The Sine Transform of Isotropic Measures

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Abstract. Sharp isoperimetric inequalities for the sine transform of even isotropic measures are established. The corresponding reverse inequalities are obtained in an asymptotically optimal form. These new inequalities have direct applications to strong volume estimates for convex bodies from data about their sections or projections.

1. Introduction

The (spherical) cosine transform plays a fundamental role in modern geometric analysis. It arises naturally in a number of different areas such as functional analysis, geometric tomography and stochastic geometry (see e.g., [**13**, **19**, **31**, **32**, **43**, **46**, **50**, **54**]). A classical theorem of Lewis [**29**] shows that each finite dimensional subspace of L_1 is isometric to a Banach space whose norm is the cosine transform of some even isotropic measure on the unit sphere. This important result of Lewis allows effortless proofs of isoperimetric inequalities, which characterize Euclidean subspaces of L_1, by applications of the Urysohn and Hölder inequalities (see [**35**] for details). The reverse inequalities, having l_1^n subspaces and their duals as extremals, would turn out to be significantly more difficult to establish. The breakthrough here was achieved by Ball and Barthe in the 1990's.

Sharp reverse isoperimetric inequalities for the unit and polar unit balls of subspaces of L_1 were established by Ball [**2**, **3**] using his ingenious reformulation of the Brascamp–Lieb inequality. The uniqueness problem for the extremals in Ball's inequalities was solved by Barthe [**7**] for discrete isotropic measures by using his newly obtained equality conditions for the Brascamp–Lieb inequality and its inverse form. Recently, Lutwak, Yang, and Zhang [**35**, **37**] settled the uniqueness questions for the extremal cases in Ball's inequalities for general isotropic measures exploiting a more direct approach based on the ideas of Ball and Barthe (see also Barthe [**8**]).

In this article we obtain sharp isoperimetric inequalities for the (spherical) *sine transform* of isotropic measures. We establish the corresponding reverse inequalities in an asymptotically optimal form using the multidimensional Brascamp–Lieb inequality and its inverse obtained by Barthe [**7**]. While not as well known as the cosine transform, its natural dual – the sine transform – appears in different guises in geometric tomography. Therefore, applications of our new inequalities lead to asymptotically sharp volume estimates for convex bodies from certain data about their sections or projections.

The setting for this article is Euclidean n-space $\mathbb{R}^n$ with $n \geq 3$. We use $\|\cdot\|$ to denote the standard Euclidean norm on $\mathbb{R}^n$ and we write $x \cdot y$ for the standard inner product of $x, y \in \mathbb{R}^n$. A non-negative finite Borel measure μ on the unit sphere S^{n-1} is said to be *isotropic* if it has the same moment of inertia about all lines through the origin or, equivalently, if for all $x \in \mathbb{R}^n$,

$$\|x\|^2 = \int_{S^{n-1}} |x \cdot u|^2 \, d\mu(u).$$

Isotropic measures have been the focus of recent studies, in particular, in relation with a variety of extremal problems for convex bodies (see, e.g., [**21–23**, **25**, **36**] and the references therein). Two basic examples of isotropic measures on S^{n-1} are (suitably normalized) spherical Lebesgue measure and the cross measures, i.e., measures concentrated uniformly on $\{\pm b_1, \ldots, \pm b_n\}$, where $b_1, \ldots, b_n$ denote orthonormal basis vectors of $\mathbb{R}^n$.

The *cosine transform* $\mathcal{C}\mu$ of a finite Borel measure μ on S^{n-1} is the continuous function defined by

$$(\mathcal{C}\mu)(x) = \int_{S^{n-1}} |x \cdot u| \, d\mu(u), \qquad x \in \mathbb{R}^n. \tag{1.1}$$

If μ is not concentrated on a great subsphere and even (i.e., it assumes the same value on antipodal sets), its cosine transform uniquely determines a norm on $\mathbb{R}^n$ whose unit ball we denote by C_μ^*.

In a highly influential paper, Bolker [**12**] has shown that a convex body is the unit ball of an n-dimensional subspace of L_1 if and only if the associated norm admits a representation of the form (1.1) for some even measure μ not concentrated on a great subsphere. Consequently, isoperimetric inequalities for the convex body C_μ^* or its polar C_μ having ellipsoids as extremals provide characterizations of Euclidean subspaces of L_1 (see [**35**]).

Optimal reverse isoperimetric inequalities for the unit balls of subspaces of L_p – having l_p^n subspaces as extremals – were established by Ball [**3**] using his normalized Brascamp–Lieb inequality. The corresponding inequalities for the polar unit balls of L_1 were also obtained by Ball [**2**] and he predicted that for $p > 1$, these inequalities would follow from an inverse form of the Brascamp–Lieb inequality. Barthe [**7**] obtained this reverse Brascamp–Lieb inequality in 1998 and used it to establish the reverse volume inequalities for the polar unit balls of subspaces of L_p. These landmark results of Ball and Barthe have had a tremendous impact on geometric analysis, see, e.g., [**1**, **4**, **6**, **9–11**, **16**, **18**, **36**, **38**].

The uniqueness questions for the extremal cases in the inequalities of Ball and Barthe were completely settled only recently by Lutwak, Yang, and Zhang [**35**] and later, independently, by Barthe [**8**]. The volume inequalities for subspaces of L_1 state the following: Among even isotropic measures, $V(C_\mu^*)$ is maximized precisely by cross measures and minimized precisely by normalized Lebesgue measure, while $V(C_\mu)$ is maximized precisely by normalized Lebesgue measure and minimized precisely by cross measures.

