

The Steiner Formula for Minkowski Valuations

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Abstract. A Steiner type formula for continuous translation invariant Minkowski valuations is established. In combination with a recent result on the symmetry of rigid motion invariant homogeneous bivaluations, this new Steiner type formula is used to obtain a family of Brunn–Minkowski type inequalities for rigid motion intertwining Minkowski valuations.

1. Introduction

The famous Steiner formula, dating back to the 19th century, expresses the volume of the parallel set of a convex body K at distance $r \geq 0$ as a polynomial in r . Up to constants (depending on the dimension of the ambient space), the coefficients of this polynomial are the intrinsic volumes of K . Steiner’s formula is among the most influential results of the early days of convex geometry. Its ramifications and many applications can be found, even today, in several mathematical areas such as differential geometry (starting from Weyl’s tube formula [65]; see e.g. [14, 20] for more recent results), geometric measure theory (going back to Federer’s seminal work on curvature measures [13]; see also [15, 16, 51, 53]), convex and stochastic geometry (see e.g. [28, 54, 56]), geometric functional analysis (see [11, 12]), and recently also in algebraic geometry (see [26, 62]).

In Euclidean space \mathbb{R}^n , the parallel set of K at distance r is equal to the sum of K and a Euclidean ball of radius r . A fundamental extension of the classical Steiner formula is Minkowski’s theorem on the polynomial expansion of the volume of a Minkowski sum of several convex bodies, leading to the theory of mixed volumes (see e.g. [54]). More recently, McMullen [48] (and later, independently, Meier [49] and Spiegel [61]) established the existence of a similar polynomial expansion for functions on convex bodies which are considerably more general than the ordinary volume, namely continuous translation invariant (real valued) *valuations*.

The origins of the notion of valuation (see Section 2 for precise definition) can be traced back to Dehn’s solution of Hilbert’s Third Problem. However, the starting point for a systematic investigation of general valuations was Hadwiger’s [27] fundamental characterization of the linear combinations of intrinsic volumes as the continuous valuations that are rigid motion invariant

(see [1, 2, 6, 39] for recent important variants). McMullen’s [48] deep result on the polynomial expansion of translation invariant valuations is among the seminal contributions to the structure theory of the space of translation invariant valuations which has been rapidly evolving over the last decade (see [2–4, 9, 17]). These recent structural insights in turn provided the means for a fuller understanding of the integral geometry of groups acting transitively on the sphere (see e.g. [3, 6, 7, 10] and the survey [8]).

While classical results on valuations were mainly concerned with real and tensor valued valuations, a very recent development explores the strong connections between *convex body valued* valuations and isoperimetric and related inequalities (see [5, 25, 37, 59]). This new line of research has its roots in the work of Ludwig [33–36] who first obtained classifications of convex and star body valued valuations which are compatible with linear transformations (see also [21–24, 38, 60, 64]). In this area, it is a major open problem whether a polynomial expansion of translation invariant convex body valued valuations is also possible (see Section 2 for details).

In this article we establish a Steiner type formula for continuous translation invariant *Minkowski valuations* (i.e. valuations taking values in the topological semigroup of convex bodies endowed with Minkowski addition). In fact, we obtain a more general polynomial expansion formula for translation invariant Minkowski valuations when the arguments are Minkowski sums of zonoids. This follows in part from a connection between Minkowski valuations and positive scalar valuations. Our new Steiner type formula gives rise to a Lefschetz operator on Minkowski valuations which we use together with a recent result on the symmetry of rigid motion intertwining homogeneous bivaluations [5] to obtain a family of Brunn–Minkowski type inequalities for intrinsic volumes of rigid motion intertwining Minkowski valuations. These new inequalities generalize a number of previous partial results [5, 44, 57, 59].

2. Statement of principal results

The setting for this article is n -dimensional Euclidean space \mathbb{R}^n with $n \geq 3$. We denote by \mathcal{K}^n the space of convex bodies in \mathbb{R}^n endowed with the Hausdorff metric. A function φ defined on \mathcal{K}^n and taking values in an

abelian semigroup is called a *valuation* if

$$\varphi(K) + \varphi(L) = \varphi(K \cup L) + \varphi(K \cap L)$$

whenever $K \cup L \in \mathcal{K}^n$. A valuation φ is said to be *translation invariant* if $\varphi(K + x) = \varphi(K)$ for all $x \in \mathbb{R}^n$ and $K \in \mathcal{K}^n$.

The most familiar real valued valuation is, of course, the ordinary volume V_n . In fact, the valuation property of volume carries over to a series of basic functions which are derived from it: By a classical result of Minkowski, the volume of a Minkowski (or vector) linear combination $\lambda_1 K_1 + \cdots + \lambda_m K_m$ of convex bodies $K_1, \dots, K_m \in \mathcal{K}^n$ with real coefficients $\lambda_1, \dots, \lambda_m \geq 0$ can be expressed as a homogeneous polynomial of degree n ,

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{j_1, \dots, j_n=1}^m V(K_{j_1}, \dots, K_{j_n}) \lambda_{j_1} \cdots \lambda_{j_n}, \quad (2.1)$$

where the coefficients $V(K_{j_1}, \dots, K_{j_n})$, called *mixed volumes* of K_{j_1}, \dots, K_{j_n} , are symmetric in the indices and depend only on K_{j_1}, \dots, K_{j_n} . Now, if $i \in \{1, \dots, n\}$ and an $(n-i)$ -tuple L_1, \dots, L_{n-i} of convex bodies is fixed, then the function $\phi : \mathcal{K}^n \rightarrow \mathbb{R}$, defined by $\phi(K) = V(K, \dots, K, L_1, \dots, L_{n-i})$, is a continuous translation invariant valuation (see e.g. [54]).

In a highly influential article, Alesker [2] showed (thereby confirming a conjecture by McMullen) that in fact every continuous translation invariant real valued valuation is a limit of linear combinations of mixed volumes. One of the crucial ingredients in the proof of Alesker's landmark result is the following significant generalization of the polynomial expansion (2.1):

Theorem 1 (McMullen [48]) *Let X be a topological vector space. Suppose that $\varphi : \mathcal{K}^n \rightarrow X$ is a continuous translation invariant valuation and let $K_1, \dots, K_m \in \mathcal{K}^n$. Then*

$$\varphi(\lambda_1 K_1 + \cdots + \lambda_m K_m), \quad \lambda_1, \dots, \lambda_m \geq 0,$$

can be expressed as a polynomial in $\lambda_1, \dots, \lambda_m$ of total degree at most n . Moreover, for each (i_1, \dots, i_m) , the coefficient of $\lambda_1^{i_1} \cdots \lambda_m^{i_m}$ is a continuous translation invariant and homogeneous valuation of degree i_j in K_j .

As a special case of Theorem 1, we note the following extension of the classical Steiner formula for volume (see Section 5): If $K \in \mathcal{K}^n$, then for

every $r \geq 0$,

$$\varphi(K + rB^n) = \sum_{j=0}^n r^{n-j} \varphi^{(j)}(K), \quad (2.2)$$

where the coefficient functions $\varphi^{(j)} : \mathcal{K}^n \rightarrow X$, $0 \leq j \leq n$, defined by (2.2), are continuous translation invariant valuations. Clearly, $\varphi^{(n)} = \varphi$.

Definition A map $\Phi : \mathcal{K}^n \rightarrow \mathcal{K}^n$ is called a Minkowski valuation if

$$\Phi(K) + \Phi(L) = \Phi(K \cup L) + \Phi(K \cap L),$$

whenever $K, L, K \cup L \in \mathcal{K}^n$ and addition on \mathcal{K}^n is Minkowski addition.

While first results on Minkowski valuations were obtained in the 1970s by Schneider [52], they have become the focus of increased interest (and acquired their name) more recently through the work of Ludwig [33, 35]. It was shown there that such central notions like projection, centroid and difference body operators can be characterized as unique Minkowski valuations compatible with affine transformations of \mathbb{R}^n (see [23, 24, 38, 60, 64] for related results).

