Volume Inequalities and Additive Maps of Convex Bodies

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Dedicated to Prof. Rolf Schneider on the occasion of his 65th birthday

Abstract. Analogs of the classical inequalities from the Brunn Minkowski Theory for rotation intertwining additive maps of convex bodies are developed. We also prove analogs of inequalities from the dual Brunn Minkowski Theory for intertwining additive maps of star bodies. These inequalities provide generalizations of results for projection and intersection bodies. As a corollary we obtain a new Brunn Minkowski inequality for the volume of polar projection bodies.

Key words. Convex bodies, Minkowski addition, Blaschke addition, mixed volumes, dual mixed volumes, radial addition, rotation intertwining map, spherical convolution, projection body, intersection body.

1. Introduction and Statement of Main Results

For $n \geq 3$ let $\mathcal{K}^n$ denote the space of convex bodies (i.e. compact, convex sets with nonempty interior) in $\mathbb{R}^n$ endowed with the Hausdorff topology. A compact, convex set K is uniquely determined by its support function $h(K, \cdot)$ on the unit sphere S^{n-1}, defined by $h(K, u) = \max\{u \cdot x : x \in K\}$. If $K \in \mathcal{K}^n$ contains the origin in its interior, the convex body $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}$ is called the polar body of K.

The projection body ΠK of $K \in \mathcal{K}^n$ is the convex body whose support function is given for $u \in S^{n-1}$ by

$$h(\Pi K, u) = \mathrm{vol}_{n-1}(K|u^{\perp}),$$

where vol_{n-1} denotes $(n-1)$-dimensional volume and $K|u^{\perp}$ is the image of the orthogonal projection of K onto the subspace orthogonal to u. Important volume inequalities for the polars of projection bodies are the Petty projection inequality [31] and the Zhang projection inequality [38]: Among bodies of given volume the polar projection bodies have maximal volume precisely for ellipsoids and minimal volume precisely for simplices. The corresponding results for the volume of the projection body itself are major open problems in convex geometry, see [28]. Projection bodies and their polars have received considerable attention over the last decades due to their connection to different areas such as geometric tomography, stereology, combinatorics, computational and stochastic geometry, see for example [1], [2], [8], [11], [12], [20], [21], [23], [27], [36].

Mixed projection bodies were introduced in the classic volume of Bonnesen-Fenchel [3]. They are related to ordinary projection bodies in the same way that mixed volumes are related to ordinary volume. In [23] and [27] Lutwak considered the volume of mixed projection bodies and their polars and established analogs of the classical mixed volume inequalities.

We will show that the following well known properties of the projection body operator $\Pi : \mathcal{K}^n \to \mathcal{K}^n$ are responsible not only for its behaviour under Minkowski linear combinations but also for most of the inequalities established in [23] and [27]:

(a) Π is continuous.

(b) Π is Blaschke Minkowski additive, i.e. $\Pi(K \# L) = \Pi K + \Pi L$ for all $K, L \in \mathcal{K}^n$.

(c) Π intertwines rotations, i.e. $\Pi(\vartheta K) = \vartheta \Pi K$ for all $K \in \mathcal{K}^n$ and all $\vartheta \in SO(n)$.

Here $\Pi K + \Pi L$ denotes the Minkowski sum of the projection bodies ΠK and ΠL and $K \# L$ is the Blaschke sum of the convex bodies K and L (see Section 2). $SO(n)$ is the group of rotations in n dimensions.

Definition 1.1 *A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ satisfying (a), (b) and (c) is called a Blaschke Minkowski homomorphism.*

In Section 4 we will see that there are many examples of Blaschke Minkowski homomorphisms, see also [10], [13] and [37]. The main purpose of this article is to extend Lutwak's Brunn Minkowski Theory for mixed projection operators to general Blaschke Minkowski homomorphisms. To this end, let in the following $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ denote a Blaschke Minkowski homomorphism.

Theorem 1.2 *There is a continuous operator*

$$\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n,$$

symmetric in its arguments such that, for $K_1, \ldots, K_m \in \mathcal{K}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$,

$$\Phi(\lambda_1 K_1 + \ldots + \lambda_m K_m) = \sum_{i_1, \ldots, i_{n-1}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Phi(K_{i_1}, \ldots, K_{i_{n-1}}), \tag{1.1}$$

where the sum is with respect to Minkowski addition.

Theorem 1.2 generalizes the notion of mixed projection bodies. We will prove a Minkowski inequality for the volume of mixed Blaschke Minkowski homomorphisms:

Theorem 1.3 *If $K, L \in \mathcal{K}^n$, then*

$$V(\Phi(K, \ldots, K, L))^{n-1} \geq V(\Phi K)^{n-2} V(\Phi L),$$

with equality if and only if K and L are homothetic.

An Aleksandrov Fenchel type inequality for the volume of mixed Blaschke Minkowski homomorphisms is provided by:

Theorem 1.4 *If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then*

$$V(\Phi(K_1, \ldots, K_{n-1}))^2 \geq V(\Phi(K_1, K_1, K_3, \ldots, K_{n-1})) V(\Phi(K_2, K_2, K_3, \ldots, K_{n-1})).$$

We also prove that the volume of a Blaschke Minkowski homomorphism satisfies a Brunn Minkowski inequality:

Theorem 1.5 *If $K, L \in \mathcal{K}^n$, then*

$$V(\Phi(K+L))^{1/n(n-1)} \geq V(\Phi K)^{1/n(n-1)} + V(\Phi L)^{1/n(n-1)},$$

with equality if and only if K and L are homothetic.

If we restrict ourselves to even operators, i.e. $\Phi K = \Phi(-K)$ for all $K \in \mathcal{K}^n$, we can also prove volume inequalities for the polars of mixed Blaschke Minkowski homomorphisms. In the following we write $\Phi^* K$ for the polar of ΦK.

Theorem $1.3_\mathbf{p}$ *If Φ is even and $K, L \in \mathcal{K}^n$, then*

$$V(\Phi^*(K, \ldots, K, L))^{n-1} \leq V(\Phi^* K)^{n-2} V(\Phi^* L),$$

with equality if and only if K and L are homothetic.

This result is again generalized by an Aleksandrov Fenchel type inequality:

Theorem $1.4_\mathbf{p}$ *If Φ is even and $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then*

$$V(\Phi^*(K_1, \ldots, K_{n-1}))^2 \leq V(\Phi^*(K_1, K_1, K_3, \ldots, K_{n-1})) V(\Phi^*(K_2, K_2, K_3, \ldots, K_{n-1})).$$

The next theorem shows that polars of even Blaschke Minkowski homomorphisms also satisfy a Brunn Minkowski inequality:

Theorem $1.5_\mathbf{p}$ *If Φ is even and $K, L \in \mathcal{K}^n$, then*

$$V(\Phi^*(K+L))^{-1/n(n-1)} \geq V(\Phi^* K)^{-1/n(n-1)} + V(\Phi^* L)^{-1/n(n-1)},$$

with equality if and only if K and L are homothetic.

Note that the special case $\Phi = \Pi$ of Theorem $1.5_\mathbf{p}$ provides a new Brunn Minkowski inequality for the volume of polar projection bodies:

Corollary $1.6_\mathbf{p}$ *If $K, L \in \mathcal{K}^n$, then*

$$V(\Pi^*(K+L))^{-1/n(n-1)} \geq V(\Pi^* K)^{-1/n(n-1)} + V(\Pi^* L)^{-1/n(n-1)},$$

with equality if and only if K and L are homothetic.

In recent years a dual theory to the Brunn Minkowski Theory of convex bodies was established. Mixed volumes are replaced by dual mixed volumes, which were introduced by Lutwak in [22]. The natural domain of dual mixed volumes is the space $\mathcal{S}^n$ of star bodies (i.e. compact sets, starshaped with respect to the origin with continuous radial functions) in $\mathbb{R}^n$ endowed with the Hausdorff topology. The radial function $\rho(L, \cdot)$ of a set L starshaped with respect to the origin is defined on S^{n-1} by $\rho(L, u) = \max\{\lambda \geq 0 : \lambda u \in L\}$.

The intersection body IL of $L \in \mathcal{S}^n$ is the star body whose radial function is given for $u \in S^{n-1}$ by

$$\rho(IL, u) = \mathrm{vol}_{n-1}(L \cap u^{\perp}).$$

Intersection bodies have attracted increased interest in recent years. They appear already in a paper by Busemann [4] but were first explicitly defined and named by Lutwak [25]. Intersection bodies turned out to be critical for the solution of the Busemann-Petty problem, see [6], [7], [9], [14], [16], [17], [40]. The fundamental volume inequality for intersection bodies is the Busemann intersection inequality [4]: Among bodies of given volume the intersection bodies have maximal volume precisely for ellipsoids centered in the origin. A corresponding result for the minimal volume of intersection bodies of a given volume is another major open problem in convex geometry.

The operator $I : \mathcal{S}^n \to \mathcal{S}^n$ has the following well known properties:

(a)$_{\mathbf{d}}$ I is continuous.

(b)$_{\mathbf{d}}$ $I(K \,\tilde{\#}\, L) = IK \,\tilde{+}\, IL$ for all $K, L \in \mathcal{S}^n$.

(c)$_{\mathbf{d}}$ I intertwines rotations.

Here $IK \,\tilde{+}\, IL$ is the radial Minkowski sum of the intersection bodies IK and IL and $K \,\tilde{\#}\, L$ is the radial Blaschke sum of the star bodies K and L (see Section 3).

Definition 1.1$_{\mathbf{d}}$ *A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is called radial Blaschke Minkowski homomorphism if it satisfies (a)$_d$, (b)$_d$ and (c)$_d$.*

As Lutwak shows in [25], see also [8], there is a duality between projection and intersection bodies, that is at present not yet well understood. We will show that there is a similar duality for general Blaschke Minkowski and radial Blaschke Minkowski homomorphisms. In analogy to the inequalities of Theorems 1.3, 1.4 and 1.5 we will establish dual inequalities for radial Blaschke Minkowski homomorphisms. To this end, let $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ denote a nontrivial radial Blaschke Minkowski homomorphism, where the operator that maps every star body to the origin is called the *trivial* radial Blaschke Minkowski homomorphism.

