What is Descriptive Geometry for ?

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This is a pleading for Descriptive Geometry. From the very first, Descriptive Geometry is a method to study 3D geometry through 2D images thus offering insight into structure and metrical properties of spatial objects, processes and principles. The education in Descriptive Geometry provides a training of the students' intellectual capability of space perception. Drawings are the guide to geometry but not the main aim.

1 Introduction

This conference is dedicated to Rudolf BEREIS, who was born exactly 100 years ago. When he started his carreer as a professor here at the Technical University Dresden in 1957, he was full of plans and ideas. All his inspiring activities were dedicated to the promotion of Descriptive Geometry (DG). He planned also a series of textbooks on this subject. However, destiny decided differently; he passed away nine years later. So only the first volume [1] has appeared.

The aim of my presentation is to explain what Descriptive Geometry is good for, a subject, which in the hierarchy of sciences is placed somewhere within or next to the field of Mathematics, but also near to Architecture, Mechanical Engineering, and Engineering Graphics. I start with definitions and continue with a few examples in order to highlight that Descriptive Geometry provides a training of the students' intellectual capability of space perception (note the diagram in [15], Fig. 5) and is therefore of incotestable importance for all engineers, physicians and natural scientists.

2 How to define Descriptive Geometry

In many American textbooks on Engineering Graphics, e.g. [2, 6], the subject Descriptive Geometry seems to be restricted only to standard constructions like the determination of the true length of a line segment or the intersection of two plane polygons in 3-space. From this point of view it must look rather strange that R. BEREIS planned a series of textbooks on this subject.

2.1 Descriptive Geometry in Europe

In order to explain the meaning of 'Descriptive Geometry' in central Europe, let us start with some definitions given in German textbooks, which appeared in the last five decades:

- J. KRAMES defined in [9]:
  "Darstellende Geometrie" ist die Hohe Schule des räumlichen Denkens und der bild-

haften Wiedergabe [in free translation: DG is the high art of spatial reasoning and its graphic representation].
This definition has also been cited by R. BEREIS in [1].

- H. BRAUNER took up a recommendation given by E. KRUPPA and preferred the designation 'Konstruktive Geometrie' [Constructional Geometry] instead of Descriptive Geometry. He defined in [4]:
  'Konstruktive Geometrie' umfaßt das Studium von Objekten des Anschauungsraumes unter Verwendung jener Methode, die an der graphisch darstellten Figur durch Konstruktion und Rechnung operiert ['Constructional Geometry' encompasses the analysis of 3D objects by means of graphical or mathematical methods applied to 2D images].

- F. HOHENBERG, whose textbook [7] focusses on applications of Descriptive Geometry in technology, formulated:
  'Konstruktive Geometrie' soll geometrische Formen und Vorgänge verstehen, vorstellen, gestalten und zeichnen lehren [Constructional Geometry teaches how to grasp, to imagine, to design and to draw geometrical shapes].

- W.-D. KLIX gives in his recent textbook [8] the following extended explanation:
  "Darstellende Geometrie" ist wie kaum ein anderes Lehrgebiet geeignet, das für jede ingenieurmäßige konstruktiv-schöpferische Tätigkeit notwendige räumliche Vorstellungsvermögen zu entwickeln sowie die Fähigkeit auszubilden, räumlich Gedachtes richtig und damit auch anderen verständlich darzustellen [DG is unique in the way how it promotes spatial reasoning, which is so fundamental for each creative activity of engineers, and how it trains the ability to express spatial ideas graphically so that they become understandable for anybody else].

As a consequence, I would like to summarize in the following way.[1]

Definition: *Darstellende Geometrie umfaßt das auf Bilder gestützte Studium von Formen, Vorgängen und Gesetzmäßigkeiten der Raumgeometrie.*
[DG is a method to study 3D geometry through 2D images. It provides insight into structure and metrical properties of spatial objects, processes and principles.]
Charakteristisch für Darstellende Geometrie ist das Wechselspiel
[Typical for DG is the interplay]

  a) *zwischen der räumlichen Situation und deren bildlicher Darstellung*
     [between the 3D situation and its 2D representation],

  b) *zwischen anschaulichem Erfassen und begrifflichem Schließen*
     [between intuitive grasping and rigorous logical reasoning].

According to this, Descriptive Geometry courses in central Europe cover not only projection theory, but also modelling techniques for curves, surfaces, and solids thus offering insight into

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[1]I continue with bilingual versions and emphasize that the German version is the original one. It might happen that my English translation does not express the exact meaning of the German statement.

a broad variety of geometric shapes. Besides, some basic differential-geometric properties of curves and surfaces and some analytic geometry is included ([12, 13]). And one aim is also to develop and to refine on the students' problem-solving skills.