Since the space of convex bodies \mathcal{K}^n does not carry a linear structure, it is an important open problem (cf. [59]) whether the Steiner type formula (2.2), or even Theorem 1, also hold for continuous translation invariant Minkowski valuations. As our main result we establish an affirmative answer to the first question:

Theorem 2 Suppose that $\Phi : \mathcal{K}^n \rightarrow \mathcal{K}^n$ is a continuous translation invariant Minkowski valuation and let $K \in \mathcal{K}^n$. Then $\Phi(K + rB^n)$, $r \geq 0$, can be expressed as a polynomial in r of degree at most n whose coefficients are convex bodies, say

$$\Phi(K + rB^n) = \sum_{j=0}^n r^{n-j} \Phi^{(j)}(K). \quad (2.3)$$

Moreover, the maps $\Phi^{(j)} : \mathcal{K}^n \rightarrow \mathcal{K}^n$, $0 \leq j \leq n$, defined by (2.3), are also continuous translation invariant Minkowski valuations.

The proof of Theorem 2 makes critical use of an embedding by Klain [31] of translation invariant continuous *even* (real valued) valuations in the space of continuous functions on the Grassmannian. In fact, our proof yields a stronger result than Theorem 2, see Corollary 4.3, where the Euclidean unit ball B^n in (2.3) can be replaced by an arbitrary zonoid (i.e. a Hausdorff limit

of finite Minkowski sums of line segments). Moreover, in Theorem 4.4 we obtain a polynomial expansion formula for continuous translation invariant Minkowski valuations when the summands are zonoids. Very recently, during the review process of this article, Wannerer and the first-named author [50] showed that a polynomial expansion (analogous to Theorem 1) of continuous translation invariant Minkowski valuations is in general *not* possible.

A special case of Theorem 2 was previously obtained by the second-named author [58], when the Minkowski valuation Φ is in addition $\text{SO}(n)$ *equivariant* and has *degree* $n - 1$, i.e. $\Phi(\vartheta K) = \vartheta\Phi(K)$ and $\Phi(\lambda K) = \lambda^{n-1}\Phi(K)$ for every $K \in \mathcal{K}^n$, $\vartheta \in \text{SO}(n)$ and real $\lambda > 0$. As an application of this particular case of Theorem 2, an array of geometric inequalities for the intrinsic volumes V_i of the *derived* Minkowski valuations $\Phi^{(j)}$ (of degree $j - 1$) was obtained in [57]. In particular, the following Brunn–Minkowski type inequality was established: If $K, L \in \mathcal{K}^n$ and $3 \leq j \leq n$, $1 \leq i \leq n$, then

$$V_i(\Phi^{(j)}(K + L))^{1/i(j-1)} \geq V_i(\Phi^{(j)}(K))^{1/i(j-1)} + V_i(\Phi^{(j)}(L))^{1/i(j-1)}. \quad (2.4)$$

It was also shown in [57] that if Φ is *non-trivial*, i.e. it does not map every convex body to the origin, equality holds in (2.4) for convex bodies K and L with non-empty interior if and only if they are homothetic.

The family of inequalities (2.4) extended at the same time previously established inequalities for projection bodies by Lutwak [44] and the famous classical Brunn–Minkowski inequalities for the intrinsic volumes (see e.g. [54] and the excellent survey [18]). We conjecture that inequality (2.4) holds in fact for all continuous translation invariant and $\text{SO}(n)$ equivariant Minkowski valuations of a given arbitrary degree $j \in \{2, \dots, n - 1\}$.

Recently, refining the techniques from the seminal work of Lutwak [44], this conjecture was confirmed in the case $i = j + 1$, first for even valuations in [59] and subsequently for general valuations in [5]. As an application of Theorem 2, we extend these results to the case $1 \leq i \leq j + 1$.

Theorem 3 *Suppose that $\Phi_j : \mathcal{K}^n \rightarrow \mathcal{K}^n$ is a non-trivial continuous translation invariant and $\text{SO}(n)$ equivariant Minkowski valuation of a given degree $j \in \{2, \dots, n - 1\}$. If $K, L \in \mathcal{K}^n$ and $1 \leq i \leq j + 1$, then*

$$V_i(\Phi_j(K + L))^{1/ij} \geq V_i(\Phi_j(K))^{1/ij} + V_i(\Phi_j(L))^{1/ij}.$$

If K and L are of class C_+^2 , then equality holds if and only if K and L are homothetic.

The proof of Theorem 3 also uses a recent result on the symmetry of rigid motion invariant homogeneous bivaluations which we describe in Section 6. For a discussion of the smoothness assumption, we refer to Section 7.

3. Background material for the proof of Theorem 2

In this section we first recall some basic facts about convex bodies and, in particular, zonoids (see, e.g. [54]). Furthermore, we collect results on translation invariant (mostly real valued) valuations needed in subsequent sections. In particular, we recall an important embedding of Klain [31] of even translation invariant continuous valuations in the space of continuous functions on the Grassmannian.

A convex body $K \in \mathcal{K}^n$ is uniquely determined by the values of its support function $h(K, x) = \max\{x \cdot y : y \in K\}$, $x \in \mathbb{R}^n$. Clearly, $h(K, \cdot)$ is positively homogeneous of degree one and subadditive for every $K \in \mathcal{K}^n$. Conversely, every function with these properties is the support function of a convex body.

A Minkowski sum of finitely many line segments is called a zonotope. A convex body that can be approximated, in the Hausdorff metric, by a sequence of zonotopes is called a *zonoid*. Over the past four decades it has become apparent that zonoids arise naturally in several different contexts (see e.g. [54, Chapter 3.5] and the references therein). It is not hard to show that a convex body $K \in \mathcal{K}^n$ is an origin-centered zonoid if and only if its support function can be represented in the form

$$h(K, x) = \int_{S^{n-1}} |x \cdot u| d\mu_K(u), \quad x \in \mathbb{R}^n,$$

with some even (non-negative) measure μ_K on S^{n-1} . In this case, the measure μ_K is unique and is called the *generating measure* of K .

We denote by \mathbf{Val} the vector space of continuous translation invariant *real valued* valuations and we use \mathbf{Val}_i to denote its subspace of all valuations of degree i . Recall that a map φ from \mathcal{K}^n to \mathbb{R} (or \mathcal{K}^n) is said to have degree i if $\varphi(\lambda K) = \lambda^i \varphi(K)$ for every $K \in \mathcal{K}^n$ and $\lambda > 0$. A valuation $\varphi \in \mathbf{Val}$ is said to be even (resp. odd) if $\varphi(-K) = (-1)^\varepsilon \varphi(K)$ with $\varepsilon = 0$ (resp. $\varepsilon = 1$) for every $K \in \mathcal{K}^n$. We write $\mathbf{Val}_i^+ \subseteq \mathbf{Val}_i$ for the subspace of even valuations of degree i and \mathbf{Val}_i^- for the subspace of odd valuations of degree i .

From the important special case $m = 1$ of Theorem 1, we deduce that if $\varphi \in \mathbf{Val}$, then there exist (unique) $\varphi_i \in \mathbf{Val}_i$, $0 \leq i \leq n$, such that

$$\varphi(\lambda K) = \varphi_0(K) + \lambda\varphi_1(K) + \cdots + \lambda^n\varphi_n(K) \quad (3.1)$$

for every $K \in \mathcal{K}^n$ and $\lambda > 0$. In fact, a simple inductive argument, shows that (3.1) is equivalent with Theorem 1. Since, clearly, every real valued valuation is the sum of an even and an odd valuation, we immediately obtain the following corollary, known as *McMullen's decomposition* of the space \mathbf{Val} :

Corollary 3.1

$$\mathbf{Val} = \bigoplus_{i=0}^n (\mathbf{Val}_i^+ \oplus \mathbf{Val}_i^-).$$

It is easy to show that the space \mathbf{Val}_0 is one-dimensional and is spanned by the Euler characteristic V_0 . The analogous non-trivial statement for \mathbf{Val}_n was proved by Hadwiger [27, p. 79]:

Lemma 3.2 *If $\varphi \in \mathbf{Val}_n$, then φ is a multiple of the ordinary volume V_n .*

Assume now that $\varphi \in \mathbf{Val}_i$ with $1 \leq i \leq n - 1$. If $K_1, \dots, K_m \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_m > 0$, then, by Theorem 1,

$$\varphi(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{j_1, \dots, j_i=1}^m \varphi(K_{j_1}, \dots, K_{j_i}) \lambda_{j_1} \cdots \lambda_{j_i},$$

where the coefficients are symmetric in the indices and depend only on K_{j_1}, \dots, K_{j_i} . Moreover, the coefficient of $\lambda_1^{i_1} \cdots \lambda_m^{i_m}$, where $i_1 + \cdots + i_m = i$, is a continuous translation invariant valuation of degree i_j in K_j , called a *mixed valuation* derived from φ . Clearly, we have $\varphi(K, \dots, K) = \varphi(K)$.