Definition The *sine transform* $\mathcal{S}\mu$ of a finite Borel measure μ on S^{n-1} is the continuous function defined by

$$(\mathcal{S}\mu)(x) = \int_{S^{n-1}} \|x|u^\perp\| \, d\mu(u), \qquad x \in \mathbb{R}^n.$$

Here, $\|x|u^\perp\|$ is the length of the orthogonal projection of x onto the hyperplane orthogonal to u. If μ is even and not concentrated on two antipodal points, its sine transform uniquely determines (see Section 2 for details) a norm on $\mathbb{R}^n$ whose unit ball we denote by S_μ^* and its polar by S_μ.

Let κ_n denote the volume of the Euclidean unit ball in $\mathbb{R}^n$ and define

$$\alpha_n := \frac{n(n-1)^{2n}}{\Gamma(n)^{1/(n-1)}} \qquad \text{and} \qquad \gamma_n := \frac{(n-1)\kappa_{n-1}^2}{\kappa_{n-2}\kappa_n}.$$

The main results of this article are the following two theorems.

Theorem 1. *If μ is an even isotropic measure on S^{n-1}, then*

$$\frac{\kappa_n}{\gamma_n^n} \leq V(S_\mu^*) \leq \frac{\kappa_n \gamma_n^n}{\alpha_n}, \tag{1.2}$$

with equality on the left if and only if μ is normalized Lebesgue measure.

While we believe that the right inequality in (1.2) is not sharp for any value of n (compare the discussion in Section 4), we will show that it is *asymptotically optimal.* More precisely, we will see in Section 4 that, up to a constant factor which tends to one as n goes to infinity, $V(S_\mu^*)$ is maximized by cross measures.

Theorem 2. *If μ is an even isotropic measure on S^{n-1}, then*

$$\frac{\kappa_n \alpha_n}{\gamma_n^n} \leq V(S_\mu) \leq \kappa_n \gamma_n^n, \tag{1.3}$$

with equality on the right if and only if μ is normalized Lebesgue measure.

We believe that the left inequality in (1.3) is also not sharp. We will show, however, that it is asymptotically optimal. More precisely, up to a factor tending to one as n goes to infinity, $V(S_\mu)$ is minimized by cross measures.

The proofs of Theorems 1 and 2 are based on applications of the Urysohn and Hölder inequalities, and the multidimensional Brascamp–Lieb inequality and its inverse respectively. In our approach we also make use of an instance of the Kantorovich duality for the Brascamp–Lieb inequality and its inverse (see Section 3 for details) which was exploited in their proof by Barthe [**7**]. This has the advantage that it will provide additional geometric insight to the dual nature of inequalities (1.2) and (1.3) (see Theorem 4.1).

The sine transform arises in geometric tomography in different contexts (see Section 5 for a detailed account). In Section 5 we will show that our main results – Theorems 1 and 2 – lead to fairly strong volume estimates for convex bodies from certain tomographic data which are dual to results of Giannopoulos and Papadimitrakis [**23**] for the cosine transform.

2. Background material

For quick later reference, we collect in this section background material regarding convex bodies. We also state some well known facts about spherical harmonics which are needed to establish basic injectivity properties of the sine transform. For a general reference the reader may wish to consult the book by Schneider [**45**].

A convex body is a non-empty compact convex subset of $\mathbb{R}^n$. We denote by $\mathcal{K}^n$ the space of convex bodies in $\mathbb{R}^n$ endowed with the Hausdorff metric. A convex body $K \in \mathcal{K}^n$ is uniquely determined by its *support function* $h(K,\cdot)$, where $h(K,x) = \max\{x \cdot y : y \in K\}$, $x \in \mathbb{R}^n$. Note that $h(K,\cdot)$ is (positively) homogeneous of degree one and convex. Conversely, each function with these properties is the support function of a unique convex body.

The polar body K^* of a convex body K containing the origin in its interior is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

Let $\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}$, $x \in \mathbb{R}^n \backslash \{0\}$, denote the radial function of K. It follows from the definitions of support functions and radial functions, and the definition of the polar body of K, that

$$\rho(K^*,\cdot) = h(K,\cdot)^{-1} \qquad \text{and} \qquad h(K^*,\cdot) = \rho(K,\cdot)^{-1}. \tag{2.1}$$

Using (2.1) and the polar coordinate formula for volume, it is easy to see that the volume of a convex body $K \in \mathcal{K}^n$ containing the origin in its interior is given by

$$V(K) = \frac{1}{n!} \int_{\mathbb{R}^n} \exp(-h(K^*, x))\, dx, \tag{2.2}$$

where integration is with respect to Lebesgue measure on $\mathbb{R}^n$.

The classical Urysohn inequality (see, e.g. [**45**, p. 318]) provides an upper bound for the volume of a convex body in terms of the average value of its support function: If $K \in \mathcal{K}^n$ has non-empty interior, then

$$\left(\frac{V(K)}{\kappa_n}\right)^{1/n} \leq \frac{1}{n\kappa_n} \int_{S^{n-1}} h(K, u)\, du, \tag{2.3}$$

with equality if and only if K is a ball. Here the integral is with respect to spherical Lebesgue measure.

For $K \in \mathcal{K}^n$ let $S(K)$ denote its surface area. The well known classical isoperimetric inequality states that among bodies of given volume, Euclidean balls have least surface area: If $K \in \mathcal{K}^n$ has non-empty interior, then

$$n^n \kappa_n V(K)^{n-1} \leq S(K)^n, \tag{2.4}$$

with equality if and only if K is a ball.

