,$$
$$p_2(\beta,k) = 16\beta^2 + 24k\beta + 32\beta + 12k^2 + 24k + 4,$$
$$p_3(\beta,k) = 8\beta + 8k + 12.$$

Summing again over all l, we arrive at

$$Q(\beta,k+2) = \frac{(k-1)\,(2\beta+k+1)}{16(k+2)\,(2\beta+k+4)}\, Q(\beta,k).$$

In terms of the multipliers this means

$$a_{k+2}^{n,1} = \frac{(k-1)\,(n+k-1)}{(k+2)\,(n+k+2)}\, a_k^{n,1}. \tag{4.20}$$

In order to solve (4.19) and (4.20), we need four initial values of $a_k^{n,1}$. We also have to calculate $a_0^{2,1}$, which was not covered by the above arguments. Using (4.14), (4.15), and (4.16), elementary integration yields

$$a_0^{2,1} = \frac{\pi^2}{4}, \quad a_2^{2,1} = -\frac{\pi^2}{32}, \quad a_3^{2,1} = -\frac{4}{45}, \quad a_0^{3,1} = \frac{2\pi}{3}, \quad a_3^{3,1} = -\frac{\pi}{24}. \tag{4.21}$$

This leads to the sequence

$$a_k^{n,1} = -\frac{\pi}{8}(n-1)\frac{\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{n+k-1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{n+k+2}{2}\right)} \tag{4.22}$$

which satisfies (4.19), (4.20), and has the initial values (4.21).

Now let $d \geq 0$ be arbitrary. For $l \geq 2$, the Legendre polynomials satisfy the recurrence relation (see, e.g., [**20**, Lemma 3.3.10])

$$(n{+}d{+}2l{-}2)(n{+}d{-}1)P_l^{n+d} = (n{+}d{+}l{-}2)(n{+}d{+}l{-}1)P_l^{n+d+2}{-}(l{-}1)lP_{l-2}^{n+d+2}.$$

From this and the fact that $P_0^m(t) = 1$, $P_1^m(t) = t$ for all $m \geq 1$, we obtain

$$\begin{aligned} a_k^{n,d+2} &= \omega_{n+d+1} \left[P_k^{n+d+2}, g_n \right]_{n+d+2} \\ &= \omega_{n+d+1} \sum_{l=0}^{\infty} \frac{N(n+d,l)}{\omega_{n+d}} a_l^{n,d} \left[P_k^{n+d+2}, P_l^{n+d} \right]_{n+d+2} \\ &= \frac{2\pi}{n+d+2k} \left(a_k^{n,d} - a_{k+2}^{n,d} \right). \end{aligned}$$

Finally, the sequence which solves this recurrence relation and has the initial values (4.17) and (4.22) is given by

$$a_k^{n,d} = -\frac{\pi^{\frac{d}{2}}(n-1)}{4} \frac{\Gamma\left(\frac{d+2}{2}\right)\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{n+k-1}{2}\right)}{\Gamma\left(\frac{d+k+1}{2}\right)\Gamma\left(\frac{n+d+k+1}{2}\right)}, \qquad k \neq 1.$$

■

We end this section with the following important definition, given rise to by Theorem 4.3.

Definition *For $2 \leq j \leq n$, let $\Box_j : C_{\mathrm{o}}^{\infty}(S^{n-1}) \rightarrow C_{\mathrm{o}}^{\infty}(S^{n-1})$ denote the linear operator which is inverse to the integral transform F_{g_j}.*

5. Generalized valuations and Minkowski valuations

In the following we recall several results on translation invariant (scalar and convex body valued) valuations, in particular, the product structure on smooth valuations and the Alesker–Poincaré duality. We also discuss basic properties of the Hard Lefschetz operators and a new isomorphism between generalized valuations of degree one and generalized functions on the sphere. At the end of this section, we state a recent representation theorem for Minkowski valuations intertwining rigid motions and give an alternative description of the classes $\mathbf{MVal}_{i,j}^{\mathrm{SO}(n)}$.

A map μ defined on convex bodies in $\mathbb{R}^n$ and taking values in an Abelian semigroup A is called a *valuation* or *additive* if

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L)$$

whenever $K \cup L$ is convex. If G is a group of affine transformations on $\mathbb{R}^n$, a valuation μ is called G-invariant if $\mu(gK) = \mu(K)$ for all $K \in \mathcal{K}^n$ and $g \in G$.

Let **Val** denote the vector space of continuous translation invariant scalar valued valuations. The structure theory of translation invariant valuations has its starting point in a classical result of McMullen [**43**], who showed that

$$\mathbf{Val} = \bigoplus_{0 \le i \le n} \mathbf{Val}_i^+ \oplus \mathbf{Val}_i^-, \tag{5.1}$$

where $\mathbf{Val}_i^+ \subseteq \mathbf{Val}$ denotes the subspace of *even* valuations (homogeneous) of degree i, and $\mathbf{Val}_i^-$ denotes the subspace of *odd* valuations of degree i. The space **Val** becomes a Banach space, when endowed with the norm

$$\|\mu\| = \sup\{|\mu(K)| : K \subseteq B\}.$$

The general linear group $\mathrm{GL}(n)$ acts on the Banach space **Val** in a natural way: For every $A \in \mathrm{GL}(n)$ and $\mu \in \mathbf{Val}$,

$$(A \cdot \mu)(K) = \mu(A^{-1}K), \qquad K \in \mathcal{K}^n.$$

Note that the subspaces $\mathbf{Val}_i^\pm$ are invariant under this $\mathrm{GL}(n)$-action. In fact, a deep result of Alesker [**3**], known as the Irreducibility Theorem, states that these subspaces are also irreducible:

Theorem 5.1 (Alesker [**3**]) *The natural representation of* $\mathrm{GL}(n)$ *on* $\mathbf{Val}_i^\pm$ *is irreducible for any* $i \in \{0, \dots, n\}$.

It follows from Theorem 5.1 that any $\mathrm{GL}(n)$-invariant subspace of translation invariant continuous valuations (of a given degree i and parity) is already dense in $\mathbf{Val}_i^\pm$.

Definition *A valuation* $\mu \in \mathbf{Val}$ *is called smooth if the map* $\mathrm{GL}(n) \to \mathbf{Val}$ *defined by* $A \mapsto A \cdot \mu$ *is infinitely differentiable.*

The subspace of smooth translation invariant valuations is denoted by $\mathbf{Val}^\infty$ and we write $\mathbf{Val}_i^{\pm,\infty}$ for smooth valuations in $\mathbf{Val}_i^\pm$. It is well known (cf. [**61**, p. 32]) that $\mathbf{Val}_i^{\pm,\infty}$ is a dense $\mathrm{GL}(n)$ invariant subspace of $\mathbf{Val}_i^\pm$. Moreover, $\mathbf{Val}^\infty$ carries a natural Fréchet space topology, called Gårding topology (see [**61**, p. 33]), which is stronger than the topology induced from **Val**. Finally, we note that the representation of $\mathrm{GL}(n)$ on $\mathbf{Val}^\infty$ is continuous.

Examples:

(a) If $L \in \mathcal{K}^n$ is strictly convex with smooth boundary, then

$$\mu_L : \mathcal{K}^n \to \mathbb{R}, \qquad \mu_L(K) = V_n(K + L),$$

is a smooth valuation.

(b) If $f \in C^\infty_o(S^{n-1})$ and $0 \leq i \leq n-1$, then $\nu_{i,f} : \mathcal{K}^n \to \mathbb{R}$, defined by

$$\nu_{i,f}(K) = \int_{S^{n-1}} f(u)\, dS_i(K,u), \tag{5.2}$$

is a smooth valuation in $\mathbf{Val}_i^\infty$.

Before we turn to generalized valuations, we recall the definition of the Alesker product of smooth translation invariant valuations.

Theorem 5.2 **([5])** *There exists a bilinear product*

$$\mathbf{Val}^\infty \times \mathbf{Val}^\infty \to \mathbf{Val}^\infty, \quad (\mu, \nu) \mapsto \mu \cdot \nu,$$

which is uniquely determined by the following two properties:

(i) The product is continuous in the Gårding topology.

(ii) If $L_1, L_2 \in \mathcal{K}^n$ are strictly convex and smooth, then

$$(\mu_{L_1} \cdot \mu_{L_2})(K) = V_{2n}(\iota(K) + L_1 \times L_2),$$

where $\iota : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is defined by $\iota(x) = (x, x)$.