We now turn to Minkowski valuations. Let \mathbf{MVal} denote the set of continuous translation invariant Minkowski valuations, and write \mathbf{MVal}_i^\pm for its subset of all even/odd Minkowski valuations of degree i .

From Lemma 3.2 and the special case $m = 1$ of Theorem 1, applied to valuations with values in the vector space $C(S^{n-1})$ of continuous functions on S^{n-1} , one can deduce the following decomposition result (cf. [55, p. 12]):

Lemma 3.3 *If $\Phi \in \mathbf{MVal}$, then for every $K \in \mathcal{K}^n$, there exist convex bodies $L_0, L_n \in \mathcal{K}^n$ such that*

$$h(\Phi(K), \cdot) = h(L_0, \cdot) + \sum_{i=1}^{n-1} g_i(K, \cdot) + V(K)h(L_n, \cdot), \quad (3.2)$$

where, for each $i \in \{1, \dots, n-1\}$:

- (i) *The function $g_i(K, \cdot)$ is a difference of support functions.*
- (ii) *The map $K \mapsto g_i(K, \cdot)$ is a continuous translation invariant valuation of degree i .*

The natural question whether for every $K \in \mathcal{K}^n$, each function $g_i(K, \cdot)$ is the support function of a convex body is equivalent to the following problem.

Problem 3.4 *Let $\Phi \in \mathbf{MVal}$ and $K \in \mathcal{K}^n$. Are there convex bodies $L_0, \Phi_1(K), \dots, \Phi_{n-1}(K), L_n \in \mathcal{K}^n$ such that*

$$\Phi(\lambda K) = L_0 + \lambda \Phi_1(K) + \dots + \lambda^{n-1} \Phi_{n-1}(K) + \lambda^n V(K) L_n \quad (3.3)$$

for every $\lambda > 0$?

During the review process of this article, Wannerer and the first-named author [50] showed that the answer to Problem 3.4 is in general negative. However, in the next section, we show that (3.3) holds for every $\Phi \in \mathbf{MVal}$ and $\lambda > 0$ if the body K is a zonoid. A crucial ingredient in the proof of this result is an embedding K_i of \mathbf{Val}_i^+ into the space $C(\text{Gr}_i)$ of continuous functions on the Grassmannian Gr_i of i -dimensional subspaces of \mathbb{R}^n constructed by Klain [31]:

Suppose that $\varphi \in \mathbf{Val}_i^+$, $1 \leq i \leq n-1$. Then, by Lemma 3.2, the restriction of φ to any subspace $E \in \text{Gr}_i$ is proportional to the i -dimensional volume vol_E on E , say

$$\varphi|_E = (K_i \varphi)(E) \text{vol}_E.$$

The continuous function $K_i \varphi : \text{Gr}_i \rightarrow \mathbb{R}$ defined in this way is called the *Klain function* of φ . The induced map

$$K_i : \mathbf{Val}_i^+ \rightarrow C(\text{Gr}_i)$$

is called the *Klain embedding*.

Theorem 3.5 (Klain [31]) *The Klain embedding is injective.*

Theorem 3.5 follows from a volume characterization of Klain [30]. Note, however, that the map K_i is not onto; its image was described in terms of the decomposition under the action of the group $SO(n)$ by Alesker and Bernstein [4].

The natural question how to reconstruct a valuation $\varphi \in \mathbf{Val}_i^+$ given its Klain function $K_i\varphi$, was answered by Klain [31] for centrally symmetric convex sets. Since we need Klain's inversion formula for zonoids only, we state just this special case. To this end, we denote by $[u_1, \dots, u_i]$ the i -dimensional volume of the parallelotope spanned by $u_1, \dots, u_i \in S^{n-1}$.

Theorem 3.6 (Klain [31]) *Suppose that $\varphi \in \mathbf{Val}_i^+$ with $1 \leq i \leq n-1$. If $Z_1, \dots, Z_i \in \mathcal{K}^n$ are zonoids with generating measures $\mu_{Z_1}, \dots, \mu_{Z_i}$, then*

$$\varphi(Z_1, \dots, Z_i) = \frac{1}{i!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} (\bar{K}_i\varphi)(u_1, \dots, u_i) [u_1, \dots, u_i] d\mu_{Z_1}(u_1) \cdots d\mu_{Z_i}(u_i),$$

where

$$(\bar{K}_i\varphi)(u_1, \dots, u_i) = \begin{cases} (K_i\varphi)(\text{span}\{u_1, \dots, u_i\}) & \text{if } [u_1, \dots, u_i] > 0, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for any zonoid $Z \in \mathcal{K}^n$, we have

$$\varphi(Z) = \frac{1}{i!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} (\bar{K}_i\varphi)(u_1, \dots, u_i) [u_1, \dots, u_i] d\mu_Z(u_1) \cdots d\mu_Z(u_i).$$

4. Proof of Theorem 2

Before we can present the proof of Theorem 2, we need the following auxiliary result.

Lemma 4.1 *For $K \in \mathcal{K}^n$, the following statements are equivalent:*

- (a) *For every non-negative $\varphi \in \mathbf{Val}$, its homogeneous components φ_i satisfy $\varphi_i(K) \geq 0$ for $0 \leq i \leq n$.*
- (b) *For every $\Phi \in \mathbf{MVal}$, there exist $L_0, L_n \in \mathcal{K}^n$ (depending only on Φ) and $\Phi_1(K), \dots, \Phi_{n-1}(K) \in \mathcal{K}^n$ such that (3.3) holds for every $\lambda > 0$.*

Proof. Let $K \in \mathcal{K}^n$ be fixed and first assume that $\varphi_i(K) \geq 0$, $0 \leq i \leq n$, for the homogeneous components φ_i of any non-negative $\varphi \in \mathbf{Val}$. Suppose that $\Phi \in \mathbf{MVal}$. Then, by Lemma 3.3, for every $L \in \mathcal{K}^n$, there are convex bodies $L_0, L_n \in \mathcal{K}^n$ and continuous functions $g_i(L, \cdot)$ such that

$$h(\Phi(\lambda L), \cdot) = h(L_0, \cdot) + \sum_{i=1}^{n-1} \lambda^i g_i(L, \cdot) + \lambda^n V(L) h(L_n, \cdot) \quad (4.1)$$

for every $\lambda > 0$. In order to prove (b), it remains to show that for each $i \in \{1, \dots, n-1\}$, the function $g_i(K, \cdot)$ is the support function of a convex body $\Phi_i(K)$. Since, by Lemma 3.3, the functions $g_i(K, \cdot)$ are positively homogeneous of degree one, it suffices to prove that

$$g_i(K, x+y) \leq g_i(K, x) + g_i(K, y) \quad (4.2)$$

for every $x, y \in \mathbb{R}^n$ and $i \in \{1, \dots, n-1\}$. To this end, fix $x, y \in \mathbb{R}^n$ and define $\psi \in \mathbf{Val}$ by

$$\psi(L) = h(\Phi(L), x) + h(\Phi(L), y) - h(\Phi(L), x+y), \quad L \in \mathcal{K}^n.$$

Since support functions are sublinear, ψ is non-negative. Moreover, by (4.1), the homogeneous components ψ_i , $1 \leq i \leq n-1$, of ψ are given by

$$\psi_i(L) = g_i(L, x) + g_i(L, y) - g_i(L, x+y).$$

Since $\psi_i(K) \geq 0$ for $0 \leq i \leq n$, we obtain (4.2). Thus, (a) implies (b).