Since convex bodies of a given volume may have arbitrarily large surface area if they are very flat, the most natural way to reverse the isoperimetric inequality is to consider affine equivalence classes of convex bodies. This leads to the following definition: The *minimal surface area* of a convex body $K \in \mathcal{K}^n$ with non-empty interior is defined by

$$\partial(K) := \min\{S(\phi K) : \phi \in \mathrm{SL}(n)\}.$$

We say K is in *surface isotropic position* if $S(K) = \partial(K)$. It was first proved by Petty [**40**] that every convex body with non-empty interior has a surface isotropic position which is unique up to orthogonal transformations.

The celebrated reverse isoperimetric inequality of Ball [**3**] can now be stated as follows: If $K \in \mathcal{K}^n$ has non-empty interior, then

$$\partial(K)^n \leq \frac{n^{3n/2}(n+1)^{(n+1)/2}}{n!} V(K)^{n-1}. \tag{2.5}$$

It was shown by Barthe [**7**] that equality holds in (2.5) if and only if K is a simplex. It was also shown by Ball [**3**] that among origin symmetric convex bodies of given volume the minimal surface area is maximized (precisely) by the cube (the uniqueness of extremals was settled by Barthe [**7**]).

A convex body $K \in \mathcal{K}^n$ with non-empty interior is also determined up to translations by its surface area measure $S_{n-1}(K, \cdot)$. Recall that for a Borel set $\omega \subseteq S^{n-1}$, $S_{n-1}(K, \omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K at which there exists a normal vector of K belonging to ω. The following result of Petty [**40**] (see also [**23**]) will allow us to apply Theorems 1 and 2 in various geometric settings (see Section 5):

Proposition 2.1. *A convex body $K \in \mathcal{K}^n$ with non-empty interior is in surface isotropic position if and only if its surface area measure $S_{n-1}(K, \cdot)$ is, up to normalization, isotropic.*

We conclude this section with a discussion of the injectivity properties of the sine transform. To this end, we need some basic facts about spherical harmonics (see e.g., Schneider [**45**, Appendix]).

Let $\mathcal{H}_k^n$ denote the finite dimensional vector space of spherical harmonics of dimension n and order k and let $N(n, k)$ denote its dimension. We use $L_2(S^{n-1})$ to denote the Hilbert space of square integrable functions on S^{n-1} with its usual inner product $(\cdot\,, \cdot)$. The spaces $\mathcal{H}_k^n$ are pairwise orthogonal with respect to this inner product. In each space $\mathcal{H}_k^n$ we choose an orthonormal basis $\{Y_{k1}, \ldots, Y_{kN(n,k)}\}$. Then $\{Y_{k1}, \ldots, Y_{kN(n,k)} : k \in \mathbb{N}\}$ forms a complete orthogonal system in $L_2(S^{n-1})$, i.e., for every $f \in L_2(S^{n-1})$, the Fourier series

$$f \sim \sum_{k=0}^{\infty} \mathrm{p}_k f$$

converges in quadratic mean to f, where $\mathrm{p}_k f$ is the orthogonal projection of f onto $\mathcal{H}_k^n$. In particular, for $f \in C(S^{n-1})$,

$$\mathrm{p}_k f = 0 \quad \text{for all } k \in \mathbb{N} \qquad \Longrightarrow \qquad f = 0. \tag{2.6}$$

Thus, $f \in C(S^{n-1})$ is uniquely determined by its series expansion.

For a finite Borel measure μ on S^{n-1}, we define

$$\mathrm{p}_k \mu = \sum_{i=1}^{N(n,k)} \int_{S^{n-1}} Y_{ki}(u)\, d\mu(u)\, Y_{ki}.$$

If $f \in C(S^{n-1})$, then

$$(f, \mathrm{p}_k\mu) = \int_{S^{n-1}} (\mathrm{p}_k f)(u)\, d\mu(u).$$

Thus, by (2.6), μ is uniquely determined by its (formal) series expansion:

$$\mathrm{p}_k\mu = 0 \quad \text{for all } k \in \mathbb{N} \qquad \Longrightarrow \qquad \mu = 0. \tag{2.7}$$

A useful tool to establish injectivity results for integral transforms is the Funk–Hecke theorem: Let g be a continuous function on $[-1, 1]$. If T_g is the transformation on the set of finite Borel measures on S^{n-1} defined by

$$(\mathrm{T}_g\mu)(u) = \int_{S^{n-1}} g(u \cdot v)\, d\mu(v), \qquad u \in S^{n-1}, \tag{2.8}$$

then there are real numbers $a_k[\mathrm{T}_g]$, the *multipliers* of T_g, such that

$$\mathrm{T}_g Y_k = a_k[\mathrm{T}_g]\, Y_k$$

for every $Y_k \in \mathcal{H}_k^n$. In particular, by Fubini's theorem,

$$\mathrm{p}_k(\mathrm{T}_g\mu) = a_k[\mathrm{T}_g]\mathrm{p}_k\mu. \tag{2.9}$$

Using (2.7) and (2.9), it follows that a transformation T_g defined on the space of finite Borel measures on S^{n-1} and satisfying (2.9) is injective if and only if all multipliers $a_k[\mathrm{T}_g]$ are non-zero.