Theorem 1.2$_{\mathbf{d}}$ *There is a continuous operator*

$$\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n,$$

symmetric in its arguments such that, for $L_1, \ldots, L_m \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$,

$$\Psi(\lambda_1 L_1 \,\tilde{+}\, \ldots \,\tilde{+}\, \lambda_m L_m) = \widetilde{\sum_{i_1,\ldots,i_{n-1}}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Psi(L_{i_1}, \ldots, L_{i_{n-1}}),$$

where the sum is with respect to radial Minkowski addition.

Mixed intersection bodies were introduced in [39]. The dual Minkowski inequality for radial Blaschke Minkowski homomorphisms is:

Theorem 1.3$_\mathbf{d}$ *If $K, L \in \mathcal{S}^n$, then*

$$V(\Psi(K,\ldots,K,L))^{n-1} \leq V(\Psi K)^{n-2} V(\Psi L),$$

with equality if and only if K and L are dilates.

Theorem 1.3$_\mathbf{d}$ is a special case of the dual Aleksandrov Fenchel inequality for the volume of radial Blaschke Minkowski homomorphisms:

Theorem 1.4$_\mathbf{d}$ *If $L_1,\ldots,L_{n-1} \in \mathcal{S}^n$, then*

$$V(\Psi(L_1,\ldots,L_{n-1}))^2 \leq V(\Psi(L_1,L_1,L_3,\ldots,L_{n-1}))V(\Psi(L_2,L_2,L_3,\ldots,L_{n-1})),$$

with equality if and only if L_1 and L_2 are dilates.

The volume of a radial Blaschke Minkowski homomorphism satisfies the following dual Brunn Minkowski inequality:

Theorem 1.5$_\mathbf{d}$ *If $K, L \in \mathcal{S}^n$, then*

$$V(\Psi(K+L))^{1/n(n-1)} \leq V(\Psi K)^{1/n(n-1)} + V(\Psi L)^{1/n(n-1)},$$

with equality if and only if K and L are dilates.

Theorems 1.3$_\mathbf{d}$, 1.4$_\mathbf{d}$ and 1.5$_\mathbf{d}$ for the intersection body operator were recently established in [18] and [19].

2. Mixed Volumes

We collect in this section the background material and notation from the Brunn Minkowski Theory that is needed in the proofs of the main theorems. As a general reference we recommend the book by Schneider [35].

The most important algebraic structure on the space $\mathcal{K}^n$ is Minkowski addition. For $K_1, K_2 \in \mathcal{K}^n$ and $\lambda_1, \lambda_2 \geq 0$, the support function of the Minkowski linear combination $\lambda_1 K_1 + \lambda_2 K_2$ is

$$h(\lambda_1 K_1 + \lambda_2 K_2, \cdot) = \lambda_1 h(K_1,\cdot) + \lambda_2 h(K_2,\cdot).$$

The volume of a Minkowski linear combination $\lambda_1 K_1 + \ldots + \lambda_m K_m$ of convex bodies $K_1,\ldots,K_m$ can be expressed as a homogeneous polynomial of degree n:

$$V(\lambda_1 K_1 + \ldots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n} V(K_{i_1},\ldots,K_{i_n})\lambda_{i_1}\cdots\lambda_{i_n}.$$

The coefficients $V(K_{i_1},\ldots,K_{i_n})$ are called mixed volumes of $K_{i_1},\ldots,K_{i_n}$. These functionals are nonnegative, symmetric and translation invariant. Moreover, they are monotone (with respect to set inclusion), multilinear with respect to Minkowski addition and their diagonal form is ordinary volume, i.e. $V(K,\ldots,K) = V(K)$.

Denote by $V_i(K,L)$ the mixed volume $V(K,\dots,K,L,\dots,L)$, where K appears $n-i$ times and L appears i times. For $0 \le i \le n-1$, write $W_i(K,L)$ for the mixed volume $V(K,\dots,K,B,\dots,B,L)$, where K appears $n-i-1$ times and the Euclidean unit ball B appears i times. The mixed volume $W_i(K,K)$ will be written as $W_i(K)$ and is called the ith quermassintegral of K. If $\mathbf{C} = (K_1,\dots,K_i)$, then $V_i(K,\mathbf{C})$ denotes the mixed volume $V(K,\dots,K,K_1,\dots,K_i)$ with $n-i$ copies of K.

Associated with $K_1,\dots,K_{n-1} \in \mathcal{K}^n$ is a Borel measure, $S(K_1,\dots,K_{n-1},\cdot)$, on S^{n-1}, called the mixed surface area measure of $K_1,\dots,K_{n-1}$. It is symmetric and has the property that, for each $K \in \mathcal{K}^n$,

$$V(K,K_1,\dots,K_{n-1}) = \frac{1}{n}\int_{S^{n-1}} h(K,u)dS(K_1,\dots,K_{n-1},u). \tag{2.1}$$

The measures $S_j(K,\cdot) := S(K,\dots,K,B,\dots,B,\cdot)$, where K appears j times and B appears $n-1-j$ times, are called the surface area measures of order j of K. Of particular importance for our purposes is the surface area measure (of order $n-1$) $S_{n-1}(K,\cdot)$ of K. It is not concentrated on any great sphere and has its center of mass in the origin. Conversely, by Minkowski's existence theorem, every measure in $\mathcal{M}_+(S^{n-1})$, the space of nonnegative Borel measures on the sphere with the weak* topology, with these properties is the surface area measure of a convex body, unique up to translation. Hence, if $K_1, K_2 \in \mathcal{K}^n$ and $\lambda_1,\lambda_2 \ge 0$ (not both 0), then there exists a convex body $\lambda_1 \cdot K_1 \# \lambda_2 \cdot K_2$, unique up to translation, such that

$$S_{n-1}(\lambda_1 \cdot K_1 \# \lambda_2 \cdot K_2,\cdot) = \lambda_1 S_{n-1}(K_1,\cdot) + \lambda_2 S_{n-1}(K_2,\cdot). \tag{2.2}$$

This addition and scalar multiplication are called Blaschke addition and scalar multiplication. For $K \in \mathcal{K}^n$ and $\lambda \ge 0$, we have $\lambda \cdot K = \lambda^{1/(n-1)}K$.

Blaschke addition is an additive structure on the space $[\mathcal{K}^n]$ of translation classes of convex bodies. Thus, the natural domain of an operator Φ with the additivity property $\Phi(K \# L) = \Phi K + \Phi L$ is the space $[\mathcal{K}^n]$. In Definition 1.1 the domain of a Blaschke Minkowski homomorphism is $\mathcal{K}^n$, because we identify operators on $[\mathcal{K}^n]$ with translation invariant operators on $\mathcal{K}^n$.

The surface area measure of a Minkowski linear combination of convex bodies $K_1,\dots,K_m$ can be expressed as a polynomial homogeneous of degree $n-1$:

$$S_{n-1}(\lambda_1K_1 + \ldots + \lambda_mK_m,\cdot) = \sum_{i_1,\dots,i_{n-1}} \lambda_{i_1}\cdots\lambda_{i_{n-1}} S(K_{i_1},\dots,K_{i_{n-1}},\cdot). \tag{2.3}$$

One of the most general and fundamental inequalities for mixed volumes is the Aleksandrov Fenchel inequality: If $K_1,\dots,K_n \in \mathcal{K}^n$ and $1 \le m \le n$, then

$$V(K_1,\dots,K_n)^m \ge \prod_{j=1}^{m} V(K_j,\dots,K_j,K_{m+1},\dots,K_n). \tag{2.4}$$

Unfortunately, the equality conditions of this inequality are, in general, unknown.

An important special case of inequality (2.4), where the equality conditions are known, is the Minkowski inequality: If $K, L \in \mathcal{K}^n$, then

$$V_1(K,L)^n \ge V(K)^{n-1}V(L), \tag{2.5}$$

with equality if and only if K and L are homothetic. In fact, a more general version of Minkowski's inequality holds: If $0 \le i \le n-2$, then

$$W_i(K,L)^{n-i} \ge W_i(K)^{n-i-1} W_i(L), \tag{2.6}$$

with equality if and only if K and L are homothetic.

A consequence of the Minkowski inequality is the Brunn Minkowski inequality: If $K, L \in \mathcal{K}^n$, then

$$V(K+L)^{1/n} \ge V(K)^{1/n} + V(L)^{1/n}, \tag{2.7}$$

with equality if and only if K and L are homothetic. This is a special case of the more general inequality: If $0 \le i \le n-2$, then

$$W_i(K+L)^{1/(n-i)} \ge W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)}, \tag{2.8}$$

with equality if and only if K and L are homothetic.

A further generalization of inequality (2.7) is also known (but without equality conditions): If $0 \le i \le n-2$, $K, L, K_1, \dots, K_i \in \mathcal{K}^n$ and $\mathbf{C} = (K_1, ..., K_i)$, then

$$V_i(K+L, \mathbf{C})^{1/(n-i)} \ge V_i(K, \mathbf{C})^{1/(n-i)} + V_i(L, \mathbf{C})^{1/(n-i)}. \tag{2.9}$$

3. Dual Mixed Volumes

In this section we summarize results from the dual Brunn Minkowski Theory, see [22]. For $L_1, L_2 \in \mathcal{S}^n$ and $\lambda_1, \lambda_2 \ge 0$, the radial Minkowski linear combination $\lambda_1 L_1 \tilde{+} \lambda_2 L_2$ is the star body defined by

$$\rho(\lambda_1 L_1 \tilde{+} \lambda_2 L_2, \cdot) = \lambda_1 \rho(L_1, \cdot) + \lambda_2 \rho(L_2, \cdot). \tag{3.1}$$

The volume of a radial Minkowski linear combination $\lambda_1 L_1 \tilde{+} \dots \tilde{+} \lambda_m L_m$ of star bodies $L_1, \dots, L_m$ can be expressed as a homogeneous polynomial of degree n:

$$V(\lambda_1 L_1 \tilde{+} \dots \tilde{+} \lambda_m L_m) = \sum_{i_1, \dots, i_n} \tilde{V}(L_{i_1}, \dots, L_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$

The coefficients $\tilde{V}(L_{i_1}, \dots, L_{i_n})$ are called dual mixed volumes of $L_{i_1}, \dots, L_{i_n}$. They are nonnegative, symmetric and monotone (with respect to set inclusion). They are also multilinear with respect to radial Minkowski addition and $\tilde{V}(L, \dots, L) = V(L)$. The following integral representation of dual mixed volumes holds:

$$\tilde{V}(L_1, \dots, L_n) = \frac{1}{n} \int_{S^{n-1}} \rho(L_1, u) \cdots \rho(L_n, u) du, \tag{3.2}$$

where du is the spherical Lebesgue measure of S^{n-1}. The definitions of $\tilde{V}_i(K, L)$, $\tilde{W}_i(K, L)$, etc. are analogous to the ones for mixed volumes in Section 2. A slight extension of the notation $\tilde{V}_i(K, L)$ is for $r \in \mathbb{R}$

$$\tilde{V}_r(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho^{n-r}(K, u) \rho^r(L, u) du. \tag{3.3}$$

Obviously we have $\tilde{V}_r(L, L) = V(L)$ for every $r \in \mathbb{R}$ and every $L \in \mathcal{S}^n$.