Assume now that (b) holds. Suppose that $\varphi \in \mathbf{Val}$ is non-negative and let φ_i , $0 \leq i \leq n$ denote its homogeneous components. Define a Minkowski valuation $\Phi \in \mathbf{MVal}$ by

$$\Phi(L) = \varphi(L) B^n, \quad L \in \mathcal{K}^n.$$

Since $\varphi \geq 0$, the valuation Φ is well defined. Using (3.1), it is easy to see that, on one hand,

$$h(\Phi(\lambda K), \cdot) = \varphi_0(K) + \lambda \varphi_1(K) + \dots + \lambda^n \varphi_n(K) \quad (4.3)$$

for every $\lambda > 0$. On the other hand, it follows from (b) that there exist $L_0, \Phi_1(K), \dots, \Phi_{n-1}(K), L_n \in \mathcal{K}^n$ such that

$$h(\Phi(\lambda K), \cdot) = h(L_0, \cdot) + \sum_{i=1}^{n-1} \lambda^i h(\Phi_i(K), \cdot) + \lambda^n V(K) h(L_n, \cdot) \quad (4.4)$$

for every $\lambda > 0$. Comparing coefficients in (4.3) and (4.4) shows that $\varphi_0(K) = h(L_0, \cdot)$, $\varphi_n(K) = V(K)h(L_n, \cdot)$ and $\varphi_i(K) = h(\Phi_i(K), \cdot)$ for $1 \leq i \leq n-1$. This is possible only if $\varphi_i(K) \geq 0$ for every $i \in \{0, \dots, n\}$. ■

Lemma 4.1 shows that Problem 3.4 is equivalent to the question whether the homogeneous components of any non-negative valuation in \mathbf{Val} are also non-negative. (A result of the latter type for monotone valuations was recently established by Bernig and Fu [10]).

Using Theorem 3.6 and Lemma 4.1, we now establish an affirmative answer to Problem 3.4 for the class of zonoids:

Theorem 4.2 *If $\Phi \in \mathbf{MVal}$, then for every zonoid $Z \in \mathcal{K}^n$, there exist convex bodies $L_0, \Phi_1(Z), \dots, \Phi_{n-1}(Z), L_n \in \mathcal{K}^n$ such that*

$$\Phi(\lambda Z) = L_0 + \lambda \Phi_1(Z) + \dots + \lambda^{n-1} \Phi_{n-1}(Z) + \lambda^n V(Z) L_n$$

for every $\lambda > 0$.

Proof. By Lemma 4.1, it suffices to show that $\varphi_i(Z) \geq 0$, $0 \leq i \leq n$, for the homogeneous components φ_i of any non-negative $\varphi \in \mathbf{Val}$ and every zonoid $Z \in \mathcal{K}^n$. To this end, first note that, by (3.1), for any $K \in \mathcal{K}^n$,

$$0 \leq \varphi(\lambda K) = \varphi_0(K) + \lambda \varphi_1(K) + \dots + \lambda^n \varphi_n(K)$$

for every $\lambda > 0$. By letting λ tend to zero, we therefore, see that φ_0 is always non-negative for any non-negative $\varphi \in \mathbf{Val}$. Similarly, dividing by λ^n and letting λ tend to infinity, it follows that φ_n is always non-negative.

It remains to show that $\varphi_i(Z) \geq 0$, $1 \leq i \leq n-1$, for any zonoid $Z \in \mathcal{K}^n$. In order to see this, let $K \in \mathcal{K}^n$ be a centrally symmetric convex body contained in an i -dimensional subspace E with $\text{vol}_E(K) > 0$. By Lemma 3.2, we have $\psi(K) = 0$ for any $\psi \in \mathbf{Val}_j$ with $j > i$. Therefore, it follows that for any non-negative $\varphi \in \mathbf{Val}$,

$$0 \leq \varphi(\lambda K) = \varphi_0(K) + \lambda \varphi_1(K) + \dots + \lambda^{i-1} \varphi_{i-1}(K) + \lambda^i \varphi_i(K)$$

for every $\lambda > 0$. Again, dividing by λ^i and letting λ tend to infinity, we see that $\varphi_i(K) \geq 0$. Let φ_i^\pm denote the even and odd parts of φ_i , respectively. Since K is centrally symmetric, we conclude $\varphi_i^-(K) = 0$ and

$$0 \leq \varphi_i(K) = \varphi_i^+(K) = (\mathbf{K}_i \varphi_i^+)(E) \text{vol}_E(K).$$

Since the subspace E was arbitrary, we see that $K_i \varphi_i^+ \geq 0$. Consequently, by Theorem 3.6, $\varphi_i^+(Z) \geq 0$ for any zonoid $Z \in \mathcal{K}^n$. Moreover, since zonoids are centrally symmetric, we have $\varphi_i^-(Z) = 0$, and thus $\varphi_i(Z) = \varphi_i^+(Z) \geq 0$. ■

Theorem 2 is now a simple consequence of Theorem 4.2. It is the special case $Z = B^n$ of the following

Corollary 4.3 *Suppose that $\Phi \in \mathbf{MVal}$ and let $K \in \mathcal{K}^n$. Then for every zonoid $Z \in \mathcal{K}^n$ there exist (unique) $\Phi_Z^{(j)} \in \mathbf{MVal}$ such that*

$$\Phi(K + rZ) = \sum_{j=0}^n r^{n-j} \Phi_Z^{(j)}(K) \quad (4.5)$$

for every $r > 0$.

Proof. Let $K \in \mathcal{K}^n$ be fixed and define $\Psi^K : \mathcal{K}^n \rightarrow \mathcal{K}^n$ by

$$\Psi^K(L) = \Phi(K + L), \quad L \in \mathcal{K}^n.$$

It is easy to see that, in fact, $\Psi^K \in \mathbf{MVal}$. Thus, by Theorem 4.2, for every zonoid Z , there exist $\Psi_0^K(Z), \dots, \Psi_n^K(Z) \in \mathcal{K}^n$ such that

$$\Phi(K + rZ) = \Psi^K(rZ) = \Psi_0^K(Z) + r\Psi_1^K(Z) + \dots + r^{n-1}\Psi_{n-1}^K(Z) + r^n\Psi_n^K(Z)$$

for every $r > 0$. Define $\Phi_Z^{(j)} : \mathcal{K}^n \rightarrow \mathcal{K}^n$ by

$$\Phi_Z^{(j)}(L) = \Psi_{n-j}^L(Z), \quad L \in \mathcal{K}^n.$$

Clearly, the maps $\Phi_Z^{(j)}$ satisfy (4.5). Moreover, from an application of the Steiner formula (2.2) to the valuation $\varphi(K) = h(\Phi(K), \cdot)$ (with values in the vector space $C(S^{n-1})$) and the uniqueness of the derived valuations $\varphi^{(j)}$, it follows that $\Phi_Z^{(j)} \in \mathbf{MVal}$. ■

We end this section with a further generalization of Theorem 4.2.

Theorem 4.4 *Suppose that $\Phi \in \mathbf{MVal}$ and let $Z_1, \dots, Z_m \in \mathcal{K}^n$ be zonoids. Then*

$$\Phi(\lambda_1 Z_1 + \dots + \lambda_m Z_m), \quad \lambda_1, \dots, \lambda_m \geq 0,$$

can be expressed as a polynomial in $\lambda_1, \dots, \lambda_m$ of total degree at most n whose coefficients are convex bodies.

