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A Way to Geometry Through Descriptive Geometry

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Summary. This is a short survey on my scientific achievements in the field of geometry and on my contributions to Descriptive Geometry education.

Descriptive Geometry in Austria

The development of Descriptive Geometry in Austria in the 20th century has been connected with famous scientists like E. Müller, E. Kruppa, J. L. Krames, F. Hohenberg, H. Brauner and W. Wunderlich. The role of Descriptive Geometry as a mathematical discipline has been promoted by E. Müller in his three volumes *"Lectures on Descriptive Geometry"* [1] and by H. Brauner's textbooks on *"Projective Geometry"* and *"Differential Geometry"* [2,3]. On the other hand, the various applications of Descriptive Geometry in technical and natural sciences have been emphasized by F. Hohenberg in his *"Constructive Geometry for Engineering"* [4] and W. Wunderlich in the two volumes *"Descriptive Geometry I, II"* [5] and his textbook on *"Plane Kinematics"* [6].

When I had been one of F. Hohenberg's students with the aim of becoming a secondary school teacher in the subjects Mathematics and Descriptive Geometrie, I became familiar with both aspects of geometry, with the theoretical side as a well as with its applicability.

Scientific Achievements

My first papers were strongly influcenced by W. Blaschke's and H. R. Müller's textbooks [7,8] on the *"kinematic mapping"* which associates the quasielliptic or elliptic 3D-geometry with the groups of planar or spherical Euclidean motions, respectively. The observation that all point paths under a planar motion can be derived from one and the same 3D-curve under a particular projection transmitted by a linear line congruence was the base for the theorem on the threefold generation of envelopes of trochoids under planetary motions in [9]. The correspondence between non-Euclidean differential geometry and the geometry of motions turned out to be fruitful for both areas, e.g., for the quasielliptic theory of curves and surfaces and the theory of one- and two-parametric motions [10]. I succeeded in solving a problem originally posed by W. Blaschke 1948 at a conference in Moscow: In [11] I determined all those surfaces which can be generated in infinitely different ways by quasielliptic Clifford translations. With the main result that there are five families of such surfaces I corrected a previous paper of the Ukrainian mathematician L.T. Motornyi (1965). The kinematic counterpart of this theorem was, e.g., subject of [12].

One of my favorite subjects of investigation are *overconstrained mechanisms*, where only the particular choice of dimensions makes a geometric structure flexible. This started in the joint paper [13] with S. Kunze concerning R. Bricard's closed hexagonal chain which turned out to be closely related to the Oloid (cf. [14]). Other examples of attractive flexible structures – all deduced from Platonic solids – were the pairs of flexible tetrahedra in [14], the Heureka polyhedron [16] (note O. Röschel's plenty of interesting generalizations summarized in his contribution for this special issue) and Grünbaum's framework consisting of the five left and five right regular tetrahedra which can be inscribed into a regular pentagondodecahedron in [17].

Some of my papers deal with the scientific base of pure Descriptive Geometry, with the *theory of projections*. [18] addresses the problem how to distinguish a photograph from a photograph of a photograph: A central axonometry is the image under a central projection if and only if the relative position of the central point and the images of the points at infinity of an orthogonal frame obey a particular trigonometric equation. A new proof for the characterization of central projections among singular collinear transformations in the n-space is presented in [20]. One result in this paper says that there might be different central projections which give similar images for any geometric object. The n-dimensional version of Pohlke's theorem and some properties of orthogonal and oblique axonometries are treated in [19]. The paper [21] reveals an unexpected relation between orthogonal projections of the n-space and a special choice of redundant coordinates in 3-space which can be useful in chemistry for representing structures of high symmetry.

The kinematic mapping treated in my early papers gave also rise to a converse of *Ivory's theorem* [22]. This brought new insight into a problem of critical configurations in satellite-geodesy [23]. However, generalisations of this theorem turned out to be essential for recent results on *infinitesimal and continuous mobility* [24] of geometric structures – like Bricard's flexible octahedra [25]. With such a converse it also seems to be possible to prove the conjecture that there are no nontrivially flexible cross-polytopes in the n-space for $n>4$ (cf. [26]). These results concerning flexibles polytopes are part of a joint INTAS-project with I. K. Sabitov (Moscow) and V. A. Alexandrov (Novosibirsk).

Educational Contributions and Software Development

My personal engagement for the geometry education in Austrian schools revealed the necessity of developing geometry software for educational purposes. In the early nineties a 2D software package *"CAD-2D"* and a 3D modelling software *"CAD-3D"* were developed under my supervision at the Institute of Geometry. Both packages are still in use in Austrian schools. This work made it necessary to study computational geometry more intensively which resulted also in PhD-theses and papers (e.g. [27,28]). The knowledge achieved with geometry software development has also been essential for consulting activities, when Austrian industries are facing us with real world problems in the fields of geometry including the theory of mechanisms. None of these contributions could be carried out without the aid of computers [29].

The algorithms developed in "CAD-3D" for carrying out the Boolean operations between polyhedra had been essential for my contribution to the joint textbook with G. Glaeser on *"Open Geometry"* [30]. This is a book about graphics programming using the programming language C++. It offers a well-documented, versatile, and robust library that is based on OpenGl®. Besides, one aim of this book is to give readers a good understanding of the geometry behind the different applications and to make readers familiar with geometrical thinking whithout having to worry about the implementation.

Descriptive Geometry for the Future

In the subject Descriptive Geometry students are taught how to look at and to represent 3D-objects and how to take the right conclusions from 2D-views. It is a constructive way to grasp spatial relations, and here the word *"constructive"* is meant in its double sense: On the one hand manual construction with traditional drawing tools or with the aid of any software, and on the other hand “constructive” in the logical sense, i.e., non confining oneselves to pure existence theorems, but really determining the explicit solution of at least presenting an algorithm for getting the solution.

This way of thinking has been essential for my personal scientific achievements in the wide field of geometry. Vice versa, scientific research in theoretical and applied geometry around Descriptive Geometry is necessary for a fruitful future of this subject and for the scientific community of graphics educators. This motivated me to join the group around the "pioneer" Steve Slaby and to work on fostering the international collaboration between graphics educators and stimulating the scientific research and teaching methodology as it takes place in the *"International Society for Geometry and Graphics" (ISGG)*. In order to provide an international forum for the publication of scientific papers which more or less are linked with Descriptive Geometry, K. Suzuki and I started 1996 to establish the *"Journal for Geometry and Graphics"* [31]. Until recent already four volumes of this refereed international scientific journal have been published.

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