If $\lambda_1, \lambda_2 \geq 0$, then the radial Blaschke linear combination $\lambda_1 \cdot L_1 \,\tilde{\#}\, \lambda_2 \cdot L_2$ of the star bodies L_1 and L_2 is the star body whose radial function satisfies

$$\rho^{n-1}(\lambda_1 \cdot L_1 \,\tilde{\#}\, \lambda_2 \cdot L_2, \cdot) = \lambda_1 \rho^{n-1}(L_1, \cdot) + \lambda_2 \rho^{n-1}(L_2, \cdot). \tag{3.4}$$

For $L_1, \ldots, L_m \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$, the function $\rho^{n-1}(\lambda_1 L_1 \,\tilde{+}\, \ldots \,\tilde{+} \lambda_m L_m, \cdot)$ can be expressed as a polynomial homogeneous of degree $n-1$:

$$\rho^{n-1}(\lambda_1 L_1 \,\tilde{+}\, \ldots \,\tilde{+}\, \lambda_m L_m, \cdot) = \sum_{i_1, \ldots, i_{n-1}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \rho(L_{i_1}, \cdot) \cdots \rho(L_{i_{n-1}}, \cdot). \tag{3.5}$$

The most general inequality for dual mixed volumes is the dual Aleksandrov Fenchel inequality: If $L_1, \ldots, L_n \in \mathcal{S}^n$ and $1 \leq m \leq n$, then

$$\tilde{V}(L_1, \ldots, L_n)^m \leq \prod_{j=1}^{m} \tilde{V}(L_j, \ldots, L_j, L_{m+1}, \ldots, L_n), \tag{3.6}$$

with equality if and only if $L_1, \ldots, L_m$ are dilates. A special case of inequality (3.6) is the dual Minkowski inequality: If $K, L \in \mathcal{S}^n$, then

$$\tilde{V}_1(K, L) \leq V(K)^{n-1} V(L), \tag{3.7}$$

with equality if and only if K and L are dilates. A more general version of the dual Minkowski inequality is: If $0 \leq i \leq n-2$, then

$$\tilde{W}_i(K, L)^{n-i} \leq \tilde{W}_i(K)^{n-i-1} \tilde{W}_i(L), \tag{3.8}$$

with equality if and only if K and L are dilates.

We will also need the following Minkowski type inequality: If $K, L \in \mathcal{S}^n$, then

$$\tilde{V}_{-1}(K, L)^n \geq V(K)^{n+1} V(L)^{-1}, \tag{3.9}$$

with equality if and only if K and L are dilates.

A consequence of the dual Minkowski inequality is the dual Brunn Minkowski inequality: If $K, L \in \mathcal{S}^n$, then

$$V(K \,\tilde{+}\, L)^{1/n} \leq V(K)^{1/n} + V(L)^{1/n}, \tag{3.10}$$

with equality if and only if K and L are dilates. Using Minkowski's integral inequality, this can be further generalized: If $0 \leq i \leq n-2$, then

$$\tilde{W}_i(K \,\tilde{+}\, L)^{1/(n-i)} \leq \tilde{W}_i(K)^{1/(n-i)} + \tilde{W}_i(L)^{1/(n-i)}, \tag{3.11}$$

with equality if and only if K and L are dilates. If $0 \leq i \leq n-2$, $K, L, L_1, \ldots, L_i \in \mathcal{S}^n$ and $\mathbf{C} = (L_1, ..., L_i)$, then

$$\tilde{V}_i(K \,\tilde{+}\, L, \mathbf{C})^{1/(n-i)} \leq \tilde{V}_i(K, \mathbf{C})^{1/(n-i)} + \tilde{V}_i(L, \mathbf{C})^{1/(n-i)}, \tag{3.12}$$

with equality if and only if K and L are dilates.

4. Blaschke Minkowski homomorphisms

In [33], [34] Schneider started a systematic investigation of Minkowski endomorphisms, i.e. continuous, rotation intertwining and Minkowski additive maps of convex bodies. Among other results he obtained a complete classification of all such maps in $\mathbb{R}^2$. Kiderlen [15] continued these investigations and extended Schneider's classification to higher dimensions under a weak monotonicity assumption. He also classified all Blaschke endomorphisms, i.e. continuous, rotation intertwining and Blaschke additive maps, in arbitrary dimension. Following the work of Schneider and Kiderlen the author studied Blaschke Minkowski homomorphisms in [37]. The main results established there are a representation theorem for general and a complete classification of all even Blaschke Minkowski homomorphisms. These results will form the main ingredients for the proofs of Theorems 1.2 to 1.5 and Theorems $1.3_{\mathbf{p}}$ to $1.5_{\mathbf{p}}$. In order to state them, we introduce further notation.

$SO(n)$ will be equipped with the invariant probability measure. As $SO(n)$ is a compact Lie group, the space $\mathcal{M}(SO(n))$ of finite Borel measures on $SO(n)$ with the weak* topology carries a natural convolution structure. The convolution $\mu * \nu$ of $\mu, \sigma \in \mathcal{M}(SO(n))$ is defined by

$$\int_{SO(n)} f(\vartheta) d(\mu * \sigma)(\vartheta) = \int_{SO(n)} f(\eta\tau) d\mu(\eta) d\sigma(\tau),$$

for every $f \in \mathcal{C}(SO(n))$, the space of continuous functions on $SO(n)$ with the uniform topology. By identifying a continuous function f with the absolute continuous measure with density f, the space $\mathcal{C}(SO(n))$ can be viewed as subspace of $\mathcal{M}(SO(n))$. Thus the convolution on $\mathcal{M}(SO(n))$ induces a convolution on $\mathcal{C}(SO(n))$. Of particular importance for us is the following Lemma, see [12], p.85.

Lemma 4.1 *Let $\mu_m, \mu \in \mathcal{M}(SO(n))$, $m = 1, 2, \dots$ and let $f \in \mathcal{C}(SO(n))$. If $\mu_m \to \mu$ weakly, then $f * \mu_m \to f * \mu$ and $\mu_m * f \to \mu * f$ uniformly.*

Identifying S^{n-1} with the homogeneous space $SO(n)/SO(n-1)$, where $SO(n-1)$ denotes the group of rotations leaving the point $\widehat{e}$ (the pole) of S^{n-1} fixed, leads to a one-to-one correspondence of $\mathcal{C}(S^{n-1})$ and $\mathcal{M}(S^{n-1})$ with right $SO(n-1)$-invariant functions and measures on $SO(n)$, see [12], [37]. Using this correspondence, the convolution structure on $\mathcal{M}(SO(n))$ carries over to $\mathcal{M}(S^{n-1})$.

In particular, the convolution $\mu * f \in \mathcal{C}(S^{n-1})$ of a measure $\mu \in \mathcal{M}(SO(n))$ and a function $f \in \mathcal{C}(S^{n-1})$ is defined by

$$(\mu * f)(u) = \int_{SO(n)} f(\vartheta^{-1} u) d\mu(\vartheta). \tag{4.1}$$

If $f = h(K, \cdot)$ is the support function of a compact, convex set K, we have $f(\vartheta^{-1}u) = h(\vartheta K, u)$ for every $u \in S^{n-1}$. Thus, if $\mu \in \mathcal{M}(SO(n))$ is a nonnegative measure, $\mu * f$ is again the support function of a compact, convex set which can be interpreted as a weighted Minkowski rotation mean of the set K.

An essential role play convolution operators on $\mathcal{C}(S^{n-1})$ and $\mathcal{M}(S^{n-1})$, which are generated by $SO(n-1)$ invariant functions and measures. A measure $\mu \in \mathcal{M}(S^{n-1})$

is called zonal, if $\vartheta\mu = \mu$ for every $\vartheta \in SO(n-1)$, where $\vartheta\mu$ is the image measure under the rotation ϑ. The set of continuous zonal functions on S^{n-1} will be denoted by $\mathcal{C}(S^{n-1}, \widehat{e})$, the definition of $\mathcal{M}(S^{n-1}, \widehat{e})$ is analogous. If $f \in \mathcal{C}(S^{n-1})$, $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ and $\eta \in SO(n)$, we have

$$(f * \mu)(\eta\widehat{e}) = \int_{S^{n-1}} f(\eta u) d\mu(u). \tag{4.2}$$

Note that, if $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$, then, by (4.2), for every $f \in \mathcal{C}(S^{n-1})$,

$$(\vartheta f) * \mu = \vartheta(f * \mu) \tag{4.3}$$

for every $\vartheta \in SO(n)$. Thus, the spherical convolution from the right is a rotation intertwining operator on $\mathcal{C}(S^{n-1})$ and $\mathcal{M}(S^{n-1})$. It is also not difficult to check from (4.2) that the convolution of zonal functions and measures is abelian.

The representation theorem for Blaschke Minkowski homomorphisms is, see [37]:

Theorem 4.2 *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a Blaschke Minkowski homomorphism, then there is a function $g \in \mathcal{C}(S^{n-1}, \widehat{e})$ such that*

$$h(\Phi K, \cdot) = S_{n-1}(K, \cdot) * g. \tag{4.4}$$

The function g is unique up to addition of a function of the form $u \mapsto x \cdot u, x \in \mathbb{R}^n$.