Proof. Using arguments as in the proof of Lemma 4.1, we see that it is enough to prove that $\varphi_i(Z_{j_1}, \dots, Z_{j_i}) \geq 0$, $1 \leq j_1, \dots, j_i \leq m$, holds for the mixed valuations derived from any non-negative valuation $\varphi_i \in \mathbf{Val}_i$ with $1 \leq i \leq n-1$. To this end, let φ_i^\pm denote the even and odd parts of φ_i , respectively. In the proof of Theorem 4.2, we have seen that $K_i \varphi_i^+ \geq 0$. Consequently, by Theorem 3.6, $\varphi_i^+(Z_{j_1}, \dots, Z_{j_i}) \geq 0$. Moreover, since zonoids are centrally symmetric, we have $\varphi_i^-(Z_{j_1}, \dots, Z_{j_i}) = 0$. Thus, we conclude $\varphi_i(Z_{j_1}, \dots, Z_{j_i}) = \varphi_i^+(Z_{j_1}, \dots, Z_{j_i}) \geq 0$. \blacksquare

5. Background material for the proof of Theorem 3

For quick reference, we state in the following the geometric inequalities (for which we refer the reader to the book by Schneider [54]) and other ingredients needed in the proof of Theorem 3.

For $K, L \in \mathcal{K}^n$ and $0 \leq i \leq n$, we write $V(K[i], L[n-i])$ for the mixed volume with i copies of K and $n-i$ copies of L . For $K, K_1, \dots, K_i \in \mathcal{K}^n$ and $\mathbf{C} = (K_1, \dots, K_i)$, we denote by $V_i(K, \mathbf{C})$ the mixed volume $V(K, \dots, K, K_1, \dots, K_i)$ where K appears $n-i$ times. For $0 \leq i \leq n-1$, we use $W_i(K, L)$ to denote the mixed volume $V(K[n-i-1], B^n[i], L)$. The mixed volume $W_i(K, K)$ will be written as $W_i(K)$ and is called the *ith quermassintegral* of K . The *ith intrinsic volume* $V_i(K)$ of K is defined by

$$\kappa_{n-i} V_i(K) = \binom{n}{i} W_{n-i}(K), \quad (5.1)$$

where κ_m is the m -dimensional volume of the Euclidean unit ball in \mathbb{R}^m . A special case of (2.1) is the *classical Steiner formula* for the volume of the parallel set of K at distance $r > 0$,

$$V(K + rB^n) = \sum_{i=0}^n r^i \binom{n}{i} W_i(K) = \sum_{i=0}^n r^{n-i} \kappa_{n-i} V_i(K).$$

Associated with a convex body $K \in \mathcal{K}^n$ is a family of Borel measures $S_i(K, \cdot)$, $0 \leq i \leq n-1$, on S^{n-1} , called the *area measures of order i* of K , such that for each $L \in \mathcal{K}^n$,

$$W_{n-1-i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u) dS_i(K, u). \quad (5.2)$$

In fact, the measure $S_i(K, \cdot)$ is uniquely determined by the property that (5.2) holds for all $L \in \mathcal{K}^n$. The existence of a polynomial expansion of the translation invariant valuation $W_{n-1-i}(\cdot, L)$, thus carries over to the surface area measures. In particular, for $r > 0$, we have the Steiner type formula

$$S_j(K + rB^n, \cdot) = \sum_{i=0}^j r^{j-i} \binom{j}{i} S_i(K, \cdot). \quad (5.3)$$

Let \mathcal{K}_o^n denote the set of convex bodies in \mathbb{R}^n with non-empty interior. One of the fundamental inequalities for mixed volumes is the general Minkowski inequality: If $K, L \in \mathcal{K}_o^n$ and $0 \leq i \leq n - 2$, then

$$W_i(K, L)^{n-i} \geq W_i(K)^{n-i-1} W_i(L), \quad (5.4)$$

with equality if and only if K and L are homothetic.

A consequence of the Minkowski inequality (5.4) is the Brunn–Minkowski inequality for quermassintegrals: If $K, L \in \mathcal{K}_o^n$ and $0 \leq i \leq n - 2$, then

$$W_i(K + L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)}, \quad (5.5)$$

with equality if and only if K and L are homothetic.

A further generalization of inequality (5.5) (where the equality conditions are not yet known) is the following: If $0 \leq i \leq n - 2$, $K, L, K_1, \dots, K_i \in \mathcal{K}^n$ and $\mathbf{C} = (K_1, \dots, K_i)$, then

$$V_i(K + L, \mathbf{C})^{1/(n-i)} \geq V_i(K, \mathbf{C})^{1/(n-i)} + V_i(L, \mathbf{C})^{1/(n-i)}. \quad (5.6)$$

The *Steiner point* $s(K)$ of $K \in \mathcal{K}^n$ is the point in K defined by

$$s(K) = n \int_{S^{n-1}} h(K, u) u \, du,$$

where the integration is with respect to the rotation invariant probability measure on S^{n-1} . It is not hard to show that s is continuous, rigid motion equivariant and Minkowski additive.

Theorem 5.1 (see e.g., [54, p. 307]) *The Steiner point map $s : \mathcal{K}^n \rightarrow \mathbb{R}^n$ is the unique vector valued rigid motion equivariant and continuous valuation.*

A convex body K is said to be of class C_+^2 if the boundary of K is a C^2 submanifold of \mathbb{R}^n with everywhere positive Gaussian curvature. The following property of bodies of class C_+^2 will be useful (see [54, p. 150]):

Lemma 5.2 *If $K \in \mathcal{K}^n$ is a convex body of class C_+^2 , then there exist a convex body $L \in \mathcal{K}^n$ and a real number $r > 0$ such that $K = L + rB^n$.*

In the remaining part of this section we recall the notion of smooth valuations as well as the definition of an important derivation operator on the space \mathbf{Val} needed in the next section. In order to do this, we first endow the space \mathbf{Val} with the norm

$$\|\varphi\| = \sup\{|\varphi(K)| : K \subseteq B^n\}, \quad \varphi \in \mathbf{Val}.$$

It is well known that \mathbf{Val} thus becomes a Banach space. The group $\mathrm{GL}(n)$ acts on \mathbf{Val} continuously by

$$(A\varphi)(K) = \varphi(A^{-1}K), \quad A \in \mathrm{GL}(n), \varphi \in \mathbf{Val}.$$

Note that the subspaces $\mathbf{Val}_i^\pm \subseteq \mathbf{Val}$ are invariant under this $\mathrm{GL}(n)$ action. Actually they are also irreducible. This important fact was established by Alesker [2]; it directly implies a conjecture by McMullen that the linear combinations of mixed volumes are dense in \mathbf{Val} . A different dense subset of \mathbf{Val} can be defined as follows:

Definition *A valuation $\varphi \in \mathbf{Val}$ is called smooth if the map $\mathrm{GL}(n) \rightarrow \mathbf{Val}$ defined by $A \mapsto A\varphi$ is infinitely differentiable.*

We use \mathbf{Val}^∞ to denote the space of continuous translation invariant and smooth valuations. For the subspaces of homogeneous valuations of given parity in \mathbf{Val}^∞ we write $\mathbf{Val}_i^{\pm, \infty}$. It is well known (cf. [63, p. 32]) that $\mathbf{Val}_i^{\pm, \infty}$ is a dense $\mathrm{GL}(n)$ invariant subspace of \mathbf{Val}_i^\pm . Moreover, from Corollary 3.1 one deduces that the space \mathbf{Val}^∞ admits a direct sum decomposition into its subspaces of homogeneous valuations of given parity.

The Steiner formula (2.2) gives rise to an important derivation operator $\Lambda : \mathbf{Val} \rightarrow \mathbf{Val}$ defined by

$$(\Lambda\varphi)(K) = \left. \frac{d}{dt} \right|_{t=0} \varphi(K + tB^n).$$

Note that Λ commutes with the action of the orthogonal group $\mathrm{O}(n)$. Moreover, if $\varphi \in \mathbf{Val}_i$ then $\Lambda\varphi \in \mathbf{Val}_{i-1}$.