2.2 G. MONGE's definition

Gaspard MONGE (1746–1818) is declared the founder of the science of Descriptive Geometry. This does not mean that he himself developed all the graphical methods. In contrary, most of them can already be found in earlier books, e.g., in those of Amédée François FREZIER.

However, G. MONGE was a most effective scientist and manager who spread his ideas of Descriptive Geometry with the publication of his *'Leçons de géométrie descriptive'* (1799) from France over whole Europe. We find in [10], p. 1, the following introductory statements:

La Géométrie descriptive a deux objets:

- **le premier**, **de donner les méthodes pour représenter** sur une feuille de dessin qui n'a que deux dimensions, savoir, longueur et largeur, tous les corps de la nature qui en ont **trois**, longueur, largeur et profondeur, pourvu néanmoins que ces corps puissent être définis rigoureusement.

- **Le second objet** est de donner **la manière de reconnaître**, d'aprés une **description exacte**, les formes des corps, et d'en **déduire toutes les vérités qui résultent et de leur forme et de leurs positions respectives.**

Figure 1: Statue of G. MONGE, Place de Monge, Beaune (birthplace), Dep. Côte-d'Or, France

This proves that the two main objectives of Descriptive Geometry — imaging and analysing 3D objects — date back to its founder. These two targets can also be found in new encyklopedias like BROCKHAUS [5]:

Darstellende Geometrie, *Teilgebiet der Mathematik, . . . Ziel der DG ist sowohl das Darstellen von dreidimensionalen Gebilden . . . als auch die Interpretation vorliegender Bilder . . .*

[DG = subject of mathematics, . . . The aim of DG is the representation of 3D objects . . . as well as the interpretation of given images . . .]

2.3   The choice of the name

It is remarkable that the word *'drawing'* does not appear in MONGE's definition. In Descriptive Geometry drawing[2] is the *guide to geometry* (compare [14]) but not the main aim; we teach geometry instead of construction techniques. Note that the French **'descriptive'** means 'describing', 'representing' but not necessarily 'graphically depicting'.

Nevertheless, in the public meaning Descriptive Geometry has falsely become synonymic for 'manually drawing images of 3D objects'. As in the last decades manual drawing with traditional instruments has been replaced by CAD or mathematical software with graphic output, 'people on the road' frequently conclude that therefore Descriptive Geometry has become obsolete. However, this is totally wrong: In contrary, only people with a profound knowledge in Descriptive Geometry are able to make extended use of CAD programs as the communication is usually based on views only. The more powerful and sophisticated a modeling software, the higher the required geometric knowledge. A poor designer will never become perfect only by using CAD instead of traditional tools. For similar reasons the importance of mathematics is still increasing though computers take over the computational part.

Another misinterpretation of Descriptive Geometry is to consider it only as a theoretical, rather 'academic' subject. F. HOHENBERG could disprove this opinion in his textbook [7] in a convincing way. In many examples he presented real world applications for which Descriptive Geometry is substantial.

In order to defend the true meaning of Descriptive Geometry, there were various attempts to rename this subject. Its applicability is stressed by using the names 'Technical Geometry' or 'Applied Geometry' instead of Descriptive Geometry. As already mentioned, another choice is 'Constructive Geometry' — 'constructive' in its figurative sense. It should indicate that not only manual drawing but also mathematical computations are used in this subject.

Anyway, the original MONGE definition 'Descriptive Geometry' with its wide meaning covers all these aspects. So, in my opinion the original name is still appropriate. Only strategic reasons (e.g., for strengthening the position of Descriptive Geometry in the curricula) can justify new and perhaps more attractive names.

And for those who like to translate 'descriptive' by 'graphically depicting' only, I add the following statement: *Descriptive Geometry is more than 'descriptive' geometry as well as 'Geometry' is more than its literal sense 'measuring the Earth'.*

3   What should remain in the students' brain

In order to estimate the educational effect of any subject included in a curriculum, one should try to figure out what remains in the student's brain after all details are already forgotten. I would like to state that even for poor students the education in Descriptive Geometry brings about the ability

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[2]It is said that Felix KLEIN once stated: *"Among all mathematicians, geometers have the advantage to see what they are studying."*

- to comprehend spatial objects from given principal views, and

- to specify particular views. Besides,

- the students get an impression of geometric idealization (abstraction), of the variety of geometric shapes, and of geometric reasoning.