We call a compact, convex set $F \subseteq \mathbb{R}^n$ a figure of revolution if F is invariant under rotations of $SO(n-1)$. A further investigation of properties of generating functions of Blaschke Minkowski homomorphisms in [37] led to the following classification of even Blaschke Minkowski homomorphisms:

Theorem 4.3 *A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is an even Blaschke Minkowski homomorphism if and only if there is a centrally symmetric figure of revolution $F \subseteq \mathbb{R}^n$, which is not a singleton, such that*

$$h(\Phi K, \cdot) = S_{n-1}(K, \cdot) * h(F, \cdot).$$

The set F is unique up to translations.

The projection body operator $\Pi : \mathcal{K}^n \to \mathcal{K}^n$ is an even Blaschke Minkowski homomorphism. Its generating figure of revolution is a dilate of the segment $[-\widehat{e}, \widehat{e}]$:

$$h(\Pi K, \cdot) = \frac{1}{2} S_{n-1}(K, \cdot) * h([-\widehat{e}, \widehat{e}], \cdot). \tag{4.5}$$

The operator Π maps polytopes to finite Minkowski linear combinations of rotated and dilated copies of the line segment $[-\widehat{e}, \widehat{e}]$. A general convex body K is mapped to a limit of such Minkowski sums of line segments.

Another well known example of an even Blaschke Minkowski homomorphism is provided by the sine transform of the surface area measure of a convex body K, see [13], [32]: Define an operator $\Theta : \mathcal{K}^n \to \mathcal{K}^n$ by

$$h(\Theta K, \cdot) = S_{n-1}(K, \cdot) * h(B \cap \widehat{e}^{\perp}, \cdot).$$

Then Θ is an even Blaschke Minkowski homomorphism whose images are (limits of) Minkowski sums of rotated and dilated copies of the disc $B \cap \widehat{e}^{\perp}$. The value $h(\Theta K, u)$ is up to a factor the integrated surface area of parallel hyperplane sections of K in the direction u.

Every map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ of the form $h(\Phi K, \cdot) = S_{n-1}(K, \cdot) * h(L, \cdot)$, for some figure of revolution L, is by (4.1) a Blaschke Minkowski homomorphism, but in general there are generating functions g of Blaschke Minkowski homomorphisms that are not support functions. An example of such a map is the (normalized) second mean section operator M_2 introduced in [10] and further investigated in [13]: Let $\mathcal{E}_2^n$ be the affine Grassmanian of two-dimensional planes in $\mathbb{R}^n$ and μ_2 its motion invariant measure, normalized such that $\mu_2(\{E \in \mathcal{E}_2^n : E \cap B^n \neq \varnothing\}) = \kappa_{n-2}$. Then

$$h(M_2 K, \cdot) = (n-1) \int_{\mathcal{E}_2^n} h(K \cap E, \cdot) d\mu_2(E) - h(\{z_{n-1}(K)\}, \cdot), \tag{4.6}$$

where $z_{n-1}(K)$ is the $(n-1)$st intrinsic moment vector of K, see [35], p.304.

An immediate consequence of Theorem 4.2 is Theorem 1.2. Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a Blaschke Minkowski homomorphism with generating function $g \in \mathcal{C}(S^{n-1}, \widehat{e})$. If we define an operator

$$\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n,$$

by

$$h(\Phi(K_1, \ldots, K_{n-1}), \cdot) = S(K_1, \ldots, K_{n-1}, \cdot) * g, \tag{4.7}$$

then (2.3) and the linearity of convolution imply (1.1). The mixed operator Φ is well defined as, by Minkowski's existence theorem, the mixed surface area measure $S(K_1, \ldots, K_{n-1}, \cdot)$ is the surface area measure (of order $n-1$) of a convex body $[K_1, \ldots, K_{n-1}]$, see [24], and thus $\Phi(K_1, \ldots, K_{n-1}) = \Phi[K_1, \ldots, K_{n-1}]$.

By Lemma 4.1 and the weak continuity of mixed surface area measures, see [35], p.276, the mixed operators defined by (4.7) are continuous and symmetric. Moreover, they have the following properties which are immediate consequences of the corresponding properties of mixed surface area measures and the convolution representation (4.7):

(i) They are multilinear with respect to Minkowski linear combinations.

(ii) Their diagonal form reduces to the Blaschke Minkowski homomorphism:

$$\Phi(K, \ldots, K) = \Phi K.$$

(iii) They intertwine simultaneous rotations, i.e. if $\vartheta \in SO(n)$, then

$$\Phi(\vartheta K_1, \ldots, \vartheta K_{n-1}) = \vartheta \Phi(K_1, \ldots, K_{n-1}).$$

For $K, L \in \mathcal{K}^n$, let $\Phi_i(K, L)$ denote the mixed operator $\Phi(K, \ldots, K, L, \ldots, L)$, with i copies of L and $n-i-1$ copies of K. For the body $\Phi_i(K, B)$ we simply write $\Phi_i K$.

The Steiner point map $s : \mathcal{K}^n \to \mathbb{R}^n$ is defined by

$$h(\{s(K)\}, \cdot) = nh(K, \cdot) * (\widehat{e} \cdot .\,).$$

The map s is the unique vector valued continuous, rigid motion intertwining and Minkowski additive map on $\mathcal{K}^n$. From the fact that $S(K_1, \ldots, K_{n-1}, \cdot) * (\widehat{e} \cdot .\,) = 0$ for $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, see [35], p.281, we obtain by (4.7) and the commutativity of zonal convolution

$$h(\{s(\Phi(K_1, \ldots, K_{n-1}))\}, \cdot) = nS(K_1, \ldots, K_{n-1}, \cdot) * (\widehat{e} \cdot .\,) * g = 0.$$

Hence,

$$s(\Phi(K_1, \ldots, K_{n-1})) = o. \tag{4.8}$$

Since $s(\Phi(K_1, \ldots, K_{n-1})) \in \operatorname{int} \Phi(K_1, \ldots, K_{n-1})$, see [35], p.43, we see that the convex body $\Phi(K_1, \ldots, K_{n-1})$ contains the origin in its interior. Thus, the polar body $\Phi^*(K_1, \ldots, K_{n-1})$, in particular $\Phi^* K$, is well defined.

For $K \in \mathcal{K}^n$ containing the origin in its interior, we have the relation $h(K, \cdot) = \rho^{-1}(K^*, \cdot)$. Thus, by (4.7), we obtain for the polar of a mixed Blaschke Minkowski homomorphism Φ with generating function $g \in \mathcal{C}(S^{n-1}, \widehat{e})$ the representation

$$\rho^{-1}(\Phi^*(K_1, \ldots, K_{n-1}), \cdot) = S(K_1, \ldots, K_{n-1}, \cdot) * g. \tag{4.9}$$

5. Radial Blaschke Minkowski homomorphisms

In the last section we collected the representation theorems on Blaschke Minkowski homomorphisms that are critical in the proofs of Theorems 1.2 to 1.5 and Theorems $1.3_{\mathbf{p}}$ to $1.5_{\mathbf{p}}$. In the following we will show that there is a corresponding characterization of radial Blaschke Minkowski homomorphisms that will be needed to prove the dual Theorems $1.3_{\mathbf{d}}$, $1.4_{\mathbf{d}}$ and $1.5_{\mathbf{d}}$.

We call a map $\Psi : \mathcal{C}(S^{n-1}) \to \mathcal{C}(S^{n-1})$ monotone, if nonnegative functions are mapped to nonnegative ones. The following theorem is a slight variation of a result by Dunkl [5]:

Theorem 5.1 *A map $\Psi : \mathcal{C}(S^{n-1}) \to \mathcal{C}(S^{n-1})$ is a monotone, linear map that intertwines rotations if and only if there is a measure $\mu \in \mathcal{M}_+(S^{n-1}, \widehat{e})$ such that*

$$\Psi f = f * \mu. \tag{5.1}$$

Proof: From the definition of spherical convolution and (4.3), it follows that mappings of the form (5.1) have the desired properties.

Conversely, let Ψ be monotone, linear and rotation intertwining. Consider the map $\psi : \mathcal{C}(S^{n-1}) \to \mathbb{R}$, $f \mapsto \Psi f(\widehat{e})$. By the properties of Ψ, the functional ψ is positive and linear on $\mathcal{C}(S^{n-1})$, thus, by the Riesz representation theorem, there is a measure $\mu \in \mathcal{M}_+(S^{n-1})$ such that

$$\psi(f) = \int_{S^{n-1}} f(u) d\mu(u).$$

Since ψ is $SO(n-1)$ invariant, the measure μ is zonal. Thus, we have for $\eta \in SO(n)$

$$\Psi f(\eta \widehat{e}) = \Psi(\eta^{-1} f)(\widehat{e}) = \psi(\eta^{-1} f) = \int_{S^{n-1}} f(\eta u) d\mu(u).$$

The theorem follows now from (4.2). ■

The following consequence of Theorem 5.1 is a dual version of Theorem 4.2:

Theorem 5.2 *A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is a radial Blaschke Minkowski homomorphism if and only if there is a nonnegative measure $\mu \in \mathcal{M}_+(S^{n-1}, \widehat{e})$ such that*

$$\rho(\Psi L, \cdot) = \rho^{n-1}(L, \cdot) * \mu. \tag{5.2}$$

Proof: From Lemma 4.1, (4.3) and the properties of spherical convolution, it is clear that mappings of the form of (5.2) are radial Blaschke Minkowski homomorphisms. Thus, we have to show that for every such operator Ψ, there is a measure $\mu \in \mathcal{M}^+(S^{n-1}, \widehat{e})$ such that (5.2) holds.