The significance of the linear operator Λ can be seen from the following Hard Lefschetz type theorem established for even valuations by Alesker [3] and for general valuations by Bernig and Bröcker [9]:

Theorem 5.3 *If $2i \geq n$, then $\Lambda^{2i-n} : \mathbf{Val}_i^\infty \rightarrow \mathbf{Val}_{n-i}^\infty$ is an isomorphism. In particular, $\Lambda : \mathbf{Val}_i^\infty \rightarrow \mathbf{Val}_{i-1}^\infty$ is injective if $2i - 1 \geq n$ and surjective if $2i - 1 \leq n$.*

Theorem 5.3 suggests that it is (sometimes) possible to deduce results on valuations of degree i from result on valuations of a degree $j > i$. We will exploit this idea in the proof of Theorem 3. To this end, we use Theorem 2 to define the derivation operator Λ for translation invariant continuous Minkowski valuations:

Corollary 5.4 *Suppose that $\Phi \in \mathbf{MVal}$. Then there exists a $\Lambda\Phi \in \mathbf{MVal}$ such that for every $K \in \mathcal{K}^n$ and $u \in S^{n-1}$,*

$$h((\Lambda\Phi)(K), u) = \left. \frac{d}{dt} \right|_{t=0} h(\Phi(K + tB^n), u).$$

Moreover, if $\Phi \in \mathbf{MVal}_i$ with $1 \leq i \leq n$, then $\Lambda\Phi \in \mathbf{MVal}_{i-1}$.

We remark that a Hard Lefschetz type theorem similar to Theorem 5.3 does *not* hold for the sets \mathbf{MVal}_i in general. This follows from results of Kiderlen [29] and the second-named author [58] on translation invariant and $\mathrm{SO}(n)$ equivariant Minkowski valuations of degree 1 and $n - 1$, respectively.

6. The symmetry of bivaluations

We recall here the notion of bivaluations and, in particular, a recent result on the symmetry of rigid motion invariant homogeneous bivaluations. As a reference for the material in this section we refer to the recent article [5].

Definition A map $\phi : \mathcal{K}^n \times \mathcal{K}^n \rightarrow \mathbb{R}$ is called a *bivaluation* if ϕ is a valuation in each argument. A bivaluation ϕ is called *translation biinvariant* if ϕ is invariant under independent translations of its arguments and ϕ is said to be *$O(n)$ invariant* if $\phi(\vartheta K, \vartheta L) = \phi(K, L)$ for all $K, L \in \mathcal{K}^n$ and $\vartheta \in O(n)$. We say ϕ has *bidegree* (i, j) if $\phi(\alpha K, \beta L) = \alpha^i \beta^j \phi(K, L)$ for all $K, L \in \mathcal{K}^n$ and $\alpha, \beta > 0$.

Important examples of invariant bivaluations can be constructed using mixed volumes and Minkowski valuations:

Examples:

- (a) For
- $0 \leq i \leq n$
- , the bivaluation
- $\phi : \mathcal{K}^n \times \mathcal{K}^n \rightarrow \mathbb{R}$
- , defined by

$$\phi(K, L) = V(K[i], L[n-i]), \quad K, L \in \mathcal{K}^n,$$

is translation biinvariant and $O(n)$ invariant and has bidegree $(i, n-i)$.

- (b) Suppose that
- $0 \leq i, j \leq n$
- and let
- $\Phi_j \in \mathbf{MVal}_j$
- be
- $O(n)$
- equivariant, i.e.
- $\Phi_j(\vartheta K) = \vartheta \Phi_j(K)$
- for all
- $K \in \mathcal{K}^n$
- and
- $\vartheta \in O(n)$
- . Then the bivaluation
- $\psi : \mathcal{K}^n \times \mathcal{K}^n \rightarrow \mathbb{R}$
- , defined by

$$\psi(K, L) = W_{n-i-1}(K, \Phi_j(L)), \quad K, L \in \mathcal{K}^n,$$

is translation biinvariant and $O(n)$ invariant and has bidegree (i, j) .

First classification results for invariant bivaluations were obtained only recently by Ludwig [38]. Systematic investigations of continuous translation biinvariant bivaluations were started in [5]. In order to describe some of the results obtained there, let \mathbf{BVal} denote the vector space of all continuous translation biinvariant (real valued) bivaluations. We write $\mathbf{BVal}_{i,j}$ for its subspace of all bivaluations of bidegree (i, j) and we use $\mathbf{BVal}^{O(n)}$ and $\mathbf{BVal}_{i,j}^{O(n)}$ to denote the respective subspaces of $O(n)$ invariant bivaluations.

From McMullen's decomposition of the space \mathbf{Val} stated in Corollary 3.1, one immediately deduces a corresponding result for the space \mathbf{BVal} :

Corollary 6.1

$$\mathbf{BVal} = \bigoplus_{i,j=0}^n \mathbf{BVal}_{i,j}.$$

Minkowski valuations arise naturally from data about projections or sections of convex bodies. For example, the projection body operator $\Pi \in \mathbf{MVal}_{n-1}$ is defined as follows: The *projection body* $\Pi(K)$ of K is the convex body defined by $h(\Pi(K), u) = \text{vol}_{n-1}(K|u^\perp)$, $u \in S^{n-1}$, where $K|u^\perp$ denotes the projection of K onto the hyperplane orthogonal to u . Introduced already by Minkowski, projection bodies have become an important tool in several areas over the last decades (see, e.g. [25, 40, 44, 45, 54, 56]; for their special role in the theory of valuations see [33, 38, 60]).

An extremely useful and well known symmetry property of the projection body operator is the following: If $K, L \in \mathcal{K}^n$, then

$$V(\Pi(K), L, \dots, L) = V(\Pi(L), K, \dots, K). \quad (6.1)$$

Variants of this identity and its generalizations have been used extensively for establishing geometric inequalities related to convex and star body valued valuations (see e.g. [19, 21, 25, 37, 40–47, 57–59]).

In [5] the following generalization of the symmetry property (6.1) was established:

Theorem 6.2 *If $\phi \in \mathbf{BVal}_{i,i}^{O(n)}$, $0 \leq i \leq n$, then*

$$\phi(K, L) = \phi(L, K)$$

for every $K, L \in \mathcal{K}^n$.

For $m = 1, 2$ let the partial derivation operators $\Lambda_m : \mathbf{BVal} \rightarrow \mathbf{BVal}$ be defined by (cf. Corollary 5.4)

$$(\Lambda_1\phi)(K, L) = \left. \frac{d}{dt} \right|_{t=0} \phi(K + tB^n, L). \quad (6.2)$$

and

$$(\Lambda_2\phi)(K, L) = \left. \frac{d}{dt} \right|_{t=0} \phi(K, L + tB^n). \quad (6.3)$$

Clearly, if $\phi \in \mathbf{BVal}_{i,j}$, then $\Lambda_1\phi \in \mathbf{BVal}_{i-1,j}$ and $\Lambda_2\phi \in \mathbf{BVal}_{i,j-1}$.

Also define an operator $T : \mathbf{BVal} \rightarrow \mathbf{BVal}$ by

$$(T\phi)(K, L) = \phi(L, K).$$

By Theorem 6.2, the restriction of T to $\mathbf{BVal}_{i,i}^{O(n)}$ acts as the identity. Thus, we obtain the following immediate consequence of Theorem 6.2:

Corollary 6.3 *Suppose that $0 \leq j \leq n$ and $0 \leq i \leq j$. Then the following diagram is commutative:*

$$\begin{array}{ccc} \mathbf{BVal}_{j,j}^{O(n)} & \xrightarrow{T = \text{Id}} & \mathbf{BVal}_{j,j}^{O(n)} \\ \Lambda_2^{j-i} \downarrow & & \downarrow \Lambda_1^{j-i} \\ \mathbf{BVal}_{j,i}^{O(n)} & \xrightarrow{T} & \mathbf{BVal}_{i,j}^{O(n)} \end{array}$$

Proof. Suppose that $\phi \in \mathbf{BVal}_{j,j}^{\mathcal{O}(n)}$, $0 \leq j \leq n$ and let $K, L \in \mathcal{K}^n$. Then, by Theorem 6.2, we have

$$\phi(L, K + tB^n) = \phi(K + tB^n, L)$$

for every $t > 0$. Consequently, by definitions (6.2) and (6.3), we obtain

$$(\mathbb{T}\Lambda_2^{j-i}\phi)(K, L) = (\Lambda_1^{j-i}\phi)(K, L). \quad \blacksquare$$

In the proof of Theorem 3 we will repeatedly make critical use of the following consequence of Corollary 6.3:

Corollary 6.4 *Let $\Phi_j \in \mathbf{MVal}_j$, $2 \leq j \leq n - 1$, be $\mathrm{SO}(n)$ equivariant. If $1 \leq i \leq j + 1$, then*

$$W_{n-i}(K, \Phi_j(L)) = \frac{(i-1)!}{j!} W_{n-1-j}(L, (\Lambda^{j+1-i}\Phi_j)(K))$$

for every $K, L \in \mathcal{K}^n$.

