UPPER SEMICONTINUITY OF SET VALUED FUNCTIONS AND A TOPOLOGICAL COUNTERPART OF BIRKHOFF'S ERGODIC THEOREM

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Abstract. Birkhoff's ergodic theorem roughly says that along orbits convergence of averages is typical. This is to be understood in the measure theoretic sense, i.e. with exceptional sets of measure zero. In the topological sense, however, averages typically, i.e. with meager exceptional sets, do not converge in an extreme way.

Such a topological counterpart of Birkhoff's theorem can be derived from semicontinuity properties of set valued functions, cf. Akin's comprehensive textbook [A 93]. The object of this note is to present this approach in a direct and self-contained way. The semicontinuity theorem in the persented version can be used to derive further old and new results of similar flavour.

1. Introduction

Preliminary remark: This version of the paper is not the final one. A relatively direct and self-contained approach for material much of which can be found in the textbook [A 93] is given. In a later version of the paper further applications to other topics are to be included.

We start by reviewing the well-known measure theoretic Birkhoff ergodic theorem in a topological context. Let X be a compact metric space, $T : X \to X$ continuous, $\mathcal{M}(X)$ the (compact metrizable) space of all Borel probability measures on X and $\mathcal{M}(X,T)$ its (nonempty and closed) subset consisting of all T-invariant $\mu \in \mathcal{M}(X)$. For $f : X \to \mathbb{R}$ let $X_{T,f} \subseteq X$ be the set of all $x \in X$ for which the limit

$$f^*(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x))$$

of time averages exists in $\mathbb{R}$. If $\mu \in \mathcal{M}(X,T)$ and $f \in \mathcal{L}_1(\mu)$ then, by Birkhoff's ergodic theorem, the set $X_{T,f}$ has full measure $\mu(X_{T,f}) = 1$. Furthermore the μ-almost everywhere defined function f^* is in $\mathcal{L}_1(\mu)$ as

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well, T-invariant (i.e. $f^* \circ T = f^*$) and has the same space average

$$\int f^* \, d\mu = \int f \, d\mu$$

as f. Since X is compact metric, the space $\mathcal{C}(X)$ of all continuous $f : X \to \mathbb{R}$ is separable. As an easy consequence, the intersection X_T of all $X_{T,f}$, $f \in \mathcal{C}(X)$, has full measure $\mu(X_T) = 1$ as well. For each $x \in X_T$ the mapping $f \mapsto f^*(x)$ is a positive linear functional on $\mathcal{C}(X)$ and, by Riesz' Theorem, corresponds to a measure $\mu^*(x) \in \mathcal{M}(X)$ (in fact $\mu^*(x) \in \mathcal{M}(X,T)$). In other words, for each $x \in X_T$ the orbit $(T^n(x))_{n\in\mathbb{N}}$ is a sequence which is uniformly distributed according to a measure $\mu^*(x) \in \mathcal{M}(X)$. Let us take a look at the mapping

$$\phi^* : X \supseteq X_T \to \mathcal{M}(X), \quad x \mapsto \mu^*(x).$$

Clearly ϕ^* is T-invariant. For an ergodic (i.e. measure theoretically irreducible) system (X,T,μ) this implies that ϕ^* has to take μ-almost everywhere the same constant value, namely μ. This means that ϕ^*, as a function on the measure space on X given by $\mu \in \mathcal{M}(X,T)$, extremely varies its appearence if μ changes. (And such a variation in fact is possible whenever T is not uniquely ergodic.)

Thus the question arises if ϕ^* has any typical behaviour in a sense which does not depend on the particular choice of $\mu \in \mathcal{M}(X,T)$. In other words, we ask if there is a notion of typical distribution behaviour of T-orbits which depends only on T. It is natural to look at the situation under a topological point of view, i.e. in terms of Baire categories[1]. This is the topic of the present article. The results show a remarkable contrast to the purely measure theoretic point of view. This contrast, roughly spoken, is the following one.

The set X_T is big in the measure theoretic sense but negligible from the topological point of view, i.e. the topologically big part X_0 is contained in the complement $X \setminus X_T$. While the typical probabilistic phenomenon (on X_T) is regular distribution (convergence to averages corresponding to a unique limit measure), the typical phenomenon in terms of Baire categories (on X_0) is a set of limit measures (accumulation points of sequences of measures) which, in a certain sense, is as big as possible. This sense, indicating an extreme irregularity of distribution behaviour, can be made precise in terms of upper semicontinuity of the set valued function

$$\phi : X \to \mathcal{M}(X), \quad x \mapsto \mathcal{M}(T,x),$$

which assigns to each $x \in X$ (not only to $x \in X_T$) the set $\mathcal{M}(T,x)$ of limit measures induced by the T-orbit of x. Upper semicontinuity of ϕ in a point

[1] cf. [O 80] as the classical reference for the analogies and differences between measure and category

$x_0 \in X$ means that $\phi(x_0)$ contains all accumulation points of the sets $\phi(x)$ with x arbitrarily close to x_0. Thus upper semicontinuity in x_0 may be interpreted in the sense that $\phi(x_0)$ is typically as big as locally possible. In a more precise formulation our main result (cf. Theorem 2, mainly based on Theorem 1) claims that ϕ is upper semicontinuous on a residual (comeager) subset of X.[2]

The contrast between measure and Baire category is particularly striking in the special case of irreducible systems. The topological irreducibility assumption which corresponds to ergodicity is that there exists a transitive and recurrent orbit. The upper semicontinuity result presented here turns out to easily imply the main result from [Wi 10]: there is a set $\mathcal{M}_0(T)$ of T-invariant measures such that $\phi(x) = \mathcal{M}_0(T)$ for all x outside a meager exceptional subset of X. The set $\mathcal{M}_0(T)$ is very big in the sense that it contains all limit measures of all dense orbits.