Endowed with this multiplicative structure, $\mathbf{Val}^\infty$ becomes an associative and commutative algebra which is graded by the degree of homogeneity and with unit given by the Euler characteristic.

The next example was computed in [**5**] and will be needed in the appendix.

Example:
Let $L_1, \ldots, L_{n-i} \in \mathcal{K}^n$ and $M_1, \ldots, M_i \in \mathcal{K}^n$ be strictly convex and smooth. If $\mu \in \mathbf{Val}_i^\infty$ and $\nu \in \mathbf{Val}_{n-i}^\infty$ are defined by

$$\mu(K) = V(K[i], L_1, \ldots, L_{n-i}) \quad \text{and} \quad \nu(K) = V(K[n-i], M_1, \ldots, M_i),$$

then

$$(\mu \cdot \nu)(K) = \binom{n}{i}^{-1} V(-L_1, \ldots, -L_{n-i}, M_1, \ldots, M_i)\, V_n(K). \tag{5.3}$$

The above example is just a special case of the more general fact that the Alesker product gives rise to a nondegenerate bilinear pairing between smooth valuations of complementary degree.

Theorem 5.3 ([**5**]) *For every $0 \le i \le n$, the continuous bilinear pairing*

$$<\cdot\,,\cdot>: \mathbf{Val}_i^\infty \times \mathbf{Val}_{n-i}^\infty \to \mathbf{Val}_n, \qquad (\mu,\nu) \mapsto \mu \cdot \nu,$$

is nondegenerate. In particular, the induced Poincaré duality map

$$\mathbf{Val}_i^\infty \to \left(\mathbf{Val}_{n-i}^\infty\right)^* \otimes \mathbf{Val}_n, \qquad \mu \mapsto <\mu, \cdot>,$$

is continuous, injective and has dense image with respect to the weak topology.

Here and in the following, for a Fréchet space X, we denote by X^* its topological dual endowed with the weak topology.

Motivated by Theorem 5.3, the notion of generalized valuations was introduced recently [**8**]. Before we state the definition, recall that by a classical theorem of Hadwiger [**26**, p. 79] the space $\mathbf{Val}_n$ is spanned by the ordinary volume V_n. In other words, if we do not refer to any Euclidean structure, then $\mathbf{Val}_n \cong \mathscr{D}(V)$, where $\mathscr{D}(V)$ denotes the vector space of all densities on an n-dimensional vector space V (see the Appendix for details).

Definition *The space of* generalized valuations *is defined by*

$$\mathbf{Val}^{-\infty} = \left(\mathbf{Val}^\infty\right)^* \otimes \mathscr{D}(V)$$

and we define the space of generalized valuations of degree $i \in \{0,\ldots,n\}$ by

$$\mathbf{Val}_i^{-\infty} = \left(\mathbf{Val}_{n-i}^\infty\right)^* \otimes \mathscr{D}(V).$$

By Theorem 5.3, we have a canonical embedding with dense image

$$\mathbf{Val}^\infty \hookrightarrow \mathbf{Val}^{-\infty}.$$

Thus, $\mathbf{Val}^{-\infty}$ can be seen as a completion of $\mathbf{Val}^\infty$ in the weak topology.

In order to establish Theorem 3, we need the following new classification of generalized valuations of degree 1. A proof of this theorem was given by Semyon Alesker and is included in the appendix.

Theorem 5.4 *The map*

$$C_{\mathrm{o}}^\infty(S^{n-1}) \to \mathbf{Val}_1^\infty, \qquad f \mapsto \left(K \mapsto \int_{S^{n-1}} f(u)\, h(K,u)\, du\right),$$

is an isomorphism of Fréchet spaces which extends uniquely by continuity in the weak topologies to an isomorphism

$$C_{\mathrm{o}}^{-\infty}(S^{n-1}) \to \mathbf{Val}_1^{-\infty}.$$

Note that, by Theorem 5.4, if $\gamma \in \mathbf{Val}_1^{-\infty}$ and $T_\gamma \in C_{\mathrm{o}}^{-\infty}(S^{n-1})$ is the corresponding distribution, then we can evaluate γ on convex bodies $K \in \mathcal{K}^n$ with smooth support function by

$$\gamma(K) := \langle h(K,\cdot), T_\gamma \rangle.$$

Next we briefly recall the Hard Lefschetz operators on smooth translation invariant scalar valuations. It is well known that McMullen's decomposition (5.1) of the space **Val** implies a general Steiner type formula for continuous translation invariant valuations which, in turn, gives rise to a derivation operator $\Lambda : \mathbf{Val} \to \mathbf{Val}$ defined by

$$(\Lambda\mu)(K) = \left.\frac{d}{dt}\right|_{t=0} \mu(K + tB).$$

Note that Λ commutes with the action of $\mathrm{O}(n)$ and that it preserves parity. Moreover, if $\mu \in \mathbf{Val}_i$, then $\Lambda\mu \in \mathbf{Val}_{i-1}$.

The importance of the operator Λ became evident from a Hard Lefschetz type theorem established by Alesker [**4**] for even valuations and by Bernig and Bröcker [**11**] for general valuations. More recently, a dual version of this fundamental result was established in [**6**, **7**]. There, the derivation operator Λ is replaced by an integration operator $\mathfrak{L} : \mathbf{Val} \to \mathbf{Val}$ defined by

$$(\mathfrak{L}\mu)(K) = (V_1 \cdot \mu)(K) = \int_{\mathrm{AGr}_{n-1,n}} \mu(K \cap E)\, dE, \tag{5.4}$$

where here and in the following $\mathrm{AGr}_{k,n}$ denotes the affine Grassmannian of k planes in $\mathbb{R}^n$ and integration is with respect to a (suitably normalized) invariant measure. The original definition of $\mathfrak{L}$ corresponds to the first equality in (5.4) and it was proved by Bernig [**10**] that the second equality holds. We also note that $\mathfrak{L}$ commutes with the action of $\mathrm{O}(n)$ and that it preserves parity. Moreover, if $\mu \in \mathbf{Val}_i$, then $\mathfrak{L}\mu \in \mathbf{Val}_{i+1}$.

In the final part of this section we turn to Minkowski valuations. Recall that the trivial Minkowski valuation maps every convex body to the set containing only the origin and that, for $0 \leq j \leq n$, we denote by $\mathbf{MVal}_j$ the set of all continuous, translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuations of degree j. In the next lemma we state basic properties of such Minkowski valuations which are well known (cf. [**9**, **47**, **55**]) and are needed in what follows.

Lemma 5.5 *If $\Phi_j \in \mathbf{MVal}_j$, $0 \leq j \leq n$, then the following statements hold:*

(a) The Steiner point of $\Phi_j K$ is at the origin, that is, $s(\Phi_j K) = o$ for every $K \in \mathcal{K}^n$.

(b) There exists $r_{\Phi_j} \geq 0$ such that

$$W_{n-1}(\Phi_j K) = r_{\Phi_j} W_{n-j}(K)$$

for every $K \in \mathcal{K}^n$. If Φ_j is non-trivial, then $r_{\Phi_j} > 0$.

(c) The $\mathrm{SO}(n-1)$ *invariant valuation $\nu_j \in \mathbf{Val}_j$, defined by*

$$\nu_j(K) = h(\Phi_j K, \bar{e})$$

uniquely determines Φ_j and is called the associated real valued valuation of $\Phi_j \in \mathbf{MVal}_j$.

Lemma 5.5 (c) motivated the following definition which first appeared in [**55**].

Definition *A Minkowski valuation $\Phi_j \in \mathbf{MVal}_j$, $0 \leq j \leq n$, is called* smooth *if its associated real valued valuation ν_j is smooth.*

Recall that smooth translation invariant scalar valuations are dense in all continuous translation invariant scalar valuations. However, this does not *directly* imply the same for Minkowski valuations but instead additional arguments were needed for the proof which was given in [**55**] for even and in [**58**] for general Minkowski valuations.

In order to state the crucial integral representation of smooth translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuations we have to briefly recall the convolution between functions and measures on S^{n-1}. First note that since $\mathrm{SO}(n)$ is a compact Lie group, the convolution $\sigma * \tau$ of signed measures σ, τ on $\mathrm{SO}(n)$ can be defined by

$$\int_{\mathrm{SO}(n)} f(\vartheta)\, d(\sigma * \tau)(\vartheta) = \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(n)} f(\eta\theta)\, d\sigma(\eta)\, d\tau(\theta), \qquad f \in C(\mathrm{SO}(n)).$$

Using the identification of the sphere S^{n-1} with the homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ leads to a one-to-one correspondence of $C(S^{n-1})$ and $\mathcal{M}(S^{n-1})$ with right $\mathrm{SO}(n-1)$ invariant functions and measures on $\mathrm{SO}(n)$, respectively. Using this correspondence, the convolution of measures on $\mathrm{SO}(n)$ induces a convolution product on $\mathcal{M}(S^{n-1})$ (cf. [**55**] for more details).