The first two items look rather elementary. However, these intellectual abilities are so fundamental that many people later on forget how hard they had to work for achieving them.

3.1 The importance of principal views

The principal views — *top view, front view, and side view* — are more or less abstract as they do not correspond to our personal visual impression. But abstraction simplifies. In the majority of cases the principal views make evident the essential properties of spatial structures, and inspecting these plane views is much easier than to concentrate on the true 3D structure.

However, it needs training to become familiar with this kind of representation and to grasp the shape of any 3D object just by looking at its principal views. Nobody questions the necessity of a permanent training for sportsmen. But in case of Descriptive Geometry, people often neglect this necessity and they speak of a purely academic subject, when, e.g., in introductory exercises two triangles in space are to intersect.

Medical doctors often hold in esteem their Descriptive Geometry education. In anatomy, they could much easier comprehend the course of blood-vessels or nerves just by sketching them in the principal views. And in orthopaedy, they were better able to grasp how human joints are operating and why mislocation cause various troubles.

A few months ago Austrian television was broadcasting a life operation at a human skull: The surgeon had to correct a mislocation of the cheek-bone, caused by a traffic accident. In a pre-operative step the required position of the cheek-bone was already marked on a screen. By an image-fusion this virtual posture was combined with the actual one. So, the surgeon's work consisted in making these two positions coincident by manual manipulations at the patient.

How did the surgeon control his work? He inspected the three principal views as they allowed to decompose the true 3D displacement into planar motions.
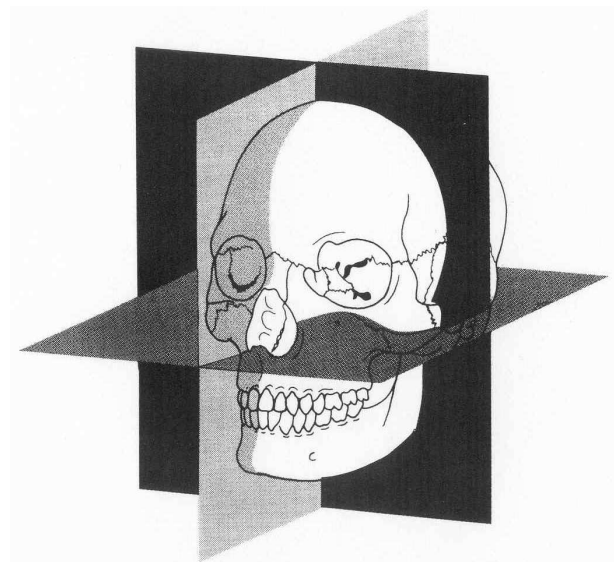

Figure 2: Explanation of principal views
in a textbook for dentists

3.2   The art of specifying particular views

Axonometric views are important and well understandable for everybody. And they are appropriate to remember on a known object or to compare with a real object nearby. However, no angle, no length, no plane shape appears in true size. Orthogonality can be figured out only because of some additional assumptions based on experience or estimation. So, these views are never sufficient for a *'description exacte'* as required in MONGE's definition.[3]

For a detailed analysis of a 3D object often *particular views* (auxiliary views) with planes in edge view or lines in point view really reveal the spatial situation. Such views often are the key to the solution of a 3D problem. In my opinion these particular views make the sublime art of Descriptive Geometry. Students learn only in the subject Descriptive Geometry what conditions can be fullfilled in particular views and how they can be specified.

The following example (Fig. 3, cf. [11]) shall demonstrate the advantage of particular views:

Example: *Where does the sun rise earlier on June 21, in Oslo or in Vienna.*

city	*eastern longitude*	*northern latitude*
Oslo	$10,6°$	$59,9°$
Vienna	$16,4°$	$48,2°$

We specify a front view in Fig. 3 with sun rays parallel to the image plane. Then we assume that this view is taken in the moment when the sun is rising in Oslo on June 21. As soon as Vienna is displayed in this view, we can see at the first glance the solution of the posed question.

The same view is also useful for clearing additional and more detailed problems like the following:

a) Can it happen over the period of a year that the sun rises simultaneously in Oslo and Vienna?

b) We increase the precision by paying attention to the fact that because of *refraction* in the atmosphere the sun is still approx. $0,6°$ under the local horizon when for the observer on earth the sun seems to rise.

c) In the *zone of astronomic dawn* the sun is between $6°$ and $18°$ under the local horizon. Inspecting the particular view presented above, it is easy to comprehend why the period of the daily dawn is shorter when the observer is nearer to the equator.