Clearly, the sine transform $\mathcal{S}$ of finite Borel measures on S^{n-1} is of the form (2.8), where

$$g(t) = \sqrt{1 - t^2}, \qquad t \in [-1, 1].$$

Thus, by the Funk–Hecke theorem, the sine transform satisfies (2.9). The multipliers $a_k[\mathcal{S}]$ have been calculated in [**24**]: For every $k \in \mathbb{N}$, we have

$$a_{2k}[\mathcal{S}] \neq 0 \qquad \text{and} \qquad a_{2k+1}[\mathcal{S}] = 0. \tag{2.10}$$

Since $f \in C(S^{n-1})$ (or a measure μ on S^{n-1}) is even if and only if $\mathrm{p}_k f = 0$ (or $\mathrm{p}_k\mu = 0$, respectively) for every odd $k \in \mathbb{N}$, (2.10) yields the following injectivity result (for a stability version see [**27**]):

Proposition 2.2. *The sine transform is injective on even measures on* S^{n-1}.

3. The Brascamp–Lieb inequality and its inverse

In the following we recall the rank $n-1$ case of the multidimensional Brascamp–Lieb inequality and its reverse form which are crucial in the proof of our main results. We also state a duality formula for these inequalities established by Barthe [**7**] which is a special case of the Kantorovich duality principle from optimal mass transportation (see e.g. [**52**, Chapters 1 & 6]).

The Brascamp–Lieb inequality [**14**, **30**] was established to prove the sharp form of Young's convolution inequality. It's multidimensional version unifies and generalizes several fundamental inequalities from geometric analysis such as the Hölder inequality and the Loomis–Whitney inequality.

Around 1990 Ball [**1**] discovered an important reformulation of the Brascamp–Lieb inequality (later generalized by Barthe [**7**]) which exploited an additional geometric hypothesis of the given data and was tailor-made for applications in convex geometry. This geometric Brascamp–Lieb inequality also allowed a simple computation of the optimal constant. We will only need (and thus only state) this powerful inequality in the rank $n-1$ case.

In the following we write π_u, $u \in S^{n-1}$, for the orthogonal projection onto the hyperplane $u^\perp$.

The Brascamp–Lieb Inequality. **(**[**30**]**)** *Let $u_1, \ldots, u_m \in S^{n-1}$, $m \geq n$, and $c_1, \ldots, c_m > 0$ such that*

$$\sum_{i=1}^{m} c_i \pi_{u_i} = \mathrm{Id}.$$

If $f_i : u_i^\perp \to [0, \infty)$, $1 \leq i \leq m$, are integrable functions, then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(x|u_i^\perp)^{c_i} dx \leq \prod_{i=1}^{m} \left(\int_{u_i^\perp} f_i \right)^{c_i}. \tag{3.1}$$

There is equality if the f_i, $1 \leq i \leq m$, are identical Gaussian densities.

The problem of characterizing all extremizers for the multidimensional Brascamp–Lieb inequality was settled only recently by Valdimarsson [**51**] after previous contributions by a number of mathematicians (see [**7**, **11**, **17**]). In order to discuss the quality of our upper bound in Theorem 1, and our lower bound in Theorem 2 respectively, we state the following special case of this characterization for the rank $n-1$ case (cf. [**51**, Theorem 12]):

Proposition 3.1. *Let $u_i \in S^{n-1}$, $c_i > 0$, $1 \leq i \leq m$, be as above. Suppose that $f_i : u_i^\perp \to [0, \infty)$, $1 \leq i \leq m$, are (non-identically-zero) integrable functions such that none of them is a Gaussian. If equality holds in (3.1), then there exist an orthonormal basis $\{b_1, \ldots, b_n\}$ of $\mathbb{R}^n$, integrable functions φ_i of one variable and constants $a_i \in \mathbb{R}$, $1 \leq i \leq m$, such that*

$$\{u_1, \ldots, u_m\} \subseteq \{\pm b_1, \ldots, \pm b_n\}$$

and

$$f_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = a_i \varphi_1(x_1) \cdots \varphi_{i-1}(x_{i-1}) \varphi_{i+1}(x_{i+1}) \cdots \varphi_n(x_n).$$

The strength of the Brascamp–Lieb inequality for volume estimates of sections of the unit ball of l_p^n was exploited by Ball (see [**1**, **3**]). He also predicted that a reverse form of the Brascamp–Lieb inequality would lead to dual estimates for projections of the unit ball of l_p^n. The breakthrough here was achieved by Barthe [**5**, **7**] who established the reverse Brascamp–Lieb inequality. In the following we state this inequality in the rank $n-1$ case which is needed in the proof of Theorem 2.

The Reverse Brascamp–Lieb Inequality. ([**7**]) *Let $u_1, \ldots, u_m \in S^{n-1}$, $m \geq n$, and $c_1, \ldots, c_m > 0$ such that*

$$\sum_{i=1}^{m} c_i \pi_{u_i} = \mathrm{Id}.$$

If $f_i : u_i^\perp \to [0, \infty)$, $1 \leq i \leq m$, are integrable functions, then

$$\int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^{m} f_i(y_i)^{c_i} : x = \sum_{i=1}^{m} c_i y_i,\ y_i \in u_i^\perp \right\} dx \geq \prod_{i=1}^{m} \left(\int_{u_i^\perp} f_i \right)^{c_i}. \tag{3.2}$$

There is equality if the f_i, $1 \leq i \leq m$, are identical Gaussian densities.