Since every positive continuous function on S^{n-1} is a radial function, the vector space $\{\rho^{n-1}(K, \cdot) - \rho^{n-1}(L, \cdot) : K, L \in \mathcal{S}^n\}$ coincides with $\mathcal{C}(S^{n-1})$. The operator $\bar{\Psi} : \mathcal{C}(S^{n-1}) \to \mathcal{C}(S^{n-1})$ defined by

$$\bar{\Psi} f = \rho(\Psi L_1, \cdot) - \rho(\Psi L_2, \cdot),$$

where $f = \rho^{n-1}(L_1, \cdot) - \rho^{n-1}(L_2, \cdot)$, is a linear extension of Ψ to $\mathcal{C}(S^{n-1})$ that intertwines rotations. Since the cone of radial functions is invariant under $\bar{\Psi}$, it is also monotone. Hence, by Theorem 5.1, there is a nonnegative measure $\mu \in \mathcal{M}_+(S^{n-1}, \widehat{e})$ such that $\bar{\Psi} f = f * \mu$. The statement now follows from $\bar{\Psi}\rho^{n-1}(L, \cdot) = \rho(\Psi L, \cdot)$. ■

The generating measure of the intersection body operator $I : \mathcal{S}^n \to \mathcal{S}^n$ is the invariant measure $\mu_{S_0^{n-2}}$ concentrated on $S_0^{n-2} := S^{n-1} \cap \widehat{e}^{\perp}$ with total mass κ_{n-1}:

$$\rho(IL, \cdot) = \rho^{n-1}(L, \cdot) * \mu_{S_0^{n-2}}.$$

Let $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ be a radial Blaschke Minkowski homomorphism with generating measure $\mu \in \mathcal{M}_+(S^{n-1}, \widehat{e})$ and define a mixed operator $\Psi : \mathcal{S}^n \times \cdots \times \mathcal{S}^n \to \mathcal{S}^n$ by

$$\rho(\Psi(L_1, \ldots, L_{n-1}), \cdot) = \rho(L_1, \cdot) \cdots \rho(L_{n-1}, \cdot) * \mu. \tag{5.3}$$

The mixed radial Blaschke Minkowski homomorphisms defined in this way are symmetric and by Lemma 4.1 continuous. Moreover, Theorem $1.2_{\mathbf{d}}$ is a direct consequence of Theorem 5.2 and (3.5). The properties (ii) and (iii) of mixed Blaschke Minkowski homomorphisms also hold for mixed radial Blaschke Minkowski homomorphisms but property (i) has to be replaced by:

(i)$_{\mathbf{d}}$ They are multilinear with respect to radial Minkowski linear combinations.

For $K, L \in \mathcal{S}^n$, the definitions of $\Psi_i(K, L)$ and $\Psi_i K$ are analogous to the ones for mixed Blaschke Minkowski homomorphisms.

6. Inequalities for Blaschke Minkowski homomorphisms

In this section we will prove Theorems 1.3, 1.4 and 1.5 as well as their polar versions. To this end, let $\Phi : \mathcal{K}^n \rightarrow \mathcal{K}^n$ always denote a Blaschke Minkowski homomorphism with generating function $g \in \mathcal{C}(S^{n-1}, \widehat{e})$. The proofs are based on techniques developed by Lutwak in [27].

It will be convenient to introduce the following notation for the canonical pairing of $f \in \mathcal{C}(S^{n-1})$ and $\mu \in \mathcal{M}(S^{n-1})$

$$\langle \mu, f \rangle = \langle f, \mu \rangle = \int_{S^{n-1}} f(u) d\mu(u).$$

One very useful tool is the following easy Lemma, see [37], p.7:

Lemma 6.1 *Let $\mu, \nu \in \mathcal{M}(S^{n-1})$ and $f \in \mathcal{C}(S^{n-1})$, then*

$$\langle \mu * \nu, f \rangle = \langle \mu, f * \nu \rangle.$$

We summarize geometric consequences of Lemma 6.1 in the following two Lemmas:

Lemma 6.2 *If $K_1, \dots, K_{n-1}, L_1, \dots, L_{n-1} \in \mathcal{K}^n$, then*

$$V(K_1, \dots, K_{n-1}, \Phi(L_1, \dots, L_{n-1})) = V(L_1, \dots, L_{n-1}, \Phi(K_1, \dots, K_{n-1})). \tag{6.1}$$

In particular, for $K, L \in \mathcal{K}^n$ and $0 \leq i, j \leq n-2$,

$$W_i(K, \Phi(L_1, \dots, L_{n-1})) = V(L_1, \dots, L_{n-1}, \Phi_i K) \tag{6.2}$$

and

$$W_i(K, \Phi_j L) = W_j(L, \Phi_i K). \tag{6.3}$$

Proof: By (2.1), we have

$$V(K_1, \dots, K_{n-1}, \Phi(L_1, \dots, L_{n-1})) = \langle h(\Phi(L_1, \dots, L_{n-1}), \cdot), S(K_1, \dots, K_{n-1}) \rangle.$$

Hence, identity (6.1) follows from (4.7) and Lemma 6.1.

For $K_1 = \ldots = K_{n-i-1} = K$ and $K_{n-i} = \ldots = K_{n-1} = B$, identity (6.1) reduces to (6.2). Finally put $L_1 = \ldots = L_{n-j-1} = L$ and $L_{n-j} = \ldots = L_{n-1} = B$ in (6.2), to obtain identity (6.3). ■

In the next Lemma we summarize further special cases of identity (6.1). These make use of the fact that the image of a ball under a Blaschke Minkowski homomorphism is again a ball. To see this, note that $dS_{n-1}(B, v) = dv$, where dv is the ordinary spherical Lebesgue measure. Thus, by Theorem 4.2,

$$h(\Phi B, u) = (S_{n-1}(B, \cdot) * g)(u) = \int_{S^{n-1}} g(v) dv =: r_\Phi.$$

So let r_Φ denote the radius of the ball ΦB.

Lemma 6.3 *If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then*

$$W_{n-1}(\Phi(K_1, \ldots, K_{n-1})) = r_\Phi V(K_1, \ldots, K_{n-1}, B). \tag{6.4}$$

In particular, for $K, L \in \mathcal{K}^n$,

$$W_{n-1}(\Phi_1(K, L)) = r_\Phi W_1(K, L), \tag{6.5}$$

and, for $0 \leq i \leq n-2$,

$$W_{n-1}(\Phi_i K) = r_\Phi W_{i+1}(K). \tag{6.6}$$

Lemma 6.2 is the critical tool in the proofs of the inequalities of Theorems 1.3, 1.4 and 1.5 without the equality conditions, compare Lutwak [24]. Lemma 6.3 will be needed to settle the cases of equality. In fact more general inequalities can be proved. The following result is a generalization of Theorem 1.3:

Theorem 6.4 *If $K, L \in \mathcal{K}^n$ and $0 \leq i \leq n-1$, then*

$$W_i(\Phi_1(K, L))^{n-1} \geq W_i(\Phi K)^{n-2} W_i(\Phi L), \tag{6.7}$$

with equality if and only if K and L are homothetic.

Proof: By (6.5) and (6.6), the case $i = n-1$ follows from inequality (2.6). Let therefore $0 \leq i \leq n-2$ and $Q \in \mathcal{K}^n$. By (6.2) and (2.4),

$$\begin{aligned} W_i(Q, \Phi_1(K, L))^{n-1} &= V(K, \ldots, K, L, \Phi_i Q)^{n-1} \geq V_1(K, \Phi_i Q)^{n-2} V_1(L, \Phi_i Q) \\ &= W_i(Q, \Phi K)^{n-2} W_i(Q, \Phi L). \end{aligned}$$

Inequality (2.6) implies

$$W_i(Q, \Phi K)^{(n-2)(n-i)} W_i(Q, \Phi L)^{n-i} \geq W_i(Q)^{(n-1)(n-i-1)} W_i(\Phi K)^{n-2} W_i(\Phi L)$$

and thus,

$$W_i(Q, \Phi_1(K, L))^{(n-1)(n-i)} \geq W_i(Q)^{(n-1)(n-i-1)} W_i(\Phi K)^{n-2} W_i(\Phi L), \tag{6.8}$$

with equality if and only if $Q, \Phi K$ and ΦL are homothetic. Setting $Q = \Phi_1(K, L)$, we obtain the desired inequality. If there is equality in (6.7), we have equality in (6.8). From the fact that the Steiner point of mixed Blaschke Minkowski homomorphisms is the origin, compare (4.8), it follows that there exist $\lambda_1, \lambda_2 > 0$ such that

$$\Phi_1(K, L) = \lambda_1 \Phi K = \lambda_2 \Phi L. \tag{6.9}$$

From the equality in (6.7), it follows that

$$\lambda_1^{n-2} \lambda_2 = 1.$$

Moreover, (6.5), (6.6) and (6.9) give

$$W_1(K, L) = \lambda_1 W_1(K) = \lambda_2 W_1(L).$$

Hence, we have

$$W_1(K,L)^{n-1} = W_1(K)^{n-2}W_1(L),$$

which implies, by (2.6), that K and L are homothetic. ■

Of course, Theorem 1.3 is the special case $i = 0$ of Theorem 6.4.

Much more general then the Minkowski inequality is the Aleksandrov Fenchel inequality for mixed operators:

Theorem 6.5 *If $K_1, \dots, K_{n-1} \in \mathcal{K}^n$ and $1 \leq m \leq n-1$, then*

$$W_i(\Phi(K_1,\dots,K_{n-1}))^m \geq \prod_{j=1}^{m} W_i(\Phi(\underbrace{K_j,\dots,K_j}_{m}, K_{m+1},\dots,K_{n-1})).$$

Proof: The case $i = n-1$ reduces by (6.4) to inequality (2.4). Hence, we can assume $i \leq n-2$. From (6.2) and (2.4), it follows that for $Q \in \mathcal{K}^n$,

$$\begin{aligned} W_i(Q,\Phi(K_1,\dots,K_{n-1}))^m &= V(K_1,\dots,K_{n-1},\Phi_i Q)^m \\ &\geq \prod_{j=1}^{m} V(K_j,\dots,K_j,K_{m+1},\dots,K_{n-1},\Phi_i Q) \\ &= \prod_{j=1}^{m} W_i(Q,\Phi(K_j,\dots,K_j,K_{m+1},\dots,K_{n-1})). \end{aligned}$$

Write $\Phi_{m'}(K_j,\mathbf{C})$ for the mixed operator $\Phi(K_j,\dots,K_j,K_{m+1},\dots,K_{n-1})$. Then, by inequality (2.6), we have

$$W_i(Q,\Phi_{m'}(K_j,\mathbf{C}))^{n-i} \geq W_i(Q)^{n-i-1}W_i(\Phi_{m'}(K_j,\mathbf{C})).$$

Hence, we obtain

$$W_i(Q,\Phi(K_1,\dots,K_{n-1}))^{m(n-i)} \geq W_i(Q)^{m(n-i-1)} \prod_{j=1}^{m} W_i(\Phi_{m'}(K_j,\mathbf{C})).$$

By setting $Q = \Phi(K_1,\dots,K_{n-1})$, this becomes the desired inequality. ■

Theorem 1.4 is the special case $m = 2$ and $i = 0$ of Theorem 6.5. If we combine the special case $m = n-2$ of Theorem 6.5 and Theorem 6.4 we obtain:

Corollary 6.6 *If $K_1, \dots, K_{n-1} \in \mathcal{K}^n$ and $0 \leq i \leq n-1$, then*

$$W_i(\Phi(K_1,\dots,K_{n-1}))^{n-1} \geq W_i(\Phi K_1)\cdots W_i(\Phi K_{n-1}),$$

with equality if and only if the K_j are homothetic.