Proof. We first note that any $\mathrm{SO}(n)$ equivariant $\Phi \in \mathbf{MVal}$ is also $\mathcal{O}(n)$ equivariant (see [5, Lemma 7.1]). Define $\phi \in \mathbf{BVal}_{j,j}^{\mathcal{O}(n)}$ by

$$\phi(K, L) = W_{n-1-j}(K, \Phi_j(L)), \quad K, L \in \mathcal{K}^n.$$

From (5.2) and (5.3), it follows that

$$W_{n-i}(K, \Phi_j(L)) = \frac{(i-1)!}{j!} (\Lambda_1^{j+1-i}\phi)(K, L).$$

Thus, applications of Corollary 6.3 and Corollary 5.4 complete the proof. \blacksquare

7. Brunn–Minkowski type inequalities

In this final section we present the proof of Theorem 3. Special cases of this result for $j = n - 1$ were obtained in [57] and for $j \in \{2, \dots, n - 1\}$ and $i = j + 1$ in [5] and [59]. The equality conditions for bodies of class C_+^2 are new for $j \leq n - 2$.

Theorem 7.1 *Let $\Phi_j \in \mathbf{MVal}_j$, $2 \leq j \leq n-1$, be $\mathrm{SO}(n)$ equivariant and non-trivial. If $K, L \in \mathcal{K}^n$ and $1 \leq i \leq j+1$, then*

$$W_{n-i}(\Phi_j(K+L))^{1/ij} \geq W_{n-i}(\Phi_j(K))^{1/ij} + W_{n-i}(\Phi_j(L))^{1/ij}. \quad (7.1)$$

If K and L are of class C_+^2 , then equality holds if and only if K and L are homothetic.

Proof. By Corollary 6.4, we have for $Q \in \mathcal{K}^n$,

$$W_{n-i}(Q, \Phi_j(K+L)) = \frac{(i-1)!}{j!} W_{n-1-j}(K+L, (\Lambda^{j+1-i}\Phi_j)(Q)). \quad (7.2)$$

From an application of inequality (5.6), we obtain

$$\begin{aligned} & W_{n-1-j}(K+L, (\Lambda^{j+1-i}\Phi_j)(Q))^{1/j} \\ & \geq W_{n-1-j}(K, (\Lambda^{j+1-i}\Phi_j)(Q))^{1/j} + W_{n-1-j}(L, (\Lambda^{j+1-i}\Phi_j)(Q))^{1/j}. \end{aligned} \quad (7.3)$$

A combination of (7.2) and (7.3) and another application of Corollary 6.4, therefore show that

$$W_{n-i}(Q, \Phi_j(K+L))^{1/j} \geq W_{n-i}(Q, \Phi_j(K))^{1/j} + W_{n-i}(Q, \Phi_j(L))^{1/j}. \quad (7.4)$$

It follows from Minkowski's inequality (5.4), that

$$W_{n-i}(Q, \Phi_j(K))^i \geq W_{n-i}(Q)^{i-1} W_{n-i}(\Phi_j(K)), \quad (7.5)$$

and, similarly,

$$W_{n-i}(Q, \Phi_j(L))^i \geq W_{n-i}(Q)^{i-1} W_{n-i}(\Phi_j(L)). \quad (7.6)$$

Plugging (7.5) and (7.6) into (7.4), and putting $Q = \Phi_j(K+L)$, now yields the desired inequality (7.1).

In order to derive the equality conditions for convex bodies of class C_+^2 , we first show that such bodies are mapped by Φ_j to bodies with non-empty interior. For the following argument, the authors are obliged to T. Wannerer. Let $Q \in \mathcal{K}^n$ be of class C_+^2 . By Lemma 5.2, there exist a convex body $M \in \mathcal{K}^n$ and a number $r > 0$ such that $Q = M + rB^n$. Using that Φ_j has degree j , we thus obtain from Theorem 2 that

$$\Phi_j(Q) = \Phi_j(M + rB^n) = r^j \Phi_j^{(0)}(M) + \cdots + \Phi_j^{(j)}(M),$$

where $\Phi_j^{(m)} \in \mathbf{MVal}_m$, $1 \leq m \leq j$, and $\Phi_j^{(0)}(M) = \Phi_j(B^n)$. Since Φ_j is $\mathrm{SO}(n)$ equivariant, we must have $\Phi_j(B^n) = tB^n$ for some $t \geq 0$. Since Φ_j is non-trivial, it follows from an application of Corollary 6.4 (with $i = 1$ and $K = B^n$) that in fact $t > 0$. Consequently, $\Phi_j(Q) \in \mathcal{K}_o^n$.

Now assume that $K, L \in \mathcal{K}^n$ are of class C_+^2 and that equality holds in (7.1). Let s be the Steiner point map. Since $Q \mapsto s(\Phi_j(Q)) + s(Q)$ is a continuous and rigid motion equivariant valuation, Theorem 5.1 implies that $s(\Phi_j(Q)) = 0$ for every $Q \in \mathcal{K}^n$. Thus, we deduce from the equality conditions of (7.5) and (7.6), that there exist $\lambda_1, \lambda_2 > 0$ such that

$$\Phi_j(K) = \lambda_1 \Phi_j(K + L) \quad \text{and} \quad \Phi_j L = \lambda_2 \Phi_j(K + L) \quad (7.7)$$

and

$$\lambda_1^{1/j} + \lambda_2^{1/j} = 1.$$

Moreover, (7.7) and another application of Corollary 6.4 (with $i = 1$ and $K = B^n$) imply that

$$W_{n-j}(K) = \lambda_1 W_{n-j}(K + L) \quad \text{and} \quad W_{n-j}(L) = \lambda_2 W_{n-j}(K + L).$$

Hence, we have

$$W_{n-j}(K + L)^{1/j} = W_{n-j}(K)^{1/j} + W_{n-j}(L)^{1/j},$$

which, by (5.5), implies that K and L are homothetic. ■

We remark that our proof shows that the smoothness assumption in the equality conditions can be omitted for bodies with non-empty interior in case Φ_j maps those bodies again to convex bodies with non-empty interior. This is always the case if $j = n - 1$ (cf. [57]), but is not known in general.

We also note that since translation invariant continuous Minkowski valuations of degree one are linear with respect to Minkowski addition (see e.g. [27]), inequality (7.1) also holds in the case $j = 1$ (this follows from (5.5)). The equality conditions, however, are different in this case.