For spherical convolution zonal functions and measures on S^{n-1} play an essential role. We denote the set of continuous zonal functions on S^{n-1} by $C(S^{n-1},\bar{e})$. For $\sigma \in \mathcal{M}(S^{n-1})$, $f \in C(S^{n-1},\bar{e})$, and $\eta \in \mathrm{SO}(n)$, it is easy to check that

$$(\sigma * f)(\eta\bar{e}) = \int_{S^{n-1}} f(\eta^{-1}u)\, d\sigma(u). \tag{5.5}$$

Note that, by (5.5), we have, for every $\vartheta \in \mathrm{SO}(n)$, that

$$(\vartheta\sigma) * f = \vartheta(\sigma * f),$$

where $\vartheta\sigma$ is the image measure of σ under the rotation $\vartheta \in \mathrm{SO}(n)$. Moreover, from the obvious identification of zonal functions on S^{n-1} with functions on $[-1,1]$, (5.5), and the Funk–Hecke Theorem, it follows that there are $a_k^n[f] \in \mathbb{R}$ such that the spherical harmonic expansion of $\sigma * f \in C(S^{n-1})$ is given by

$$\sigma * f \sim \sum_{k=0}^{\infty} a_k^n[f]\, \pi_k\sigma.$$

Hence, convolution from the right induces a multiplier transformation. It is also not difficult to check from (5.5) that the convolution of zonal functions and measures is Abelian.

Another property of spherical convolution which is going to be critical for us is the fact that the convolution is selfadjoint, in particular, we have for all $\sigma,\tau \in \mathcal{M}(S^{n-1})$ and every $f \in C(S^{n-1},\bar{e})$,

$$\int_{S^{n-1}} (\sigma * f)(u)\, d\tau(u) = \int_{S^{n-1}} (\tau * f)(u)\, d\sigma(u). \tag{5.6}$$

We are now in a position to state a recent Hadwiger type theorem for smooth Minkowski valuations which is the key to the proof of Theorem 3.

Theorem 5.6 **([58])** *If $\Phi_j \in \mathbf{MVal}_j^\infty$, $j \in \{1,\ldots,n-1\}$, then there exists a unique $f \in C_{\mathrm{o}}^\infty(S^{n-1},\bar{e})$, called the* generating function *of Φ_j, such that for every $K \in \mathcal{K}^n$,*

$$h(\Phi_j K,\cdot) = S_j(K,\cdot) * f. \tag{5.7}$$

Examples:

(a) Kiderlen [**27**] proved (in a slightly different form) that if $\Phi_1 \in \mathbf{MVal}_1^\infty$, then there exists a unique $g \in C_o^\infty(S^{n-1}, \bar{e})$ such that for every $K \in \mathcal{K}^n$,

$$h(\Phi_1 K, \cdot) = h(K, \cdot) * g. \tag{5.8}$$

In order see how (5.8) is related to Theorem 5.6, we use (4.10) and the fact that $\square_n : C_o^\infty(S^{n-1}) \to C_o^\infty(S^{n-1})$ is a bijective multiplier transformation, to obtain a function $f \in C^\infty(S^{n-1}, \bar{e})$ with $\square_n f = g$ and conclude that

$$h(\Phi_1 K, \cdot) = h(K, \cdot) * g = h(K, \cdot) * \square_n f = \square_n h(K, \cdot) * f = S_1(K, \cdot) * f.$$

(b) The case $j = n - 1$ of Theorem 5.6 was first proved (in a more general form) in [**54**]. Moreover, it was also shown there that $\Phi_{n-1} \in \mathbf{MVal}_{n-1}^\infty$ is *even* if and only if there exists an o-symmetric body of revolution $L \in \mathcal{K}^n$ with smooth support function such that for every $K \in \mathcal{K}^n$,

$$h(\Phi_{n-1} K, \cdot) = S_{n-1}(K, \cdot) * h(L, \cdot).$$

(c) For $i \in \{1, \dots, n-1\}$, the support function of the projection body map of order i, $\Pi_i \in \mathbf{MVal}_i$, is given by

$$h(\Pi_i K, u) = V_i(K|u^\perp) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| \, dS_i(K, v), \qquad u \in S^{n-1}.$$

Note that Π_i is continuous but *not* smooth. Its (merely) continuous generating function is given by $f(u) = \frac{1}{2}|u \cdot \bar{e}|$, $u \in S^{n-1}$.

(d) For $i \in \{2, \dots, n\}$, the (normalized) mean section operator of order i, $\mathrm{M}_i \in \mathbf{MVal}_{n+1-i}$, was first defined in [**18**] by

$$h(\mathrm{M}_i K, \cdot) = \int_{\mathrm{AGr}_{i,n}} h(\mathrm{J}(K \cap E), \cdot) \, dE.$$

Here, $\mathrm{J} \in \mathbf{MVal}_1$ is defined by $\mathrm{J}K = K - s(K)$, where $s : \mathcal{K}^n \to \mathbb{R}^n$ is the Steiner point map. Recently, Goodey and Weil [**19**] proved that the generating functions of the mean section operators are up to normalization the zonal functions $\breve{g}_i \in L_1(S^{n-1}, \bar{e})$ determined by C. Berg's functions g_i on $[-1, 1]$. More precisely,

$$h(\mathrm{M}_i K, \cdot) = p_{n,i} \, S_{n+1-i}(K, \cdot) * \breve{g}_i, \tag{5.9}$$

with constants $p_{n,i}$ which were explicitly determined in [**19**].

The integration operator $\mathfrak{L}$ on translation invariant scalar valuations can be extended to Minkowski valuations by using (5.4):

$$h((\mathfrak{L}\Phi)(K),\cdot)=\int_{\mathrm{AGr}_{n-1,n}} h(\Phi(K\cap E),\cdot)\,dE.$$

It was proved in [**47**] that also the derivation operator Λ can be extended to continuous translation invariant Minkowski valuations:

$$h((\Lambda\Phi)(K),\cdot)=\left.\frac{d}{dt}\right|_{t=0} h(\Phi(K+tB),\cdot).$$

Note that in this case it is not trivial that the right hand side actually defines the support function of a convex body; this was proved in [**47**].

If $\Phi_j\in\mathbf{MVal}_j^\infty$, $1\leq j\leq n-1$, with associated real valued valuation $\nu_j\in\mathbf{Val}_j^\infty$, then the associated real valued valuations of $\mathfrak{L}\Phi_j$ and $\Lambda\Phi_j$ are given by $\mathfrak{L}\nu_j\in\mathbf{Val}_{j+1}^\infty$ and $\Lambda\nu_j\in\mathbf{Val}_{j+1}^\infty$, respectively. In particular, we have $\mathfrak{L}\Phi_j\in\mathbf{MVal}_{j+1}^\infty$ and $\Lambda\Phi_j\in\mathbf{MVal}_{j-1}^\infty$.

In view of Theorem 5.6, it is a natural problem to determine the induced action of the $\mathrm{SO}(n)$ equivariant operators Λ and $\mathfrak{L}$ on the generating functions of smooth Minkowski valuations. This was done in [**57**] and is the content of the following theorem.

Theorem 5.7 **([57])** *Suppose that $\Phi_j\in\mathbf{MVal}_j^\infty$ and let $f\in C_{\mathrm{o}}^\infty(S^{n-1},\bar{e})$ be the generating function of Φ_j.*

(a) If $2\leq j\leq n-1$, then the generating function of $\Lambda\Phi_j$ is given by jf.

(b) If $1\leq j\leq n-2$, then there exists a constant $c_{n,j}>0$ such that the generating function of $\mathfrak{L}\Phi_j$ is given by $c_{n,j}\,\Box_{n-j+1}f*\breve{g}_{n-j}$.

In particular, $\Lambda:\mathbf{MVal}_j^\infty\to\mathbf{MVal}_{j-1}^\infty$ is injective for all $2\leq j\leq n-1$ and $\mathfrak{L}:\mathbf{MVal}_j^\infty\to\mathbf{MVal}_{j+1}^\infty$ is injective for all $1\leq j\leq n-2$.