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[3]The same is true for pictures *shaded* like photographs; they can be extremely cheating. Pure *line graphics* look less attractive; they are more abstract. But often this is an advantage as much more information is included, and line graphics allow to concentrate on essentials.
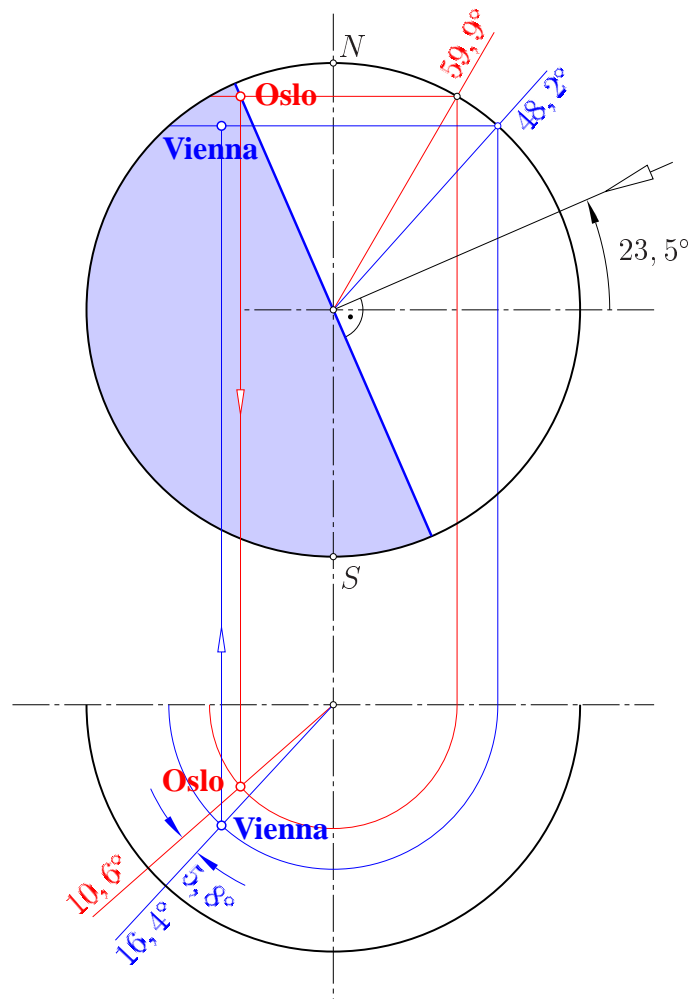
Figure 3: Where does the sun rise earlier on June 21, in Oslo or in Vienna ?

3.3 Views are a guide to spatial geometry

I don't know if anybody is able to manipulate virtual 3D objects — without any tools — in his imagination only, and to figure out how these objects look like in different postures. Maybe, sculptors or pilots have this ability. Actually, I myself don't; and the *rhombic dodecahedron* serves for me as a convincing example:

This convex polyhedron can be built by erecting quadratic pyramides with $45°$ inclined planes over each face of a cube (see Fig. 4). As any two coplanar triangles can be glued together forming a rhomb, this polyhedron has 12 congruent faces and seems to be well understood. Nevertheless, I'm not able to imagine (with closed eyes) how this polyhedron looks like from above when it is resting with one face on a table. Fortunately, a simple freehand sketch helps to figure out this view as well as other remarkable properties like:
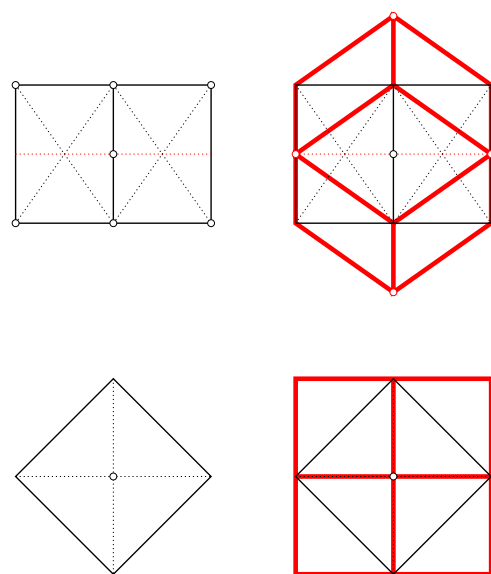
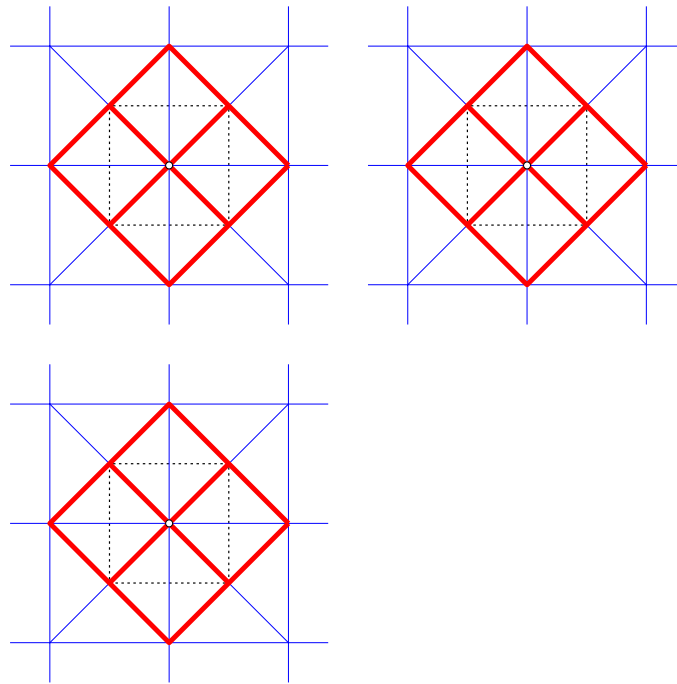