The special case $K_1 = \dots = K_{n-1-j} = K$ and $K_{n-j} = \dots = K_{n-1} = L$ of Corollary 6.6 leads to a further generalization of Theorem 1.3:

Corollary 6.7 *If $K, L \in \mathcal{K}^n$ and $0 \le i \le n-1$, $1 \le j \le n-2$, then*

$$W_i(\Phi_j(K,L))^{n-1} \ge W_i(\Phi K)^{n-j-1} W_i(\Phi L)^j,$$

with equality if and only if K and L are homothetic.

The following theorem provides a general Brunn Minkowski inequality for the operators Φ_j.

Theorem 6.8 *If $K, L \in \mathcal{K}^n$ and $0 \le i \le n-1$, $0 \le j \le n-3$, then*

$$W_i(\Phi_j(K+L))^{1/(n-i)(n-j-1)} \ge W_i(\Phi_j K)^{1/(n-i)(n-j-1)} + W_i(\Phi_j L)^{1/(n-i)(n-j-1)}, \quad (6.10)$$

with equality if and only if K and L are homothetic.

Proof: By (6.3) and (2.9), we have for $Q \in \mathcal{K}^n$,

$$\begin{aligned} W_i(Q, \Phi_j(K+L))^{1/(n-j-1)} &= W_j(K+L, \Phi_i Q)^{1/(n-j-1)} \\ &\ge W_j(K, \Phi_i Q)^{1/(n-j-1)} + W_j(L, \Phi_i Q)^{1/(n-j-1)} \\ &= W_i(Q, \Phi_j K)^{1/(n-j-1)} + W_i(Q, \Phi_j L)^{1/(n-j-1)}. \end{aligned}$$

By inequality (2.6),

$$W_i(Q, \Phi_j K)^{n-i} \ge W_i(Q)^{n-i-1} W_i(\Phi_j K),$$

with equality if and only if Q and $\Phi_j K$ are homothetic, and

$$W_i(Q, \Phi_j L)^{n-i} \ge W_i(Q)^{n-i-1} W_i(\Phi_j L),$$

with equality if and only if Q and $\Phi_j L$ are homothetic. Thus, we obtain

$$\begin{aligned} &W_i(Q, \Phi_j(K+L))^{1/(n-j-1)} W_i(Q)^{-(n-i-1)/(n-i)(n-j-1)} \\ &\qquad \ge W_i(\Phi_j K)^{1/(n-i)(n-j-1)} + W_i(\Phi_j L)^{1/(n-i)(n-j-1)}, \end{aligned}$$

with equality if and only if Q, $\Phi_j K$ and $\Phi_j L$ are homothetic. If we set $Q = \Phi_j(K+L)$, we obtain (6.10). If there is equality in (6.10), then, by (4.8), there exist $\lambda_1, \lambda_2 > 0$ such that

$$\Phi_j K = \lambda_1 \Phi_j(K+L) \qquad \text{and} \qquad \Phi_j L = \lambda_2 \Phi_j(K+L). \quad (6.11)$$

From equality in (6.10), it follows that

$$\lambda_1^{1/(n-j-1)} + \lambda_2^{1/(n-j-1)} = 1.$$

Moreover, (6.6) and (6.11) imply

$$W_{j+1}(K) = \lambda_1 W_{j+1}(K+L) \qquad \text{and} \qquad W_{j+1}(L) = \lambda_2 W_{j+1}(K+L).$$

Hence, we have

$$W_{j+1}(K+L)^{1/(n-j-1)} = W_{j+1}(K)^{1/(n-j-1)} + W_{j+1}(L)^{1/(n-j-1)},$$

which implies, by (2.8), that K and L are homothetic. ■

We turn now to the proofs of Theorems $1.3_{\mathbf{p}}$, $1.4_{\mathbf{p}}$ and $1.5_{\mathbf{p}}$. To this end, we will *restrict* ourselves to Blaschke Minkowski homomorphisms Φ with a generating function of the form $g = h(F, \cdot)$, where $F \subseteq \mathbb{R}^n$ is a figure of revolution which is not a singleton. Note that, by (4.1), every function of that form is generating function of a Blaschke Minkowski homomorphism. In particular, by Theorem 4.3, every even Blaschke Minkowski homomorphism has a generating function of that type.

We now associate with each such Blaschke Minkowski homomorphism Φ a new operator $M_\Phi : \mathcal{S}^n \to \mathcal{K}^n$, defined by

$$h(M_\Phi L, \cdot) = \rho^{n+1}(L, \cdot) * h(F, \cdot). \qquad (6.12)$$

By (4.1), the operator M_Φ is well defined. Note that M_Φ depends, in contrast to Φ, on the position of F but that by Theorem 4.2, we may assume that $s(F) = o$. In this way we associate to each Blaschke Minkowski homomorphism a unique operator M_Φ.

The next lemma will play the role of Lemma 6.2.

Lemma $6.2_{\mathbf{p}}$ *If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$ and $L \in \mathcal{S}^n$, then*

$$\tilde{V}_{-1}(L, \Phi^*(K_1, \ldots, K_{n-1})) = V(K_1, \ldots, K_{n-1}, M_\Phi L). \qquad (6.13)$$

In particular, for $K \in \mathcal{K}^n$,

$$\tilde{V}_{-1}(L, \Phi_i^* K) = W_i(K, M_\Phi L). \qquad (6.14)$$

Proof: By (3.3), we have

$$\tilde{V}_{-1}(K, \Phi^*(K_1, \ldots, K_{n-1})) = \langle \rho^{n+1}(K, \cdot), \rho^{-1}(\Phi^*(K_1, \ldots, K_{n-1}), \cdot) \rangle.$$

Hence, identity (6.13) follows from (4.9) and Lemma 6.1. For $K_1 = \ldots = K_{n-i-1} = K$ and $K_{n-i} = \ldots = K_{n-1} = B$, identity (6.13) reduces to (6.14). ■

We now immediately get the following Minkowski type inequality for the volume of polar Blaschke Minkowski homomorphisms Φ with a generating function of the form $g = h(F, \cdot)$. This, in particular, proves Theorem $1.3_{\mathbf{p}}$:

Theorem $6.4_{\mathbf{p}}$ *If $K, L \in \mathcal{K}^n$, then*

$$V(\Phi_1^*(K, L))^{n-1} \leq V(\Phi^* K)^{n-2} V(\Phi^* L), \qquad (6.15)$$

with equality if and only if K and L are homothetic.

Proof: Let $Q \in \mathcal{S}^n$. Then, by (6.13) and (2.4),

$$\begin{aligned} \tilde{V}_{-1}(Q, \Phi_1^*(K, L))^{n-1} &= V(K, \ldots, K, L, M_\Phi Q)^{n-1} \geq V_1(K, M_\Phi Q)^{n-2} V_1(L, M_\Phi Q) \\ &= \tilde{V}_{-1}(Q, \Phi^* K)^{n-2} \tilde{V}_{-1}(Q, \Phi^* L). \end{aligned}$$

By inequality (3.9), we have

$$\tilde{V}_{-1}(Q, \Phi^* K)^{(n-2)n} \tilde{V}_{-1}(Q, \Phi^* L)^n \geq V(Q)^{(n+1)(n-1)} V(\Phi^* K)^{-(n-2)} V(\Phi^* L)^{-1},$$

and thus,

$$\tilde{V}_{-1}(Q, \Phi_1^*(K,L))^{(n-1)n} \geq V(Q)^{(n+1)(n-1)} V(\Phi^* K)^{-(n-2)} V(\Phi^* L)^{-1},$$

with equality if and only if $Q, \Phi^* K$ and $\Phi^* L$ are dilates. Setting $Q = \Phi_1^*(K,L)$, we obtain the desired inequality. If there is equality in (6.15), then there exist $\lambda_1, \lambda_2 > 0$ such that

$$\Phi_1^*(K,L) = \lambda_1 \Phi^* K = \lambda_2 \Phi^* L. \tag{6.16}$$

For every convex body $K \in \mathcal{K}^n$ containing the origin and for every $\lambda > 0$, we have $(\lambda K)^* = \lambda^{-1} K^*$, and thus

$$\Phi_1(K,L) = \lambda_1^{-1} \Phi K = \lambda_2^{-1} \Phi L.$$

From the equality in (6.15), it follows that

$$\lambda_1^{-(n-2)} \lambda_2^{-1} = 1.$$

By (6.5), (6.6) and (6.16), we obtain

$$W_1(K,L) = \lambda_1^{-1} W_1(K) = \lambda_2^{-1} W_1(L).$$

Hence, we have

$$W_1(K,L)^{n-1} = W_1(K)^{n-2} W_1(L),$$

which implies, by (2.6), that K and L are homothetic. ■

Theorem 6.5$_{\mathbf{p}}$ *If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$ and $1 \leq m \leq n-1$, then*

$$V(\Phi^*(K_1, \ldots, K_{n-1}))^m \leq \prod_{j=1}^m V(\Phi^*(\underbrace{K_j, \ldots, K_j}_{m}, K_{m+1}, \ldots, K_{n-1})).$$