The constants $c_{n,j}$ from Theorem 5.7 (b) were explicitly determined in [**57**]. We also note that, by Theorem 5.7, for $i>j$, the map

$$\Lambda^{j-i}:\mathbf{MVal}_{j,i}^\infty\to\mathbf{MVal}_i^\infty$$

is well defined. From Theorem 5.7 (a) and Examples (a) and (b) above, we can also deduce more information about the classes $\mathbf{MVal}_{j,i}^\infty$.

Corollary 5.8

(a) Suppose that $1 \leq i, j \leq n-1$, $\Phi_j \in \mathbf{MVal}_j^\infty$, and let $f \in C_{\mathrm{o}}^\infty(S^{n-1}, \bar{e})$ be the generating function of Φ_j. Then $\Phi_j \in \mathbf{MVal}_{j,i}^\infty$ if and only if $S_i(K, \cdot) * f$ is a support function for every $K \in \mathcal{K}^n$.

(b) $\mathbf{MVal}_{1,n-1}^\infty \subsetneq \mathbf{MVal}_1^\infty$.

Proof. Statement (a) is a direct consequence of the definition of $\mathbf{MVal}_{j,i}^\infty$ and Theorem 5.7 (a).

In order to prove (b) let $\Phi_1 \in \mathbf{MVal}_1^\infty$ be even and let $f \in C_{\mathrm{o}}^\infty(S^{n-1}, \bar{e})$ be the generating function of Φ_1. Then, by (a) and Example (b) from above, $\Phi_1 \in \mathbf{MVal}_{1,n-1}^\infty$ if and only if $f = h(L, \cdot)$ for some o-symmetric body of revolution $L \in \mathcal{K}^n$. In this case, we have

$$h(\Phi_1 K, \cdot) = S_1(K, \cdot) * h(L, \cdot) = \Box_n h(K, \cdot) * h(L, \cdot) = h(K, \cdot) * s_1(L, \cdot),$$

where $s_1(L, \cdot) = \Box_n h(L, \cdot)$ is the smooth density of $S_1(L, \cdot)$. It was proved by Kiderlen [**27**] that for any (even) non-negative $g \in C_{\mathrm{o}}^\infty(S^{n-1}, \bar{e})$, (5.8) defines an (even) Minkowski valuation in $\mathbf{MVal}_1^\infty$. Since the set of area measures of order 1 is nowhere dense in $\mathcal{M}_{\mathrm{o}}$, this proves the claim. ■

Note that, by Corollary 5.8 (b), in general $\mathbf{MVal}_{j,i}^\infty \subsetneq \mathbf{MVal}_j^\infty$ for $i > j$. Explicit examples of Minkowski valuations in $\mathbf{MVal}_j$ with generating functions which do not generate a Minkowski valuation in $\mathbf{MVal}_{n-1}$ are provided by the mean section operators. This follows from (5.9) and the case $i = n-1$ of Theorem 5.6 for continuous Minkowski valuations established in [**54**], where it was proved that $\Phi_{n-1} \in \mathbf{MVal}_{n-1}$ is generated by a *continuous* function $f \in C_{\mathrm{o}}(S^{n-1}, \bar{e})$. However, C. Berg's functions g_i are not continuous on $[-1, 1]$ for $i \geq 5$.

We end this section with another remark concerning Corollary 5.8 (a): Generating functions or earlier versions of Theorem 5.6, respectively, were the critical tool used in the proofs of the first Brunn–Minkowski type inequalities for Minkowski valuations. In the next section we will see that the Hard Lefschetz operators on Minkowski valuations (which were introduced only recently) and Theorem 5.6 both naturally lead to the same classes $\mathbf{MVal}_{j,i}$ for which we can establish such inequalities.

6. Proofs of main results

After these preparations, we are now in a position to complete the proofs of Theorems 3, 4, and 5. We begin with Theorem 3 which we first recall.

Theorem 6.1 *Let $\Phi_j \in \mathbf{MVal}_j^\infty$, $2 \leq j \leq n-1$. For $j+2 \leq i \leq n$ and every $L \in \mathcal{K}^n$, there exists $\gamma_{i,j}(L,\cdot) \in \mathbf{Val}_1^{-\infty}$ such that*

$$W_{n-i}(K, \Phi_j L) = \gamma_{i,j}(L, (\mathfrak{L}^{i-j-1}\Phi_j)(K))$$

for every $K \in \mathcal{K}^n$. Moreover, if $\Phi_j \in \mathbf{MVal}_{j,i-1}^\infty$, then

$$\gamma_{i,j}(L, (\mathfrak{L}^{i-j-1}\Phi_j)(K)) = \frac{(i-1)!}{j!} W_{n-1-j}(L, (\Lambda^{j+1-i}\Phi_j)(K)).$$

Proof. First define an isomorphism $\Theta_j : C_{\mathrm{o}}^\infty(S^{n-1}) \to C_{\mathrm{o}}^\infty(S^{n-1})$ by

$$\Theta_j \zeta = c_{n,j} \,\Box_{n-j+1}\zeta * \breve{g}_{n-j} = c_{n,j}\, \zeta * \Box_{n-j+1}\breve{g}_{n-j},$$

where the constant $c_{n,j} > 0$ is as in Theorem 5.7 (b). Here, the second equality follows from the fact that multiplier transformations commute and $\Box_{n-j+1}\breve{g}_{n-j}$ is to be understood in the sense of distributions, where we use the canonical extension of the selfadjoint operator $\Box_{n-j+1}$ to $C_{\mathrm{o}}^{-\infty}(S^{n-1})$.

Let $\tau_{\bar{e}} = \delta_{\bar{e}} - \pi_1\delta_{\bar{e}} \in \mathcal{M}_{\mathrm{o}}(S^{n-1})$, where $\delta_{\bar{e}}$ is the Dirac measure supported in $\bar{e} \in S^{n-1}$. Then, by (5.5), $\zeta * \tau_{\bar{e}} = \zeta$ for every $\zeta \in C_{\mathrm{o}}^\infty(S^{n-1})$. Now since $\Box_k \breve{g}_k = \tau_{\bar{e}}$, it follows from Theorem 5.7 (b) that if $f \in C_{\mathrm{o}}^\infty(S^{n-1}, \bar{e})$ is the generating function of Φ_j, then $\mathfrak{L}^{i-j-1}\Phi_j \in \mathbf{MVal}_{i-1}^\infty$ is generated by

$$\Theta_{i-2}\Theta_{i-1}\cdots\Theta_{j+1}\Theta_j f = q_{n,i,j}\,\Box_{n-j+1} f * \breve{g}_{n-i+2}, \tag{6.1}$$

where $q_{n,i,j} = \prod_{m=j}^{i-2} c_{n,m} > 0$. Note that the inverse of the isomorphism (6.1) is, for $\zeta \in C_{\mathrm{o}}^\infty(S^{n-1})$, given by

$$q_{n,i,j}^{-1}\,\Box_{n-i+2}\zeta * \breve{g}_{n-j+1}.$$

For every $L \in \mathcal{K}^n$ we define a distribution $T_{i,j}(L) \in C_{\mathrm{o}}^{-\infty}(S^{n-1})$ by

$$\langle \zeta, T_{i,j}(L)\rangle = q_{n,i,j}^{-1} \int_{S^{n-1}} (\Box_{n-i+2}\zeta * \breve{g}_{n-j+1})(u)\, dS_j(L,u)$$

for $\zeta \in C_{\mathrm{o}}^\infty(S^{n-1})$. Let $\gamma_{i,j}(L,\cdot) \in \mathbf{Val}_1^{-\infty}$ be the generalized valuation corresponding to $T_{i,j}(L)$ determined by Theorem 5.4.