The proof of the reverse Brascamp–Lieb inequality by Barthe relies on the existence and uniqueness of a certain measure preserving map, the socalled Brenier map, between two sufficiently regular probability measures (see e.g. [**15**, **39**]). Barthe's proof also exploited a classical principle dating back to Kantorovich which states that the problem of optimal mass transportation admits two dual formulations. In particular, this duality principle made it possible to derive both the Brascamp–Lieb inequality and its inverse from a single inequality which is stated in the following theorem.

Theorem 3.2. *Let $u_1, \ldots, u_m \in S^{n-1}$, $m \geq n$, and $c_1, \ldots, c_m > 0$ such that*

$$\sum_{i=1}^{m} c_i \pi_{u_i} = \mathrm{Id}.$$

If $f_i, g_i : u_i^{\perp} \to [0, \infty)$, $1 \leq i \leq m$, are integrable functions such that

$$\int_{u_i^{\perp}} f_i = \int_{u_i^{\perp}} g_i = 1,$$

then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(x|u_i^{\perp})^{c_i} dx \leq \int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^{m} g_i(y_i)^{c_i} : x = \sum_{i=1}^{m} c_i y_i, y_i \in u_i^{\perp} \right\} dx. \quad (3.3)$$

Note that equality in (3.3) can only hold if the f_i are extremizers for the Brascamp–Lieb inequality and the g_i are extremizers for the reverse Brascamp–Lieb inequality.

Inequality (3.3) will provide a convenient way to obtain the upper bound in Theorem 1 and the lower bound in Theorem 2 from a single inequality.

4. Proof of the main results

After these preparations, we are now in a position to prove our main theorems. In fact we will establish stronger results since we consider in this section arbitrary (and not necessarily even) isotropic measures.

The following two results, which directly imply Theorems 1 and 2, make use of Theorem 3.2 in our context:

Theorem 4.1. *If μ is an isotropic measure on S^{n-1}, then*

$$V(S_\mu^*) \leq V(S_\mu)/\alpha_n.$$

Proof: First assume that μ is discrete, say $\operatorname{supp} \mu = \{u_1, \ldots, u_m\}$ and $\mu(\{u_i\}) =: \bar{c}_i > 0$. Since μ is isotropic, it follows that $\mu(S^{n-1}) = \sum_{i=1}^{m} \bar{c}_i = n$. Therefore, using $\pi_u = \mathrm{Id} - u \otimes u$, we have

$$\frac{1}{n-1} \sum_{i=1}^{m} \bar{c}_i \, \pi_{u_i} = \sum_{i=1}^{m} \bar{c}_i \, u_i \otimes u_i = \mathrm{Id}. \quad (4.1)$$

From (2.2) and the definition of the sine transform, it follows that

$$V(S_\mu^*) = \frac{1}{n!}\int_{\mathbb{R}^n}\prod_{i=1}^m \exp(-(n-1)\|x|u_i^\perp\|)^{c_i}\,dx, \tag{4.2}$$

where $c_i := \bar{c}_i/(n-1)$, $i = 1, \ldots, m$. Let B denote the Euclidean unit ball in $\mathbb{R}^n$. Since $\|x|u^\perp\| = h(B|u^\perp, x)$, we have

$$S_\mu = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^m \bar{c}_i y_i,\ y_i \in B|u_i^\perp \right\}.$$

Consequently, we obtain

$$V(S_\mu) = \int_{\mathbb{R}^n} \sup\left\{ \prod_{i=1}^m \mathbf{1}_{[0,n-1]}(\|y_i\|)^{c_i} : x = \sum_{i=1}^m c_i y_i,\ y_i \in u_i^\perp \right\} dx. \tag{4.3}$$

Define functions $f_i, g_i : u_i^\perp \to [0,\infty), 1 \le i \le m$, by

$$f_i(y) = \frac{(n-1)^{n-1}}{\Gamma(n)\kappa_{n-1}} \exp(-(n-1)\|y\|) \tag{4.4}$$

and

$$g_i(y) = \frac{1}{(n-1)^{n-1}\kappa_{n-1}} \mathbf{1}_{[0,n-1]}(\|y\|). \tag{4.5}$$

Note that the normalizations are chosen such that

$$\int_{u_i^\perp} f_i = \int_{u_i^\perp} g_i = 1.$$

Since $\sum_{i=1}^m c_i = n/(n-1)$, we obtain, by (4.1) – (4.3) and Theorem 3.2,

$$\begin{aligned} V(S_\mu^*) &= \frac{(\Gamma(n)\kappa_{n-1})^{n/(n-1)}}{n!(n-1)^n} \int_{\mathbb{R}^n} \prod_{i=1}^m f_i(x|u_i^\perp)^{c_i}\,dx \\ &\le \frac{(\Gamma(n)\kappa_{n-1})^{n/(n-1)}}{n!(n-1)^n} \int_{\mathbb{R}^n} \sup\left\{ \prod_{i=1}^m g_i(y_i)^{c_i} : x = \sum_{i=1}^m c_i y_i, y_i \in u_i^\perp \right\} dx \qquad (4.6) \\ &= \frac{\Gamma(n)^{1/(n-1)}}{n(n-1)^{2n}} V(S_\mu) = V(S_\mu)/\alpha_n. \end{aligned}$$

Now let μ be an arbitrary isotropic measure on S^{n-1}. As in [**8**, pp. 55–56] construct a sequence μ_k, $k \in \mathbb{N}$, of discrete isotropic measures such that μ_k converges weakly to μ as $k \to \infty$. It follows that $\lim_{k\to\infty} h(S_{\mu_k}, v) = h(S_\mu, v)$ for every $v \in S^{n-1}$. Since the pointwise convergence of support functions implies the convergence of the respective convex bodies in the Hausdorff metric (see, e.g., [**45**, Chapter 1]), the continuity of volume and polarity on convex bodies containing the origin in their interiors finishes the proof. ■

Our next result completes the proof of Theorems 1 and 2:

Theorem 4.2. *If μ is an isotropic measure on S^{n-1}, then*

$$\frac{\kappa_n}{\gamma_n^n} \le V(S_\mu^*) \qquad \text{and} \qquad V(S_\mu) \le \kappa_n \gamma_n^n.$$

If μ is even, then there is equality in either inequality if and only if μ is normalized Lebesgue measure.