Proof: From (6.13), it follows that for $Q \in \mathcal{S}^n$,

$$\begin{aligned} \tilde{V}_{-1}(Q, \Phi^*(K_1, \ldots, K_{n-1}))^m &= V(K_1, \ldots, K_{n-1}, M_\Phi Q)^m \\ &\geq \prod_{j=1}^m V(K_j, \ldots, K_j, K_{m+1}, \ldots, K_{n-1}, M_\Phi Q) \\ &= \prod_{j=1}^m \tilde{V}_{-1}(Q, \Phi^*(K_j, \ldots, K_j, K_{m+1}, \ldots, K_{n-1})). \end{aligned}$$

Write $\Phi_{m'}^*(K_j, \mathbf{C})$ for the mixed operator $\Phi^*(K_j, \ldots, K_j, K_{m+1}, \ldots, K_{n-1})$. Then, by inequality (3.9), we have

$$\tilde{V}_{-1}(Q, \Phi_{m'}^*(K_j, \mathbf{C}))^n \geq V(Q)^{n+1} V(\Phi_{m'}^*(K_j, \mathbf{C}))^{-1}.$$

Hence, we obtain

$$V(Q, \Phi^*(K_1, \ldots, K_{n-1}))^{mn} \geq V(Q)^{m(n+1)} \prod_{j=1}^m V(\Phi_{m'}^*(K_j, \mathbf{C}))^{-1}.$$

Setting $Q = \Phi^*(K_1, \ldots, K_{n-1})$, this becomes the desired inequality. ■

Theorem $1.3_{\mathbf{p}}$ is the special case $m = 2$ of Theorem 6.5 for even Blaschke Minkowski homomorphisms. Combine the special case $m = n-2$ of Theorem $6.5_{\mathbf{p}}$ and Theorem $6.4_{\mathbf{p}}$, to obtain:

Corollary $6.6_{\mathbf{p}}$ *If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then*

$$V(\Phi^*(K_1, \ldots, K_{n-1}))^{n-1} \leq V(\Phi^* K_1) \cdots V(\Phi^* K_{n-1}),$$

with equality if and only if the K_j are homothetic.

The special case, $K_1 = \ldots = K_{n-1-j} = K$ and $K_{n-j} = \ldots = K_{n-1} = L$, of Corollary $6.6_{\mathbf{p}}$ leads to a generalization of Theorem $6.4_{\mathbf{p}}$:

Corollary $6.7_{\mathbf{p}}$ *If $K, L \in \mathcal{K}^n$ and $1 \leq j \leq n-2$, then*

$$V(\Phi_j^*(K, L))^{n-1} \leq V(\Phi^* K)^{n-j-1} V(\Phi^* L)^j,$$

with equality if and only if K and L are homothetic.

The last theorem in this section provides a Brunn Minkowski inequality for the volume of the polar Blaschke Minkowski homomorphisms under consideration:

Theorem $6.8_{\mathbf{p}}$ *If $K, L \in \mathcal{K}^n$ and $0 \leq j \leq n-3$, then*

$$V(\Phi_j^*(K+L))^{-1/n(n-j-1)} \geq V(\Phi_j^* K)^{-1/n(n-j-1)} + V(\Phi_j^* L)^{-1/n(n-j-1)}, \qquad (6.17)$$

with equality if and only if K and L are homothetic.

Proof: By (6.14) and (2.9), we have for $Q \in \mathcal{S}^n$,

$$\begin{aligned} \tilde{V}_{-1}(Q, \Phi_j^*(K+L))^{1/(n-j-1)} &= W_j(K+L, M_\Phi Q)^{1/(n-j-1)} \\ &\geq W_j(K, M_\Phi Q)^{1/(n-j-1)} + W_j(L, M_\Phi Q)^{1/(n-j-1)} \\ &= \tilde{V}_{-1}(Q, \Phi_j^* K)^{1/(n-j-1)} + \tilde{V}_{-1}(Q, \Phi_j^* L)^{1/(n-j-1)}. \end{aligned}$$

By inequality (3.9),

$$\tilde{V}_{-1}(Q, \Phi_j^* K)^n \geq V(Q)^{n+1} V(\Phi_j^* K)^{-1},$$

with equality if and only if Q and $\Phi_j^* K$ are dilates, and

$$\tilde{V}_{-1}(Q, \Phi_j^* L)^n \geq V(Q)^{n+1} V(\Phi_j^* L)^{-1},$$

with equality if and only if Q and $\Phi_j^* L$ are dilates. Thus, we obtain

$$\begin{aligned} &\tilde{V}_{-1}(Q, \Phi_j^*(K+L))^{1/(n-j-1)} V(Q)^{-(n+1)/n(n-j-1)} \\ &\qquad \geq V(\Phi_j^* K)^{-1/n(n-j-1)} + V(\Phi_j^* L)^{-1/n(n-j-1)}, \end{aligned}$$

with equality if and only if $Q, \Phi_j K$ and $\Phi_j L$ are dilates. If we set $Q = \Phi_j^*(K+L)$, we obtain (6.17). Suppose equality holds in (6.17), then there exist $\lambda_1, \lambda_2 > 0$ such that

$$\Phi_j^* K = \lambda_1 \Phi_j^*(K+L) \qquad \text{and} \qquad \Phi_j^* L = \lambda_2 \Phi_j^*(K+L),$$

and thus,

$$\Phi_j K = \lambda_1^{-1}\Phi_j(K+L) \qquad \text{and} \qquad \Phi_j L = \lambda_2^{-1}\Phi_j(K+L). \tag{6.18}$$

From the equality in (6.17), it follows that

$$\lambda_1^{-1/(n-j-1)} + \lambda_2^{-1/(n-j-1)} = 1,$$

and (6.6) and (6.18) imply

$$W_{j+1}(K) = \lambda_1^{-1} W_{j+1}(K+L) \qquad \text{and} \qquad W_{j+1}(L) = \lambda_2^{-1} W_{j+1}(K+L).$$

Hence, we have

$$W_{j+1}(K+L)^{1/(n-j-1)} = W_{j+1}(K)^{1/(n-j-1)} + W_{j+1}(L)^{1/(n-j-1)},$$

which implies, by (2.8), that K and L are homothetic. ■

7. Inequalities for radial Blaschke Minkowski homomorphisms

The main tools in the proofs of Theorems 1.3, 1.4 and 1.5 are Lemmas 6.2 and 6.3. These were immediate consequences of the convolution representation of Blaschke Minkowski homomorphisms provided by Theorem 4.2. In Section 5, we have shown that there is a corresponding representation for radial Blaschke Minkowski homomorphisms, which will now lead to dual versions of Lemmas 6.2 and 6.3. In the following let $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ denote a nontrivial radial Blaschke Minkowski homomorphism. In the same way as Lemmas 6.2 and 6.3 were consequences of Theorem 4.2 and Lemma 6.1, we obtain from Theorem 5.2:

Lemma 7.1 *If $K_1, \ldots, K_{n-1}, L_1, \ldots, L_{n-1} \in \mathcal{S}^n$, then*

$$\tilde{V}(K_1, \ldots, K_{n-1}, \Psi(L_1, \ldots, L_{n-1})) = \tilde{V}(L_1, \ldots, L_{n-1}, \Psi(K_1, \ldots, K_{n-1})). \tag{7.1}$$

In particular, for $K, L \in \mathcal{S}^n$ and $0 \le i, j \le n-2$,

$$\tilde{W}_i(K, \Psi(L_1, \ldots, L_{n-1})) = \tilde{V}(L_1, \ldots, L_{n-1}, \Psi_i K) \tag{7.2}$$

and

$$\tilde{W}_i(K, \Psi_j L) = \tilde{W}_j(L, \Psi_i K). \tag{7.3}$$

It follows from Theorem 5.2 that the image of the Euclidean unit ball under a radial Blaschke Minkowski homomorphism Ψ is again a ball. Let r_Ψ denote the radius of this ball. Then the dual version of Lemma 6.3 is:

Lemma 7.2 *If $L_1, \ldots, L_{n-1} \in \mathcal{S}^n$, then*

$$\tilde{W}_{n-1}(\Psi(L_1, \ldots, L_{n-1})) = r_\Psi \tilde{V}(L_1, \ldots, L_{n-1}, B). \tag{7.4}$$

In particular, for $K, L \in \mathcal{S}^n$,

$$\tilde{W}_{n-1}(\Psi_1(K, L)) = r_\Psi \tilde{W}_1(K, L) \tag{7.5}$$

and, for $0 \le i \le n-2$,

$$\tilde{W}_{n-1}(\Psi_i L) = r_\Psi \tilde{W}_{i+1}(L). \tag{7.6}$$

The proofs of Theorems $1.3_\mathbf{d}$, $1.4_\mathbf{d}$ and $1.5_\mathbf{d}$ are now analogous to the proofs of Theorems 1.3, 1.4 and 1.5. We just have to replace Lemmas 6.2 and 6.3 by Lemmas 7.1 and 7.2, and use the inequalities for dual mixed volumes from Section 3 instead of the inequalities for mixed volumes from Section 2. For this reason we will omit all the proofs except one in this section:

Theorem 7.3 *If $L_1, \ldots, L_{n-1} \in \mathcal{S}^n$ and $2 \leq m \leq n-1$, then*

$$\tilde{W}_i(\Psi(L_1, \ldots, L_{n-1}))^m \leq \prod_{j=1}^m \tilde{W}_i(\Psi(\underbrace{L_j, \ldots, L_j}_{m}, L_{m+1}, \ldots, L_{n-1})),$$

with equality if and only if $L_1, \ldots, L_m$ are dilates.