Since $\mathfrak{L}^{i-j-1}\Phi_j$ is smooth, it follows that $h((\mathfrak{L}^{i-j-1}\Phi_j)(K),\cdot)$ is smooth for every $K \in \mathcal{K}^n$. Hence, we can evaluate $\gamma_{i,j}(L,\cdot)$ on $(\mathfrak{L}^{i-j-1}\Phi_j)(K)$. Using that

$$h((\mathfrak{L}^{i-j-1}\Phi_j)(K),\cdot) = q_{n,i,j}\, S_{i-1}(K,\cdot) * (\Box_{n-j+1} f * \breve{g}_{n-i+2}),$$

we obtain

$$\begin{aligned} \gamma_{i,j}(L,(\mathfrak{L}^{i-j-1}\Phi_j)(K)) &= \langle h((\mathfrak{L}^{i-j-1}\Phi_j)(K),\cdot), T_{i,j}(L)\rangle \\ &= \int_{S^{n-1}} (S_{i-1}(K,\cdot) * f)(u)\, dS_j(L,u). \end{aligned}$$

Now on one hand it follows from (5.6), that

$$\gamma_{i,j}(L,(\mathfrak{L}^{i-j-1}\Phi_j)(K)) = \int_{S^{n-1}} (S_j(L,\cdot) * f)(u)\, dS_{i-1}(K,u) = W_{n-i}(K,\Phi_j L).$$

On the other hand, if $\Phi_j \in \mathbf{MVal}_{j,i-1}^{\infty}$, then, by Theorem 5.7 (a),

$$S_{i-1}(K,\cdot) * f = \frac{(i-1)!}{j!} h((\Lambda^{j+1-i}\Phi_j)(K),\cdot)$$

and, thus,

$$\gamma_{i,j}(L,(\mathfrak{L}^{i-j-1}\Phi_j)(K)) = \frac{(i-1)!}{j!} W_{n-1-j}(L,(\Lambda^{j+1-i}\Phi_j)(K))$$

which completes the proof. ■

Note that, by Theorem 4.4 and Lemma 4.2, $\Box_{n-i+2}\breve{g}_{n-j+1} \in C_{\mathrm{o}}^{-\infty}(S^{n-1})$ and that if $f \in C_{\mathrm{o}}(S^{n-1})$, then also

$$f * \Box_{n-i+2}\breve{g}_{n-j+1} = \Box_{n-i+2} f * \breve{g}_{n-j+1} \in C_{\mathrm{o}}^{-\infty}(S^{n-1}). \tag{6.2}$$

However, in general (6.2) does *not* define a continuous function on S^{n-1} if f is merely continuous.

Next, we note that using Theorem 5.6 we can also give a new and short proof of Theorem 2: If $\Phi_j \in \mathbf{MVal}_j^{\infty}$, $2 \le j \le n-1$, has generating function $f \in C_{\mathrm{o}}^{\infty}(S^{n-1})$ and $1 \le i \le j+1$, then, by (5.6) and Theorem 5.7 (a),

$$\begin{aligned} W_{n-i}(K,\Phi_j L) &= \int_{S^{n-1}} (S_{i-1}(K,\cdot) * f)(u)\, dS_j(L,u) \\ &= \frac{(i-1)!}{j!}\, W_{n-j-1}(L,(\Lambda^{j+1-i}\Phi_j)(K)) \end{aligned}$$

for every $K, L \in \mathcal{K}^n$.

Putting together Theorem 2 and Theorem 6.1 we obtain the following.

Corollary 6.2 *For $1 \leq i \leq n$, $2 \leq j \leq n-1$, and $\Phi_j \in \mathbf{MVal}_{j,i-1}^{\infty}$, we have*

$$W_{n-i}(K, \Phi_j L) = \frac{(i-1)!}{j!} W_{n-1-j}(L, (\Lambda^{j+1-i}\Phi_j)(K)) \tag{6.3}$$

for every $K, L \in \mathcal{K}^n$.

For the proof of Theorem 5 and in order to establish the equality cases in Theorem 4, we need the following monotonicity property of Minkowski valuations:

Lemma 6.3 *Suppose that $1 \leq i \leq n$, $2 \leq j \leq n-1$, and let $\Phi_j \in \mathbf{MVal}_{j,i-1}$ be non-trivial. If $K, L \in \mathcal{K}^n$ have non-empty interiors, then $K \subseteq L$ implies that*

$$W_{n-i}(\Phi_j K) \leq W_{n-i}(\Phi_j L) \tag{6.4}$$

with equality if and only if $K = L$. In particular, $W_{n-i}(\Phi_j K) > 0$ for every $K \in \mathcal{K}^n$ with non-empty interior.

Proof. We first assume that $i \geq 2$, that Φ_j is smooth and that K and L are of class C^2_+. In this case, it was proved in [**47**, p. 992] that $\Phi_j K$ and $\Phi_j L$ also have non-empty interiors. Moreover, by (6.3) and the monotonicity of mixed volumes, we have for every $Q \in \mathcal{K}^n$,

$$\begin{aligned} W_{n-i}(Q, \Phi_j L) &= \frac{(i-1)!}{j!} W_{n-1-j}(L, (\Lambda^{j+1-i}\Phi_j)(Q)) \\ &\geq \frac{(i-1)!}{j!} W_{n-1-j}(K, (\Lambda^{j+1-i}\Phi_j)(Q)) = W_{n-i}(Q, \Phi_j K). \end{aligned}$$

Thus, taking $Q = \Phi_j L$ and using inequality (3.3), yields

$$W_{n-i}(\Phi_j L)^i \geq W_{n-i}(\Phi_j L, \Phi_j K)^i \geq W_{n-i}(\Phi_j L)^{i-1} W_{n-i}(\Phi_j K)$$

which implies (6.4) since $W_{n-i}(\Phi_j L) > 0$. If $\Phi_j \in \mathbf{MVal}_{j,i-1}$ is not smooth and K and L are arbitrary, (6.4) follows by approximation.

In order to establish the equality conditions first note that, by the $\mathrm{SO}(n)$ equivariance of Φ_j, the convex body $\Phi_j B$ must be an o-symmetric ball. Moreover, from Lemma 5.5 (b), it follows that $\Phi_j B = r_{\Phi_j} B$, where $r_{\Phi_j} > 0$. Thus, since K and L have non-empty interiors, we conclude from (6.4) that $W_{n-i}(\Phi_j K), W_{n-i}(\Phi_j L) > 0$ or, equivalently, that $\Phi_j K$ and $\Phi_j L$ have dimension at least i holds for all $K, L \in \mathcal{K}^n$ with non-empty interiors.

Assume now that equality holds in (6.4). Then, by the equality conditions of (3.3) and Lemma 5.5 (a), there exists an $\alpha > 0$ such that $\Phi_j K = \alpha\, \Phi_j L$. It follows from equality in (6.4) that $\alpha = 1$. Thus, by Lemma 5.5 (b), we have

$$W_{n-j}(K) = r_{\Phi_j}^{-1} W_{n-1}(\Phi_j K) = r_{\Phi_j}^{-1} W_{n-1}(\Phi_j L) = W_{n-j}(L). \tag{6.5}$$

Using again the monotonicity of mixed volumes and (3.3), we obtain

$$W_{n-j}(L)^j = W_{n-j}(L, L)^j \geq W_{n-j}(L, K)^j \geq W_{n-j}(L)^{j-1} W_{n-j}(K).$$

From (6.5) and the equality conditions of inequality (3.3), we conclude that K is a translate of L. But since $K \subseteq L$, we must have $K = L$.

Inequality (6.4) for $i = 1$ follows directly from Lemma 5.5 (b) and the monotonicity of quermassintegrals. If equality holds in (6.4) for $i = 1$, then we have (6.5) and therefore, as before, obtain that $K = L$. ■

In contrast to Lemma 6.3, we note that not every Minkowski valuation $\Phi_j \in \mathbf{MVal}_{j,i-1}$ is monotone with respect to set inclusion (cf. [**27**]). However, all known examples of Minkowski valuations $\Phi_j \in \mathbf{MVal}_j$, $1 \leq j \leq n-1$ are *weakly monotone*, that is, for every pair of convex bodies $K, L \in \mathcal{K}^n$ such that $K \subseteq L$, there exists a vector $x(K, L) \in \mathbb{R}^n$ such that

$$\Phi_j K \subseteq \Phi_j L + x(K, L).$$

It is an open problem whether all translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuations are weakly monotone. Using arguments as in the proof of Lemma 6.3, we can show the following.

Proposition 6.4 *Suppose that* $2 \leq j \leq n-1$. *If* $\Phi_j \in \mathbf{MVal}_{j,n-1}$, *then* Φ_j *is weakly monotone.*

Proof. Without loss of generality we may assume that Φ_j is smooth. If $K, L \in \mathcal{K}^n$ such that $K \subseteq L$, then, as in Lemma 6.3, it follows from (6.3) and the monotonicity of mixed volumes that for every $Q \in \mathcal{K}^n$,

$$\begin{aligned} W_0(Q, \Phi_j L) &= \frac{(n-1)!}{j!} W_{n-1-j}(L, (\Lambda^{j+1-n}\Phi_j)(Q)) \\ &\geq \frac{(n-1)!}{j!} W_{n-1-j}(K, (\Lambda^{j+1-n}\Phi_j)(Q)) = W_0(Q, \Phi_j K). \end{aligned}$$

But it is well known (cf. [**54**, Corollary 4.3]) that $W_0(Q, \Phi_j K) \leq W_0(Q, \Phi_j L)$ for every $Q \in \mathcal{K}^n$ implies $\Phi_j K \subseteq \Phi_j L + x$ for some $x \in \mathbb{R}^n$. ■

We return now to the proof of Theorem 4 which we also first recall.