Proof: By the polar coordinate formula for volume, (2.1), and the Hölder inequality, we have

$$\left(\frac{V(S_\mu^*)}{\kappa_n}\right)^{-1/n} = \left(\frac{1}{n\kappa_n}\int_{S^{n-1}} h(S_\mu, u)^{-n}\, du\right)^{-1/n} \le \frac{1}{n\kappa_n}\int_{S^{n-1}} h(S_\mu, u)\, du$$

with equality if and only if $h(S_\mu, \cdot)$ is constant, i.e. S_μ is a ball. From the definition of the sine transform and Fubini's theorem, we obtain

$$\begin{aligned}\frac{1}{n\kappa_n}\int_{S^{n-1}} h(S_\mu, u)\, du &= \frac{1}{n\kappa_n}\int_{S^{n-1}}\int_{S^{n-1}} \sqrt{1-(u\cdot v)^2}\, du\, d\mu(v)\\ &= (n-1)\frac{\kappa_{n-1}}{\kappa_n}\int_{-1}^{1}(1-t^2)^{n/2-1}dt = \gamma_n.\end{aligned}$$

Consequently,

$$\left(\frac{V(S_\mu^*)}{\kappa_n}\right)^{-1/n} \le \gamma_n$$

with equality if and only if S_μ is a ball. Proposition 2.2 now yields the equality conditions for even isotropic measures.

In order to establish the second inequality, we apply the classical Urysohn inequality (2.3) to obtain

$$\left(\frac{\mathrm{vol}(S_\mu)}{\kappa_n}\right)^{1/n} \le \frac{1}{n\kappa_n}\int_{S^{n-1}} h(S_\mu, u)\, du = \gamma_n$$

with equality if and only if S_μ is a ball. Again, the equality conditions for even isotropic measures follow from Proposition 2.2. ∎

We do not believe that our upper bound in Theorem 1 and our lower bound in Theorem 2 are sharp: For equality to hold in these inequalities, we must have equality in Theorem 4.1. If the isotropic measure μ is discrete, this is equivalent to equality in (4.6). But from the remark after Theorem 3.2, Proposition 3.1 and the specific form of the functions f_i, g_i defined in (4.4) and (4.5), it follows that equality can not hold in (4.6). Hence, for discrete measures we can not have equality in Theorem 4.1. In order to deduce the same fact for arbitrary isotropic measures, we need a continuous analogue of Proposition 3.1. Such a result was proved by Barthe [**8**, Theorem 2] for the rank 1 case of the Brascamp–Lieb inequality using the equality conditions for a determinant inequality by Ball which were obtained by Lutwak, Yang, and Zhang [**8**, p. 168]. Unfortunately, neither Barthe's continuous analogue of Proposition 3.1 nor the equality conditions of Ball's determinant inequality are known in (the more complex) rank $n-1$ case.

The following result shows, however, that our upper bound in Theorem 1 and our lower bound in Theorem 2, respectively, are asymptotically optimal in a strong sense:

Theorem 4.3. *If ν_n, $n \geq 3$, are cross measures on S^{n-1}, then*

$$\lim_{n\to\infty} \frac{\alpha_n}{\kappa_n \gamma_n^n} V(S_{\nu_n}^*) = \lim_{n\to\infty} \frac{\gamma_n^n}{\kappa_n \alpha_n} V(S_{\nu_n}) = 1.$$

Proof: Let $\operatorname{supp}\nu_n = \{\pm e_1, \ldots, \pm e_n\}$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$. By the definition of the sine transform, the support function of S_{ν_n} is given by

$$h(S_{\nu_n}, v) = \sum_{i=1}^n \|\pi_{e_i} v\|, \qquad v \in S^{n-1}.$$

A simple computation shows that

$$\max_{v\in S^{n-1}} \sum_{i=1}^n \|\pi_{e_i} v\| = n\sqrt{1-\frac{1}{n}}.$$

(The maximum is attained precisely at the points $(\pm\frac{1}{\sqrt{n}}, \ldots, \pm\frac{1}{\sqrt{n}})$.) Hence, we have the inclusion

$$S_{\nu_n} \subseteq n\sqrt{1-\frac{1}{n}}\, B. \tag{4.7}$$

Theorem 1 and Theorem 2 together with (4.7) now immediately yield the following volume bounds for $S^*_{\nu_n}$ and S_{ν_n}, respectively:

$$\frac{\kappa_n}{n^n}\left(1-\frac{1}{n}\right)^{-n/2} \leq V(S^*_{\nu_n}) \leq \frac{\kappa_n \gamma_n^n}{\alpha_n} \tag{4.8}$$

and

$$\frac{\kappa_n \alpha_n}{\gamma_n^n} \leq V(S_{\nu_n}) \leq \kappa_n n^n \left(1-\frac{1}{n}\right)^{n/2}. \tag{4.9}$$