Figure 4: Cube and rhombic dodecahedron

Figure 5: The rhombic dodecahedron as the intersection of three quadratic prisms

- There are two types of vertices at the rhombic dodecahedron: 8 vertices belong to the initial cube; the other 6 are mirror images of the cube's center under reflection in the faces.

- The rhombic dodecahedron is the intersection of 3 quadratic prisms with pairwise orthogonal axes (see Fig. 5).

- The rhombic dodecahedron is the intersection of hexagonal prisms with axes in direction of the cube-diagonals. There are chains of 6 adjacent faces (note shaded rhombs in Fig. 6) which are located on the same hexagonal prism.

- The side and back walls of a honey comb belong to a rhombic dodecahedron.

- Each dihedral angle makes $120°$, and there is an in-sphere (contacting all edges of the initial cube).

- The rhombic dodecahedron [4] is dual to the cuboctahedron.

- The rhombic dodecahedron is a space-filling polyhedron. This can be figured out by starting with a '3D-chessboard' built from cubes. Then the 'white' cubes can be partitioned into 6 quadratic pyramids. Each can be added to the adjacent 'black' cube thus enlarging it to a rhombic dodecahedron.

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[4]More strictly, it should be called *first* rhombic dodecahedron. Due to S. BILINSKI [3] there is a *second* one: In this case the dihedral angle is $144°$. This polyhedron is obtained from the triacontahedron, the dual of the icosidodecahedron, by removing two prismatic zones and bringing the remaining pieces together. The author is grateful to H. MARTINI for pointing his attention to this fact.
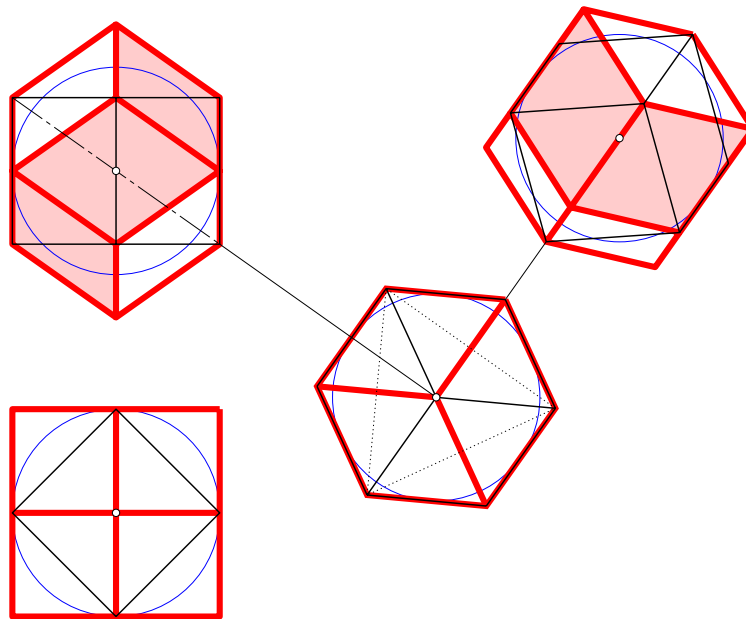
Figure 6: Different views of the rhombic dodecahedron

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