Theorem 6.5 *Let* $1 \leq i \leq n$ *and let* $\Phi_j \in \mathbf{MVal}_{j,i-1}$, $2 \leq j \leq n-1$, *be non-trivial. If* $K, L \in \mathcal{K}^n$ *have non-empty interiors, then for all* $\lambda \in (0,1)$,

$$W_{n-i}(\Phi_j((1-\lambda)K + \lambda L)) \geq W_{n-i}(\Phi_j K)^{1-\lambda} W_{n-i}(\Phi_j L)^{\lambda}, \qquad (6.6)$$

with equality if and only if K *and* L *are translates of each other.*

Proof. First we assume that $i \geq 2$ and that Φ_j is smooth. We also use the abbreviations $K_\lambda = (1-\lambda)K + \lambda L$ and $Q = \Phi_j K_\lambda$. Then, by (6.3),

$$W_{n-i}(\Phi_j K_\lambda) = W_{n-i}(Q, \Phi_j K_\lambda) = \frac{(i-1)!}{j!} W_{n-1-j}(K_\lambda, (\Lambda^{j+1-i}\Phi_j)(Q)).$$

From an application of inequality (3.5), we therefore obtain

$$W_{n-i}(\Phi_j K_\lambda) \geq \frac{(i-1)!}{j!} W_{n\text{-}1\text{-}j}(K, (\Lambda^{j+1-i}\Phi_j)(Q))^{1-\lambda} W_{n\text{-}1\text{-}j}(L, (\Lambda^{j+1-i}\Phi_j)(Q))^{\lambda}.$$

Thus, using (6.3) again, we obtain

$$W_{n-i}(\Phi_j K_\lambda)^i \geq W_{n-i}(Q, \Phi_j K)^{i(1-\lambda)} W_{n-i}(Q, \Phi_j L)^{i\lambda}.$$

Now, if $\Phi_j \in \mathbf{MVal}_{j,i-1}$ is not smooth, then this inequality still follows by approximation. Hence, using (3.3) and the fact that, by Lemma 6.3, $W_{n-i}(Q) > 0$, we arrive at

$$W_{n-i}(\Phi_j K_\lambda)^i \geq W_{n-i}(Q)^{i-1} W_{n-i}(\Phi_j K)^{1-\lambda} W_{n-i}(\Phi_j L)^{\lambda}$$

which, by the definitions of $\Phi_j K_\lambda$ and Q, is the desired inequality (6.6).

In order to establish the equality conditions, first note that by Lemma 6.3, $\Phi_j K$, $\Phi_j L$, and $\Phi_j K_\lambda$ all have dimension at least i. Therefore, the equality conditions of inequality (3.3) imply that $\Phi_j K$ is homothetic to $\Phi_j K_\lambda$, which is in turn homothetic to $\Phi_j L$. In fact, by Lemma 5.5 (a), they have to be dilates of one another, that is, there exist $t_1, t_2 > 0$ such that

$$t_1 \Phi_j K = \Phi_j K_\lambda = t_2 \Phi_j L,$$

where $1 = t_1^{1-\lambda} t_2^{\lambda}$, by the equality in (6.6). Moreover, an application of Lemma 5.5 (b) yields $t_1 W_{n-j}(K) = W_{n-j}(K_\lambda) = t_2 W_{n-j}(L)$. Consequently, we have

$$W_{n-j}(K_\lambda) = W_{n-j}(K)^{1-\lambda} W_{n-j}(L)^{\lambda}.$$

By the equality conditions of inequality (3.4), this is possible only if K and L are translates. This completes the proof for $i \geq 2$. If $i = 1$, then the statement is an immediate consequence of Lemma 5.5 (b) and (3.4). ■

It remains to complete the proof of Theorem 5.

Theorem 6.6 *Let $\varphi \in \Theta_1$, $1 \leq i \leq n$, and let $\Phi_j \in \mathbf{MVal}_{j,i-1}$, $2 \leq j \leq n-1$, be non-trivial. If $K, L \in \mathcal{K}^n$ contain the origin, then for all $\lambda \in (0,1)$,*

$$W_{n-i}(\Phi_j(K +_{\varphi,\lambda} L)) \geq W_{n-i}(\Phi_j K)^{1-\lambda} W_{n-i}(\Phi_j L)^{\lambda}. \tag{6.7}$$

When φ is strictly convex and K and L have non-empty interiors, equality holds if and only if $K = L$.

Proof. First note that inequality (6.7) follows from Lemma 3.2, Lemma 6.3, and Theorem 6.5.

In order to establish the equality conditions, let φ be strictly convex and let K and L have non-empty interiors. It follows from the equality conditions of Lemma 6.3 that

$$K +_{\varphi,\lambda} L = (1-\lambda)K + \lambda L. \tag{6.8}$$

We want to show that this is possible only if $K = L$ or, equivalently, if $h(K,u) = h(L,u)$ for all $u \in S^{n-1}$. If $h(K,u) = h(L,u) = 0$, then there is nothing to prove. Therefore, we may assume that $h(K +_{\varphi,\lambda} L, u) > 0$. Now from the definition of the Orlicz convex combination, (6.8), together with the convexity of φ and our assumption that $\varphi(1) = 1$, we obtain

$$\varphi\left(\frac{(1-\lambda)h(K,u) + \lambda h(L,u)}{h(K +_{\varphi,\lambda} L, u)}\right) = 1.$$

But since we have assumed that φ is strictly convex, this implies that $h(K,u) = h(L,u)$. ∎

Like the classical inequality (3.4), Theorem 6.5 as well as Theorem 6.6 in case of a homogeneous addition are equivalent to corresponding additive versions. Here we state one such additive version for L_p Minkowski addition.

Corollary 6.7 *Let $p > 1$, $1 \leq i \leq n$, and let $\Phi_j \in \mathbf{MVal}_{j,i-1}$, $2 \leq j \leq n-1$, be non-trivial. If $K, L \in \mathcal{K}^n$ contain the origin in their interiors, then*

$$V_i(\Phi_j((1-\lambda) \cdot K +_p \lambda \cdot L))^{\frac{p}{ij}} \geq (1-\lambda) V_i(\Phi_j K)^{\frac{p}{ij}} + \lambda V_i(\Phi_j L)^{\frac{p}{ij}},$$

with equality if and only if K and L are dilates of each other.

We finally remark that the special case $j = n-1$ of Corollary 6.7 was recently obtained by Wang [**63**].

Appendix

by Semyon Alesker

The purpose of this appendix is to provide a proof of Theorem 5.4. To this end we first show that all valuations in $\mathbf{Val}_1^\infty$ and $\mathbf{Val}_{n-1}^\infty$ are of the form (5.2). In order to prove this, we want to apply the Irreducibility Theorem as well as a deep result from representation theory by Casselmann–Wallach [**14**]. Therefore, we need to rewrite the valuations $\nu_{1,f}$ and $\nu_{n-1,f}$ in $\mathrm{GL}(n)$ invariant terms without referring to a Euclidean structure.

Recall that a *line bundle* over a smooth manifold M consists of a smooth manifold E and a surjective smooth map $\pi : E \to M$ satisfying the following:

- For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ is a 1-dimensional vector space.
- Every $p \in M$ has an open neighborhood U in M for which there exists a diffeomorphism $\varrho : \pi^{-1}(U) \to U \times \mathbb{R}$ such that, for each $q \in U$, $\varrho(E_q) \subseteq \{q\} \times \mathbb{R}$ and $\varrho|_{E_q} : E_q \to \{q\} \times \mathbb{R}$ is a linear isomorphism.

For more information on line bundles, in particular, the definitions of the dual of a line bundle and the tensor product of line bundles needed in the following, see, e.g., [**60**, p. 4].

A *section* of a line bundle $\pi : E \to M$ is a continuous map $h : M \to E$ such that $h(p) \in E_p$ for every $p \in M$. We denote by $C(M, E)$ the vector space of all sections of E and by $C^\infty(M, E)$ the space of smooth sections of E endowed with the natural locally convex topology which makes it a Fréchet space (see, e.g, [**21**, Chapter 3]). A sequence of smooth sections converges in this topology if and only if in local coordinates all the derivatives converge uniformly on compact subsets.