Proof: The case $i = n-1$ reduces by (6.13) to inequality (3.6). Hence, assume $i \leq n-2$. From (7.1), it follows that for $Q \in \mathcal{S}^n$,

$$\begin{aligned}
\tilde{W}_i(Q, \Psi(L_1, \ldots, L_{n-1}))^m &= \tilde{V}(L_1, \ldots, L_{n-1}, \Psi_i Q)^m \\
&\leq \prod_{j=1}^m \tilde{V}(L_j, \ldots, L_j, L_{m+1}, \ldots, L_{n-1}, \Psi_i Q) \\
&= \prod_{j=1}^m \tilde{W}_i(Q, \Psi(L_j, \ldots, L_j, L_{m+1}, \ldots, L_{n-1})),
\end{aligned}$$

with equality if and only if $L_1, \ldots, L_m$ are dilates. Let $\Psi_{m'}(L_j, \mathbf{C})$ denote the body $\Psi(L_j, \ldots, L_j, L_{m+1}, \ldots, L_{n-1})$. Then, by inequality (3.8), we have

$$\tilde{W}_i(Q, \Psi_{m'}(L_j, \mathbf{C}))^{n-i} \leq \tilde{W}_i(Q)^{n-i-1} \tilde{W}_i(\Psi_{m'}(L_j, \mathbf{C})),$$

with equality if and only if Q and $\Psi_{m'}(L_j, \mathbf{C})$ are dilates. Hence,

$$\tilde{W}_i(Q, \Psi^*(L_1, \ldots, L_{n-1}))^{m(n-i)} \leq \tilde{W}_i(Q)^{m(n-i-1)} \prod_{j=1}^m \tilde{W}_i(\Psi_{m'}(L_j, \mathbf{C})).$$

By setting $Q = \Psi(L_1, \ldots, L_{n-1})$, the statement follows. ∎

Theorem $1.3_\mathbf{d}$ and $1.4_\mathbf{d}$ are now just special cases of Theorem 7.3. Further consequences are the dual versions of Corollaries 6.6 and 6.7:

Corollary 7.4 *If $L_1, \ldots, L_{n-1} \in \mathcal{S}^n$ and $0 \leq i \leq n-1$, then*

$$\tilde{W}_i(\Psi(L_1, \ldots, L_{n-1}))^{n-1} \leq \tilde{W}_i(\Psi L_1) \cdots \tilde{W}_i(\Psi L_{n-1}),$$

with equality if and only if the L_j are dilates.

Corollary 7.5 *If $K, L \in \mathcal{S}^n$ and $0 \leq i \leq n-1$, $1 \leq j \leq n-2$, then*

$$\tilde{W}_i(\Psi_j(K, L))^{n-1} \leq \tilde{W}_i(\Psi K)^{n-j-1} \tilde{W}_i(\Psi L)^j,$$

with equality if and only if K and L are dilates.

The dual counterpart of Theorem 6.8 is:

Theorem 7.6 *If $K, L \in \mathcal{S}^n$ and $0 \leq i \leq n-1$, $0 \leq j \leq n-3$, then*

$$\tilde{W}_i(\Psi_j(K+L))^{1/(n-i)(n-j-1)} \leq \tilde{W}_i(\Psi_j K)^{1/(n-i)(n-j-1)} + \tilde{W}_i(\Phi_j L)^{1/(n-i)(n-j-1)}, \quad (7.7)$$

with equality if and only if K and L are dilates.

8. Final remarks

In Theorems $6.4_{\mathbf{p}}$, $6.5_{\mathbf{p}}$ and $6.8_{\mathbf{p}}$, we restrict ourselves to Blaschke Minkowski homomorphisms Φ with generating functions g that are support functions. We did this to ensure that star bodies are mapped to convex bodies by the operators M_Φ defined in (6.12). An example of a Blaschke Minkowski homomorphism whose generating function is not a support function is the second mean section operator M_2, see (4.6). A natural question is whether Theorems $6.4_{\mathbf{p}}$, $6.5_{\mathbf{p}}$ and $6.8_{\mathbf{p}}$ hold for general Blaschke Minkowski homomorphisms.

If Φ is the projection body operator, the map M_Φ becomes a multiple of the moment body operator which is (up to volume normalization) the well known centroid body operator $\Gamma : \mathcal{S}^n \to \mathcal{K}^n$. Centroid bodies were defined and investigated by Petty [30]. They have proven to be an important tool in establishing fundamental affine isoperimetric inequalities, see [8], [26], [29], [31]. The Busemann-Petty centroid inequality, for example, states that

$$V(\Gamma K) \geq \left(\frac{2\kappa_{n-1}}{(n+1)\kappa_n} \right)^n V(K), \tag{8.1}$$

where κ_n is the volume of the Euclidean unit ball in n dimensions. Inequality (8.1) is critical for the proof of Petty's projection inequality

$$V(K)^{n-1} V(\Pi^* K) \leq \left(\frac{\kappa_n}{\kappa_{n-1}} \right)^n. \tag{8.2}$$

It is the author's belief that an inequality corresponding to (8.1) holds for all operators M_Φ. This would immediately provide a generalization of Petty's inequality to general Blaschke Minkowski homomorphisms and would show that the affine invariant inequality (8.2) holds in a more general setting.

Acknowledgements. The work of the author was supported by the Austrian Science Fund (FWF), within the scope of the project "Affinely associated bodies", Project Number: P16547-N12 and the project "Phenomena in high dimensions" of the European Community, Contract Number: MRTN-CT-2004-511953. The author is obliged to Monika Ludwig for her helpful remarks.

References

[1] J. Bourgain and J. Lindenstrauss, *Projection bodies*, Geometric aspects of functional analysis (1986/87), Springer, Berlin (1988), 250-270.

[2] E.D. Bolker, *A class of convex bodies*, Trans. Amer. Math. Soc. **145** (1969), 323-345.

[3] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Springer, Berlin, 1934.

[4] H. Busemann, *Volume in terms of concurrent cross-sections*, Pacific J. Math. **3** (1953), 1-12.

[5] C.F. Dunkl, *Operators and harmonic analysis on the sphere*, Trans. Amer. Math. Soc. **125** (1966), 250-263.

[6] R. Gardner, *Intersection bodies and the Busemann-Petty problem*, Trans. Amer. Math. Soc. **342** (1994), 435-445.

[7] R. Gardner, *A positive answer to the Busemann-Petty problem in three dimensions*, Ann. of Math. (2) **140** (1994), 435-447.

[8] R. Gardner, *Geometric Tomography*, Cambridge University Press, 1995.

[9] R. Gardner, A. Koldobsky and T. Schlumprecht, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, Ann. of Math. (2) **149** (1999), 691-703.

[10] P. Goodey and W. Weil, *The determination of convex bodies from the mean of random sections*, Math. Proc. Camb. Phil. Soc. **112** (1992), 419-430.

[11] P. Goodey and W. Weil, *Zonoids and generalizations*, Handbook of Convex Geometry (P.M. Gruber and J.M. Wills, eds.), North-Holland, Amsterdam, 1993, 1297-1326.

[12] E. Grinberg and G. Zhang, *Convolutions, transforms, and convex bodies*, Proc. London Math. Soc. (3) **78** (1999), 77-115.

[13] D. Hug and R. Schneider, *Stability results involving surface area measures of convex bodies*, Rendiconti Del Circolo Matematica Di Palermo **70** (2002), 21-51.

[14] N. J. Kalton and A. Koldobsky, *Intersection bodies and L_p spaces*, Adv. Math., to appear.

[15] M. Kiderlen, *Blaschke- and Minkowski-Endomorphisms of convex bodies*, Trans. Amer. Math. Soc., to appear.

[16] A. Koldobsky, *Intersection bodies, positive definite distributions, and the Busemann-Petty problem*, Amer. J. Math. **120** (1998), 827-840.

[17] A. Koldobsky, *A functional analytic approach to intersection bodies*, Geom. Funct. Anal. **10** (2000), 1507-1526.

[18] G. Leng and C. Zhao, *Inequalities for dual quermassintegrals of mixed intersection bodies*, Proc. Indian Acad. Sci. **115** (2003), no. 1, 79-91.

[19] G. Leng and C. Zhao, *Brunn-Minkowski inequality for mixed intersection bodies*, J. Math. Anal. Appl. **301** (2005), 115-123.

[20] M. Ludwig, *Projection bodies and valuations*, Adv. Math. **172** (2002), 158-168.

[21] M. Ludwig, *Minkowski valuations*, Trans. Amer. Math. Soc. **357** (2005), no. 10, 4191-4213.

[22] E. Lutwak, *Dual mixed volumes*, Pacific J. Math. **58** (1975), 531-538.

[23] E. Lutwak, *Mixed projection inequalities*, Trans. Amer. Math. Soc. **287** (1985), no. 1, 91-105.

[24] E. Lutwak, *Volume of mixed bodies*, Trans. Amer. Math. Soc. **294** (1986), no. 2, 487-500.

[25] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232-261.

[26] E. Lutwak, *Centroid bodies and dual mixed volumes*, Proc. London Math. Soc. (3) **60** (1990), no. 2, 365-391.

[27] E. Lutwak, *Inequalities for mixed projection bodies*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 901-916.

[28] E. Lutwak, *Selected affine isoperimetric inequalities*, Handbook of Convex Geometry, Vol. A (P.M. Gruber and J.M. Wills, eds.), North-Holland, Amsterdam, 1993, 151-176.

[29] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space*, Geometric aspects of functional analysis (1987-88), Lecture Notes in Math., vol. 1376, Springer, Berlin, 1989, 64-104.

[30] C.M. Petty, *Centroid surfaces*, Pacific J. Math. **11** (1961), 1535-1547.

[31] C.M. Petty, *Isoperimetric problems*, Proc. Conf. on Convexity and Combinatorial Geometry, Univ. of Oklahoma, June 1971 (1972), 26-41.

[32] R. Schneider, *Über eine Integralgleichung in der Theorie der konvexen Körper*, Math. Nachr. **44** (1970), 55-75.

[33] R. Schneider, *Equivariant endomorphisms of the space of convex bodies*, Trans. Amer. Math. Soc. **194** (1974), 53-78.

[34] R. Schneider, *Bewegungsäquivariante, additive und stetige Transformationen konvexer Bereiche*, Arch. Math. **25** (1974), 303-312.

[35] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, 1993.

[36] R. Schneider and W. Weil, *Zonoids and related topics*, Convexity and its applications, Birkhäuser, Basel, 1983, 296-317.

[37] F.E. Schuster, *Convolutions and Multiplier Transformations of Convex Bodies*, Trans. Amer. Math. Soc., to appear.

[38] G. Zhang, *Restricted chord projection and affine inequalities*, Geom. Dedicata **53** (1999), 183-202.

[39] G. Zhang, *Centered bodies and dual mixed volumes*, Trans. Am. Math. Soc. **345** (1994), 777-801.

[40] G. Zhang, *A positive solution to the Busemann-Petty problem in* $\mathbb{R}^4$, Ann. of Math. (2) **149** (1999), 535-543.

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