Using Stirling's formula and the definition of the constants α_n and γ_n, it is easy to show that

$$\lim_{n\to\infty} \frac{\alpha_n}{n^n \gamma_n^n}\left(1-\frac{1}{n}\right)^{-n/2} = 1.$$

Consequently, we also have

$$\lim_{n\to\infty} \frac{n^n \gamma_n^n}{\alpha_n}\left(1-\frac{1}{n}\right)^{n/2} = 1$$

which completes the proof in view of (4.8) and (4.9). ■

In view of Theorem 4.3, we formulate the following

Conjecture. *Among even isotropic measures, $V(S^*_\mu)$ is maximized precisely by cross measures, while $V(S_\mu)$ is minimized precisely by cross measures.*

5. The sine transform in geometric tomography

In this last section we briefly recall several tomographic operators on convex bodies induced by the sine transform. As applications of our main results, we then present asymptotically optimal volume inequalities for these operators. Our results are dual to volume estimates due to Giannopoulos and Papadimitrakis [**23**] for projection bodies which we also recall.

The *projection body* ΠK of $K \in \mathcal{K}^n$ is the convex body defined by

$$h(\Pi K, v) = \mathrm{vol}_{n-1}(K|v^\perp) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v|\, dS_{n-1}(K, v), \qquad v \in S^{n-1}.$$

Here, the second equation is the well known Cauchy projection formula.

Projection bodies were introduced by Minkowski at the turn of the previous century and have since become an important tool in the study of projections of convex bodies (see e.g. [**2**, **12**, **13**, **28**, **43**, **47–49**]). It was first proved by Petty [**41**] that for all $\phi \in \mathrm{GL}(n)$ and all $K \in \mathcal{K}^n$,

$$\Pi(\phi K) = |\det \phi\,| \phi^{-\mathrm{T}}\, \Pi K. \tag{5.1}$$

In particular, (5.1) shows that the volume of projection bodies and their polars is invariant under volume preserving linear transformations. The fundamental affine isoperimetric inequalities for polar projection bodies are the Petty [**42**] and the Zhang [**55**] projection inequalities (for an important recent generalization of Petty's projection inequality, see [**33**]): If $K \in \mathcal{K}^n$ has non-empty interior, then

$$\frac{(2n)!}{n^n(n!)^2} \leq V(\Pi^* K)V(K)^{n-1} \leq \left(\frac{\kappa_n}{\kappa_{n-1}}\right)^n. \tag{5.2}$$

There is equality in the left inequality if and only if K is a simplex and equality in the right inequality if and only if K is an ellipsoid. It is a major open problem to determine the corresponding inequalities for the volume of the projection body itself.

In [**23**] Giannopoulos and Papadimitrakis first observed that the volume inequalities of Ball for unit and polar unit balls of subspaces of L_1 admit an elegant reformulation using projection bodies (cf. the proof of Theorem 5.3):

Theorem 5.1. ([**23**]) *If $K \in \mathcal{K}^n$ has non-empty interior, then*

$$\kappa_n \left(\frac{n\kappa_n}{\kappa_{n-1}}\right)^n \leq V(\Pi^* K)\partial(K)^n \leq \frac{4^n n^n}{n!}$$

and

$$\frac{1}{n^n} \leq V(\Pi K)/\partial(K)^n \leq \left(\frac{\kappa_{n-1}}{n\kappa_n}\right)^n \kappa_n. \tag{5.3}$$

Note that all the inequalities of Theorem 5.1 are sharp; consider e.g. ellipsoids and parallelotopes. For centrally-symmetric convex bodies the equality conditions were settled by Lutwak, Yang, and Zhang in [**35**].

The inequalities (5.3) together with the isoperimetric inequality (2.4) and its exact reverse form (2.5) immediately provide asymptotically optimal reverse forms of the Petty and Zhang projection inequalities (5.2):

Corollary 5.2. *If $K \in \mathcal{K}^n$ has non-empty interior, then*

$$n^{-1/2} \leq \left[V(\Pi K)/V(K)^{n-1}\right]^{1/n} \leq e^{3/2}.$$

Up to a constant multiple, both inequalities in Corollary 5.2 are best possible; consider e.g. ellipsoids and simplices (see also [**34**]).

The sine transform of surface area measures also arises naturally in geometric tomography in a number of different guises.

Examples:

(a) If $K \in \mathcal{K}^n$ and $v \in S^{n-1}$, then it was shown by Schneider [**44**] that

$$\int_{-\infty}^{\infty} V_{n-2}(K \cap (v^{\perp} + tv))\, dt = \frac{1}{2(n+1)} \int_{S^{n-1}} \|v|u^{\perp}\|\, dS_{n-1}(K,u),$$

where $2V_{n-2}(L)$ denotes the $(n-2)$-dimensional surface area of an $(n-1)$-dimensional convex body L. Thus, the sine transform of the surface area measure of K is, up to a factor, the integrated surface area of parallel hyperplane sections of K.