Important examples of line bundles are density bundles of manifolds (see, e.g., [**29**, p. 429]). Recall that a *density* on an n-dimensional vector space V is a function on the n-fold product of V, $\delta : V \times \cdots \times V \to \mathbb{R}$, such that if $A : V \to V$ is any linear map, then

$$\delta(Av_1, \ldots, Av_n) = |\det A|\, \delta(v_1, \ldots, v_n).$$

We denote by $\mathscr{D}(V)$ the vector space of all densities on V and, as usual, write $\Lambda^k(V)$ for the kth exterior power of V.

Proposition A.1 **(**[**29**, p. 428]**)** *The vector space $\mathscr{D}(V)$ is 1-dimensional and spanned by $|\omega|$ for any nonzero $\omega \in \Lambda^n(V^*)$.*

The *density bundle* $\pi : \mathscr{D}M \to M$ of a smooth manifold M is defined by

$$\mathscr{D}M = \coprod_{p \in M} \mathscr{D}(T_pM),$$

where π is the natural projection map taking each element of $\mathscr{D}(T_pM)$ to p.

A *density* on M is a section of $\mathscr{D}M$. By Proposition A.1, any nonvanishing n-form ω on M determines a positive density $|\omega|$ on M. In fact, if ω is a nonvanishing n-form on an open subset $U \subseteq M$, then any density δ on U is of the form $\delta = f\,|\omega|$ for some continuous function f on U. From this, it is now straightforward to define the integral over M of a compactly supported density on M. We refer to [**29**, p. 431ff] for the details.

If M is a compact smooth manifold and $\pi : E \to M$ a line bundle over M, then, using integration of densities on M, one can define a canonical and nondegenerate bilinear pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle : C(M,E) \times C(M, E^* \otimes \mathscr{D}M) \to \mathbb{R}, \quad (f,g) \mapsto \int_M [f,g]. \tag{A.1}$$

Here, $C(M, E^* \otimes \mathscr{D}M)$ is the space of sections of the line bundle

$$E^* \otimes \mathscr{D}M \cong \operatorname{Hom}(E, \mathscr{D}M),$$

whose fiber at $p \in M$ is the space of all linear maps $E_p \to \mathscr{D}(T_pM)$, and

$$[\,\cdot\,,\,\cdot\,] : C(M,E) \times C(M, E^* \otimes \mathscr{D}M) \to C(M, \mathscr{D}M)$$

is pointwise just the evaluation map.

Now let V be an n-dimensional vector space and denote by $\mathbb{P}_+(V^*)$ the oriented projectivized cotangent bundle, that is, the compact smooth manifold given by

$$\mathbb{P}_+(V^*) = (V^* \backslash \{0\})/\mathbb{R}^+.$$

In the following we write $[\xi] := \operatorname{span} \xi$ for the 1-dimensional linear span of $\xi \in V^* \backslash \{0\}$ and we use $[\xi]_+$ to denote the elements of $\mathbb{P}_+(V^*)$. Note that if we choose a Euclidean structure on V, then we can identify V^* with V and $\mathbb{P}_+(V^*)$ is diffeomorphic to S^{n-1}. However, in contrast to S^{n-1}, we have a natural $\mathrm{GL}(V)$ action on $\mathbb{P}_+(V^*)$ given by

$$A \cdot [\xi]_+ = [A \cdot \xi]_+, \qquad A \in \mathrm{GL}(V),$$

where $A \cdot \xi \in V^*$ is defined by $(A \cdot \xi)(v) = \xi(A^{-1}v)$ for $v \in V$. Also note that $\mathrm{GL}(V)$ acts naturally on $\mathscr{D}(V)$ by

$$(A \cdot \delta)(v_1, \ldots, v_n) = \delta(A^{-1}v_1, \ldots, A^{-1}v_n), \qquad A \in \mathrm{GL}(V), v_j \in V.$$

Theorem A.2

(a) The map $C_{\mathrm{o}}^{\infty}(S^{n-1}) \to \mathbf{Val}_1^{\infty}$, $f \mapsto \nu_{1,f}$, *where*

$$\nu_{1,f}(K) = \int_{S^{n-1}} f(u)\, dS_1(K,u), \tag{A.2}$$

is an isomorphism of Fréchet spaces.

(b) The map $C_{\mathrm{o}}^{\infty}(S^{n-1}) \to \mathbf{Val}_{n-1}^{\infty}$, $f \mapsto \nu_{n-1,f}$, *where*

$$\nu_{n-1,f}(K) = \int_{S^{n-1}} f(u)\, dS_{n-1}(K,u), \tag{A.3}$$

is an isomorphism of Fréchet spaces.

Proof. First we note that, by the density properties of area measures, both maps $f \mapsto \nu_{1,f}$ and $f \mapsto \nu_{n-1,f}$ are injective. Moreover, it is not difficult to show that they are also both continuous in the respective Fréchet topologies.

In order to prove (a), we use (4.10) and the fact that $\Box_n$ is self-adjoint, to rewrite (A.2) to

$$\nu_{1,f}(K) = \int_{S^{n-1}} \Box_n f(u)\, h(K,u)\, du.$$

Since $\Box_n : C_{\mathrm{o}}^{\infty}(S^{n-1}) \to C_{\mathrm{o}}^{\infty}(S^{n-1})$ is an isomorphism, it suffices to show that

$$f \mapsto \left(K \mapsto \int_{S^{n-1}} f(u)\, h(K,u)\, du \right) \tag{A.4}$$

is an isomorphism between $C_{\mathrm{o}}^{\infty}(S^{n-1})$ and $\mathbf{Val}_1^{\infty}$. To this end, recall that the support function of a convex body $K \in \mathcal{K}^n$ is a 1-homogeneous function on V^* and, thus, can be identified with a section of a line bundle E over $\mathbb{P}_+(V^*)$ whose fiber over $[\xi]_+ \in \mathbb{P}_+(V^*)$ is given by $E_{[\xi]_+} = \mathscr{D}([\xi])$. To be more precise, we identify $h(K,\cdot)$ with the section $\bar{h}(K,\cdot) \in C(\mathbb{P}_+(V^*), E)$ defined by

$$\bar{h}(K,[\xi]_+)(c\xi) = |c| h(K,\xi), \qquad \xi \in V^*,\ c \in \mathbb{R}.$$

Note that if we choose a Euclidean structure on V, then $C(\mathbb{P}_+(V^*), E)$ can be identified with $C(S^{n-1})$. In the same way, the Fréchet space of smooth sections $C^{\infty}(\mathbb{P}_+(V^*), E^* \otimes \mathscr{D}\,\mathbb{P}_+(V^*))$ is isomorphic to $C^{\infty}(S^{n-1})$ and we let $C_{\mathrm{o}}^{\infty}(\mathbb{P}_+(V^*), E^* \otimes \mathscr{D}\,\mathbb{P}_+(V^*))$ denote the subspace isomorphic to $C_{\mathrm{o}}^{\infty}(S^{n-1})$.

Using these identifications and the pairing defined in (A.1), the integral in (A.4) can be rewritten as

$$\int_{S^{n-1}} f(u)\, h(K,u)\, du = \int_{\mathbb{P}_+(V^*)} [\bar{h}(K,\cdot), \bar{f}] = \langle \bar{h}(K,\cdot), \bar{f} \rangle, \tag{A.5}$$

where $\bar{f} \in C_{\mathrm{o}}^\infty(\mathbb{P}_+(V^*), E^* \otimes \mathscr{D}\,\mathbb{P}_+(V^*))$ denotes the section corresponding to $f \in C_{\mathrm{o}}^\infty(S^{n-1})$. Finally, note that the group $\mathrm{GL}(V)$ acts on the spaces $C(\mathbb{P}_+(V^*), E)$ and $C_{\mathrm{o}}^\infty(\mathbb{P}_+(V^*), E^* \otimes \mathscr{D}\,\mathbb{P}_+(V^*))$ by left translation, that is,

$$(A \cdot \bar{f})([\xi]_+) = \bar{f}(A^{-1} \cdot [\xi]_+), \qquad A \in \mathrm{GL}(V),\ [\xi]_+ \in \mathbb{P}_+(V^*),$$

and the pairing (A.5) is invariant under these actions. Thus, the map

$$C_{\mathrm{o}}^\infty(\mathbb{P}_+(V^*), E^* \otimes \mathscr{D}\,\mathbb{P}_+(V^*)) \to \mathbf{Val}_1^\infty, \quad \bar{f} \mapsto \langle \bar{h}(K,\cdot), \bar{f} \rangle \tag{A.6}$$

is $\mathrm{GL}(V)$ equivariant. Hence, its image is a $\mathrm{GL}(V)$ invariant subspace of $\mathbf{Val}_1^\infty$ and, therefore, dense by the Irreducibility Theorem. However, by the Casselmann–Wallach theorem [**14**], this image is also closed which proves (a).