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References

- [1] S. Alesker, *Continuous rotation invariant valuations on convex sets*, Ann. of Math. (2) **149** (1999), 977–1005.
- [2] S. Alesker, *Description of translation invariant valuations on convex sets with solution of P. McMullen’s conjecture*, Geom. Funct. Anal. **11** (2001), 244–272.
- [3] S. Alesker, *Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations*, J. Differential Geom. **63** (2003), 63–95.
- [4] S. Alesker and J. Bernstein, *Range characterization of the cosine transform on higher Grassmannians*, Adv. Math. **184** (2004), 367–379.
- [5] S. Alesker, A. Bernig and F.E. Schuster, *Harmonic analysis of translation invariant valuations*, Geom. Funct. Anal. **21** (2011), 751–773.
- [6] A. Bernig, *A Hadwiger-type theorem for the special unitary group*, Geom. Funct. Anal. **19** (2009), 356–372.
- [7] A. Bernig, *Invariant valuations on quaternionic vector spaces*, J. Inst. Math. Jussieu, in press.
- [8] A. Bernig, *Algebraic integral geometry*, Preprint, arXiv:1004.3145.
- [9] A. Bernig and L. Bröcker, *Valuations on manifolds and Rumin cohomology*, J. Differential Geom. **75** (2007), 433–457.
- [10] A. Bernig and J.H.G. Fu, *Hermitian integral geometry*, Ann. of Math. (2) **173** (2011), 907–945.
- [11] A. Colesanti, *A Steiner type formula for convex functions*, Mathematika **44** (1997), 195–214.
- [12] A. Colesanti and D. Hug, *Steiner type formulae and weighted measures of singularities for semi-convex functions*, Trans. Amer. Math. Soc. **352** (2000), 3239–3263.
- [13] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–91.
- [14] J.H.G. Fu, *Tubular neighborhoods in Euclidean spaces*, Duke Math. J. **52** (1985), 1025–1046.
- [15] J.H.G. Fu, *Curvature measures and Chern classes of singular varieties*, J. Differential Geom. **39** (1994), 251–280.
- [16] J.H.G. Fu, *Curvature measures of subanalytic sets*, Amer. J. Math. **116** (1994), 819–880.
- [17] J.H.G. Fu, *Structure of the unitary valuation algebra*, J. Differential Geom. **72** (2006), 509–533.
- [18] R.J. Gardner, *The Brunn–Minkowski inequality*, Bull. Am. Math. Soc. **39** (2002), 355–405.
- [19] R.J. Gardner, *Geometric tomography*, Second ed., Cambridge University Press, Cambridge, 2006.
- [20] A. Gray, *Tubes*, Addison-Wesley, Redwood City, 1990.
- [21] C. Haberl, *L_p intersection bodies*, Adv. Math. **217** (2008), 2599–2624.
- [22] C. Haberl, *Star body valued valuations*, Indiana Univ. Math. J. **58** (2009), 2253–2276.
- [23] C. Haberl, *Blaschke valuations*, Amer. J. Math. **133** (2011), 717–751.

- [24] C. Haberl and M. Ludwig, *A characterization of L_p intersection bodies*, Int. Math. Res. Not. (2006), Article ID 10548, 29 pages.
- [25] C. Haberl and F.E. Schuster, *General L_p affine isoperimetric inequalities*, J. Differential Geom. **83** (2009), 1–26.
- [26] M. Henk and M.A. Hernández Cifre, *Notes on the roots of Steiner polynomials*, Rev. Mat. Iberoam. **24** (2008), 631–644.
- [27] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, Berlin, 1957.
- [28] D. Hug, G. Last, and W. Weil, *A local Steiner-type formula for general closed sets and applications*, Math. Z. **246** (2004), 237–272.
- [29] M. Kiderlen, *Blaschke- and Minkowski-Endomorphisms of convex bodies*, Trans. Amer. Math. Soc. **358** (2006), 5539–5564.
- [30] D.A. Klain, *A short proof of Hadwiger’s characterization theorem*, Mathematika **42** (1995), 329–339.
- [31] D.A. Klain, *Even valuations on convex bodies*, Trans. Amer. Math. Soc. **352** (2000), 71–93.
- [32] D.A. Klain and G.-C. Rota, *Introduction to geometric probability*, Cambridge University Press, Cambridge, 1997.
- [33] M. Ludwig, *Projection bodies and valuations*, Adv. Math. **172** (2002), 158–168.
- [34] M. Ludwig, *Ellipsoids and matrix valued valuations*, Duke Math. J. **119** (2003), 159–188.
- [35] M. Ludwig, *Minkowski valuations*, Trans. Amer. Math. Soc. **357** (2005), 4191–4213.
- [36] M. Ludwig, *Intersection bodies and valuations*, Amer. J. Math. **128** (2006), 1409–1428.
- [37] M. Ludwig, *General affine surface areas*, Adv. Math. **224** (2010), 2346–2360.
- [38] M. Ludwig, *Minkowski areas and valuations*, J. Differential Geom. **86** (2010), 133–161.
- [39] M. Ludwig and M. Reitzner, *A classification of $SL(n)$ invariant valuations*, Ann. of Math. (2) **172** (2010), 1223–1271.
- [40] E. Lutwak, *Mixed projection inequalities*, Trans. Amer. Math. Soc. **287** (1985), 91–105.
- [41] E. Lutwak, *On some affine isoperimetric inequalities*, J. Differential Geom. **23** (1986), 1–13.
- [42] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232–261.
- [43] E. Lutwak, *Centroid bodies and dual mixed volumes*, Proc. London Math. Soc. **60** (1990), 365–391.
- [44] E. Lutwak, *Inequalities for mixed projection bodies*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 901–916.
- [45] E. Lutwak, D. Yang, and G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111–132.

- [46] E. Lutwak, D. Yang, and G. Zhang, *A new ellipsoid associated with convex bodies*, Duke Math. J. **104** (2000), 375–390.
- [47] E. Lutwak and G. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom. **47** (1997), 1–16.
- [48] P. McMullen, *Valuations and Euler-type relations on certain classes of convex polytopes*, Proc. London Math. Soc. **35** (1977), 113–135.
- [49] C. Meier, *Multilinearität bei Polyederaddition*, Arch. Math. **29** (1977), 210–217.
- [50] L. Parapatits and T. Wannerer, *On the stability of the Klain map*, in preparation.
- [51] J. Rataj and M. Zähle, *Normal cycles of Lipschitz manifolds by approximation with parallel sets*, Differential Geom. Appl. **19** (2003), 113–126.
- [52] R. Schneider, *Equivariant endomorphisms of the space of convex bodies*, Trans. Amer. Math. Soc. **194** (1974), 53–78.
- [53] R. Schneider, *Bestimmung konvexer Körper durch Krümmungsmaße*, Comment. Math. Helv. **54** (1979), 42–60.
- [54] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, Cambridge University Press, 1993.
- [55] R. Schneider and F.E. Schuster, *Rotation equivariant Minkowski valuations*, Int. Math. Res. Not. (2006), Article ID 72894, 20 pages.
- [56] R. Schneider and W. Weil, *Stochastic and Integral Geometry*, Springer, Berlin, 2008.
- [57] F.E. Schuster, *Volume inequalities and additive maps of convex bodies*, Mathematika **53** (2006), 211–234.
- [58] F.E. Schuster, *Convolutions and multiplier transformations*, Trans. Amer. Math. Soc. **359** (2007), 5567–5591.
- [59] F.E. Schuster, *Crofton Measures and Minkowski Valuations*, Duke Math. J. **154** (2010), 1–30.
- [60] F.E. Schuster and T. Wannerer, *$GL(n)$ contravariant Minkowski valuations*, Trans. Amer. Math. Soc. **364** (2012), 815–826.
- [61] W. Spiegel, *Ein Beitrag über additive, translationsinvariante, stetige Eikörperfunktionale*, Geom. Dedicata **7** (1978), 9–19.
- [62] B. Teissier, *Bonnesen-type inequalities in algebraic geometry. I. Introduction to the problem*, in Seminar on Differential Geometry, 85–105. Annals of Mathematical Studies **102**, Princeton University Press, Princeton, N.J., 1982.
- [63] N.R. Wallach, *Real reductive groups. I*, Pure and Applied Mathematics **132**, Academic Press, Inc., Boston, MA, 1988.
- [64] T. Wannerer, *$GL(n)$ equivariant Minkowski valuations*, Indiana Univ. Math. J., in press.
- [65] H. Weyl, *On the volume of tubes*, Amer. J. Math. **61** (1939), 461–472.

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