(b) For $i \in \{1,\dots,n-1\}$, the i-th mean section operator $\mathrm{M}_i : \mathcal{K}^n \to \mathcal{K}^n$, introduced by Goodey and Weil in [**24**], is defined by

$$h(\mathrm{M}_i K, \cdot) = \int_{\mathrm{AGr}_{i,n}} h(K \cap E, \cdot)\, d\sigma_i(E).$$

Here, $\mathrm{AGr}_{i,n}$ is the affine Grassmannian of i-dimensional planes in $\mathbb{R}^n$ and σ_i is its (suitably normalized) motion invariant measure. It was shown in [**24**] that for origin-symmetric convex bodies

$$h(\mathrm{M}_2 K, \cdot) = \frac{\kappa_2^2 \kappa_{n-2}}{n(n-1)\kappa_n} \int_{S^{n-1}} \|v|u^{\perp}\|\, dS_{n-1}(K,u).$$

(c) For $i \in \{0,\dots,n\}$, let $V_i(K)$ denote the i-th intrinsic volume of $K \in \mathcal{K}^n$. The projection body $\Pi_i K$ of order i of K is defined by

$$h(\Pi_i K, v) = V_i(K|v^{\perp}), \qquad v \in S^{n-1}.$$

A direct computation shows that

$$h(\Pi_1 \Pi K, v) = \frac{\kappa_{n-2}}{n-1} \int_{S^{n-1}} \|v|u^{\perp}\|\, dS_{n-1}(K,u).$$

An important part of geometric tomography deals with the estimation of the volume (and other quantities) of a convex or star body from data about the projections or the sections of the body (see e.g. [**2**, **20**, **26**, **43**, **53**, **54**] and, in particular, [**19**, Chapter 9] and the references therein). The rest of this section is devoted to establishing volume inequalities for the examples above which are (in some sense) dual to Theorem 5.1 and Corollary 5.2. In order to allow for an immediate comparison with the results for projection bodies, we will introduce yet another operator $\Psi : \mathcal{K}^n \to \mathcal{K}^n$, defined by

$$h(\Psi K, v) = \frac{\kappa_{n-2}}{(n-1)\kappa_{n-1}} \int_{S^{n-1}} \|v|u^{\perp}\| \, dS_{n-1}(K, u).$$

Here, the normalization is chosen such that $\Pi B = \Psi B$.

It is important to note that while Ψ still commutes with orthogonal transformations, it does *not* intertwine affine transformations like the projection body map Π. (The very special role of the projection body operator in affine convex geometry has only been demonstrated recently by Ludwig [**31**, **32**].) Consequently, the quantities $V(\Psi K)$ and $V(\Psi^* K)$ are rigid motion invariant but not invariant under volume preserving linear transformations. In fact, for a convex body K of given volume, $V(\Psi K)$ may be arbitrarily large and $V(\Psi^* K)$ arbitrarily small, respectively. We will therefore fix a position of the body, to be more precise, the surface isotropic position, to bound the quantities $V(\Psi K)$ and $V(\Psi^* K)$.

The following result is a reformulation of the slightly more general versions of Theorems 1 and 2 proved in Section 4:

Theorem 5.3. *If $K \in \mathcal{K}^n$ is in surface isotropic position, then*

$$\kappa_n \left(\frac{n\kappa_n}{\kappa_{n-1}} \right)^n \leq V(\Psi^* K) \partial(K)^n \leq \left(\frac{n}{\kappa_n} \right)^{n-1} \frac{\kappa_{n-1}^{3n} \Gamma(n)^{1/(n-1)}}{\kappa_{n-2}^{2n}},$$

with equality among centrally-symmetric convex bodies in the left inequality if and only if K is a ball, and

$$\frac{\kappa_{n-2}^{2n} \kappa_n^2}{\kappa_{n-1}^{3n} \Gamma(n)^{1/(n-1)}} \left(\frac{\kappa_n}{n} \right)^{n-1} \leq V(\Psi K)/\partial(K)^n \leq \left(\frac{\kappa_{n-1}}{n\kappa_n} \right)^n \kappa_n,$$

with equality among centrally-symmetric convex bodies in the right inequality if and only if K is a ball.

Proof: Define the non-negative Borel measure μ on S^{n-1} by

$$\mu = \frac{n}{\partial(K)} S_{n-1}(K, \cdot).$$

Since K is in surface isotropic position, it follows from Proposition 2.1 that μ is isotropic. Clearly, by the definitions of S_μ and the map Ψ, we have

$$S_\mu^* = \frac{\partial(K)\kappa_{n-2}}{n(n-1)\kappa_{n-1}} \Psi^* K \qquad \text{and} \qquad S_\mu = \frac{n(n-1)\kappa_{n-1}}{\partial(K)\kappa_{n-2}} \Psi K.$$

Applications of Theorem 4.1 and Theorem 4.2, now complete the proof. ■

A combination of the inequalities of Theorem 5.3 with the isoperimetric inequality (2.4) and its exact reverse form (2.5), now yields

Corollary 5.4. *If $K \in \mathcal{K}^n$ is in surface isotropic position, then*

$$(en)^{-1} \leq \left[V(\Psi^* K) V(K)^{n-1}\right]^{1/n} \leq e^{3/2} n^{-1/2}$$

and

$$n^{-1/2} \leq \left[V(\Psi K)/V(K)^{n-1}\right]^{1/n} \leq e^{3/2}.$$

Note that both bounds are, up to a constant multiple, best possible (consider e.g. Euclidean balls and cubes) and that they are precisely of the same order as the corresponding bounds for projection bodies given by (5.2) and Corollary 5.2.

We finally remark that it is easy to show that among convex bodies of given volume there exists an upper bound for the quantity $V(\Psi^* K)$, and a lower bound for $V(\Psi K)$ respectively. It is the authors believe that both bounds are attained precisely by Euclidean balls. For convex bodies in surface isotropic position, Corollary 5.4 confirms these conjectures asymptotically.

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