For the proof of (b), first note that if $P \in \mathcal{K}^n$ is a polytope, then by the definition of $S_{n-1}(P,\cdot)$, we have

$$\nu_{n-1,f}(P) = \sum_{F \in \mathcal{F}_{n-1}(P)} f(u_F)\, \mathrm{vol}_{n-1}(F), \tag{A.7}$$

where $\mathcal{F}_{n-1}(P)$ is the set of all facets of P and u_F is the outer unit normal vector of the facet F. Conversely, it is well known that if $f \in C_{\mathrm{o}}(S^{n-1})$, then any function on polytopes in $\mathbb{R}^n$ of the form (A.7) has a unique extension to a valuation in $\mathbf{Val}_{n-1}$ (see, e.g., [**52**, Chapter 6.4]). Moreover, if $f \in C_{\mathrm{o}}^\infty(S^{n-1})$, then this valuation is smooth and given by (A.3). In order to rewrite $\nu_{n-1,f}$ in $\mathrm{GL}(V)$ invariant terms, it therefore suffices to rewrite (A.7).

To this end, let $\mathbb{P}_+^\vee(V)$ denote the compact manifold of all cooriented $n-1$ dimensional subspaces in V. (Recall that if H is such a subspace, an orientation of V/H is called a coorientation of H.) Note that there is a natural diffeomorphism between $\mathbb{P}_+^\vee(V)$ and $\mathbb{P}_+(V^*)$ which is equivariant under the action of $\mathrm{GL}(V)$ on both manifolds. Let S denote the line bundle over $\mathbb{P}_+^\vee(V)$ whose fiber over $H \in \mathbb{P}_+^\vee(V)$ is given by $S_H = \mathscr{D}(H)$. If we choose a Euclidean structure, $C^\infty(\mathbb{P}_+^\vee(V), S)$ is clearly isomorphic to $C^\infty(S^{n-1})$ and we write again $C_{\mathrm{o}}^\infty(\mathbb{P}_+^\vee(V), S)$ for the subspace isomorphic to $C_{\mathrm{o}}^\infty(S^{n-1})$.

We now rewrite (A.7) in the form

$$\nu_{n-1,f}(P) = \sum_{F \in \mathcal{F}_{n-1}(P)} \int_{\hat{F}} \bar{f}(\hat{F}), \tag{A.8}$$

where $\bar{f} \in C^\infty_{\mathrm{o}}(\mathbb{P}^\vee_+(V), S)$ is the section corresponding to $f \in C^\infty_{\mathrm{o}}(S^{n-1})$ and $\hat{F}$ is the cooriented subspace parallel to the facet F. Using the identification of $\mathbb{P}^\vee_+(V)$ with $\mathbb{P}_+(V^*)$, we can identify the line bundle S over $\mathbb{P}^\vee_+(V)$ with the line bundle $E \otimes \mathscr{D}(V)$ over $\mathbb{P}_+(V^*)$, where E is the line bundle from part (a) of the proof. Thus, there is a canonical isomorphism of Fréchet spaces

$$C^\infty_{\mathrm{o}}(\mathbb{P}_+(V^*), E) \otimes \mathscr{D}(V) \to C^\infty_{\mathrm{o}}(\mathbb{P}^\vee_+(V), S) \tag{A.9}$$

which is $\mathrm{GL}(V)$ equivariant. Together, (A.8) and (A.9) determine a continuous $\mathrm{GL}(V)$ equivariant map

$$C^\infty_{\mathrm{o}}(\mathbb{P}_+(V^*), E) \otimes \mathscr{D}(V) \to \mathbf{Val}^\infty_{n-1} \tag{A.10}$$

whose image is dense by the Irreducibility Theorem and closed by the Casselmann–Wallach theorem [**14**]. ■

Now let M again be a compact smooth manifold and $\pi : E \to M$ a line bundle over M. In the same way the Poincaré duality map motivated the definition of generalized valuations, the pairing (A.1) motivates the following definition of the space of *generalized sections* of E:

$$C^{-\infty}(M, E) = C^\infty(M, E^* \otimes \mathscr{D}M)^*.$$

Note that the pairing (A.1) yields a canonical embedding

$$C^\infty(M, E) \hookrightarrow C^{-\infty}(M, E). \tag{A.11}$$

Finally, we are in a position to proof the main result of this appendix.

Proof of Theorem 5.4. The first part of the statement was already established in the proof of Theorem A.2. In order to prove the second statement, we first compute explicitly the Poincaré duality map $\mathbf{Val}^\infty_1 \to \mathbf{Val}^{-\infty}_1$. To this end, let $\mu \in \mathbf{Val}^\infty_1$ and $\nu \in \mathbf{Val}^\infty_{n-1}$ be given by

$$\mu(K) = V(K, L, \dots, L) \qquad \text{and} \qquad \nu(K) = V(K, \dots, K, M)$$

for some strictly convex bodies $L, M \in \mathcal{K}^n$ with smooth boundary. By the Irreducibility Theorem, linear combinations of valuations of this form are dense in $\mathbf{Val}_1^\infty$ and $\mathbf{Val}_{n-1}^\infty$, respectively. From (5.3), it follows that

$$<\mu, \nu>= \frac{1}{n} V(-L, \ldots, -L, M)\, V_n = \frac{V_n}{n^2} \int_{S^{n-1}} h(M, u)\, dS_{n-1}(-L, u).$$

Thus, if $\bar{f} \in C_{\mathrm{o}}^\infty(\mathbb{P}_+(V^*), E^* \otimes \mathscr{D}\, \mathbb{P}_+(V^*))$ is the section corresponding to the valuation $\mu \in \mathbf{Val}_1^\infty$ according to (A.6) and $\bar{g} \in C_{\mathrm{o}}^\infty(\mathbb{P}_+(V^*), E) \otimes \mathscr{D}(V)$ is the section corresponding to $\nu \in \mathbf{Val}_{n-1}^\infty$ according to (A.10), then

$$\int_{S^{n-1}} h(M, u)\, dS_{n-1}(-L, u) = \int_{\mathbb{P}_+(V^*)} [\bar{f} \circ a, \bar{g}],$$

where $a : \mathbb{P}_+(V^*) \to \mathbb{P}_+(V^*)$ denotes the antipodal involution on $\mathbb{P}_+(V^*)$, that is, the change of orientation. Now, noting that, by (A.10) and the fact that $\mathscr{D}(V)^* \otimes \mathscr{D}(V)$ is trivial,

$$\mathbf{Val}_1^{-\infty} = \left(\mathbf{Val}_{n-1}^\infty\right)^* \otimes \mathscr{D}(V) \cong C_{\mathrm{o}}^{-\infty}(\mathbb{P}_+(V^*), E^* \otimes \mathscr{D}\, \mathbb{P}_+(V^*))$$

and using again the isomorphism (A.6), we see that the Poincaré duality map induces a map $C_{\mathrm{o}}^\infty(\mathbb{P}_+(V^*), E^* \otimes \mathscr{D}\, \mathbb{P}_+(V^*)) \to C_{\mathrm{o}}^{-\infty}(\mathbb{P}_+(V^*), E^* \otimes \mathscr{D}\, \mathbb{P}_+(V^*))$, given by

$$\bar{f} \mapsto \frac{1}{n^2} \bar{f} \circ a.$$

Here we have used the embedding (A.11). This map obviously extends to an isomorphism of topological vector spaces equipped with weak topologies

$$C_{\mathrm{o}}^{-\infty}(\mathbb{P}_+(V^*), E^* \otimes \mathscr{D}\, \mathbb{P}_+(V^*)) \to C_{\mathrm{o}}^{-\infty}(\mathbb{P}_+(V^*), E^* \otimes \mathscr{D}\, \mathbb{P}_+(V^*)) \cong \mathbf{Val}_1^{-\infty}.$$

However, if we endow V with a Euclidean structure, this map becomes an isomorphism

$$C_{\mathrm{o}}^{-\infty}(S^{n-1}) \to \mathbf{Val}_1^{-\infty}$$

which, when restricted to smooth functions, is just given by

$$f \mapsto \left(K \mapsto \frac{1}{n^2} \int_{S^{n-1}} f(-u)\, h(K, u)\, du \right).$$

Clearly, this implies the desired statement. ■

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