Our results have several further consequences describing the topologically typical irregularity of the distribution behaviour of sequences. Some of these consequences as well as related results can also be found in [D 53], [H 56], [G 83], [Wi 97], [GSW 00], [GSW 07] and [TZ 10] and also in the textbook [DGS 76]. Thus the approach via semicontinuity as presented here seems to provide the natural unified viewpoint to understand such phenomena.

The paper is organized as follows. In Section 2 we introduce some notation and collect auxiliary tools on semicontinuity. Section 3 is devoted to what is here called set limits of continuous functions (Theorem 1) and provides the general and common background for Section 4, which collects applications to dynamical systems (Theorem 2) and related situations.

2. Notation and auxiliary results on semicontinuity

Throughout the paper the symbols $X, T, \mathcal{M}(X)$ and $\mathcal{M}(X,T)$ are used as in the introduction. For each $x \in X$, let $\delta_x \in \mathcal{M}(X)$ denote the point probability measure concentrated in $x \in X$ and $\mathcal{M}(T,x)$ the set of limit measures of the orbit of x. More explicitly:

$$\mathcal{M}(T,x) = \bigcap_{N \in \mathbb{N}} \overline{\{\mu_n(T,x) : \ n \geq N\}}$$

is the set of accumulation points of the sequence of the measures

$$\mu_n(T,x) = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k x} \in \mathcal{M}(X).$$

Furthermore we use standard terminology in the context of Baire categories: Countable unions of nowhere dense sets are called meager or of first Baire category while their complements are called comeager or residual. A

[2]Semicontinuity of set valued functions plays an important role also in the work on topological ergodic decompositions of E. Glasner and others, cf. for instance [AG 98], first paragraph.

property depending on $x \in X$ holds residually if it holds for all x from a residual subset of X. Alternatively we say that a (topologically) typical x has this property.

If d is a metric for the topology on X, the ball with center x_0 and radius $\varepsilon > 0$ is denoted by $B(x_0, \varepsilon) = \{x \in X : d(x, x_0) < \varepsilon\}$. The neighbourhood filter $\mathcal{U}(x)$ consists of all $U \subseteq X$ such that $B(x, \varepsilon) \subseteq U$ for some $\varepsilon > 0$. $\mathcal{K}(X)$ denotes the set of all nonempty closed (hence compact) subsets of X,

$$d(A, B) = \inf_{a \in A, b \in B} d(a, b)$$

the distance between two nonempty sets $A, B \subseteq X$. For disjoint $A, B \in \mathcal{K}(X)$ one has $d(A, B) > 0$. There is another notion of distance which makes $\mathcal{K}(X)$ a compact metric space, the Hausdorff metric d_H. In order to define d_H let, for $A \subseteq X$ and $\varepsilon > 0$,

$$A^\varepsilon = \bigcup_{a \in A} B(a, \varepsilon).$$

By compactness of X there is, for arbitrary $A, B \in \mathcal{K}(X)$, an $\varepsilon > 0$ such that $A \subseteq B^\varepsilon$. Let the infimum over all such ε be denoted by $d_0(A, B)$. Then

$$d_H(A, B) = \max\{d_0(A, B), d_0(B, A)\}$$

defines the Hausdorff metric on $\mathcal{K}(X)$.[3] Using this notation we turn to semicontinuity of set valued functions.

Let X and Y be topological spaces, Y compact metrizable. Then a mapping $\phi : X \to \mathcal{K}(Y)$ is called a **set valued function** on X. ϕ is called **upper** respectively **lower semicontinuous** in $x_0 \in X$ if, for every $\varepsilon > 0$, there is a neighbourhood $U \in \mathcal{U}(x_0)$ of x_0 such that $\phi(x) \subseteq \phi(x_0)^\varepsilon$ respectively $\phi(x_0) \subseteq \phi(x)^\varepsilon$ for all $x \in U$. ϕ is called upper respectively lower semicontinuous on X if it is upper respectively lower semicontinuous in x for all $x \in X$. For given ϕ, the function $\overline{\phi} : X \to \mathcal{K}(Y)$ is defined by

$$\overline{\phi}(x_0) = \bigcap_{U \in \mathcal{U}(x_0)} \overline{\bigcup_{x \in U} \phi(x)}$$

and is called the **upper regularization** of ϕ.

We refer the reader to [Ku 66] for more material on semicontinuity of set valued functions. The following facts will be used in the sequel. For the convenience of the reader and for keeping the presentation self-contained proofs are included for those statements which are not obvious.

Proposition 1. *Let X and Y be compact metric spaces.*

(1) *A set valued function $\phi : X \to \mathcal{K}(Y)$ is continuous in $x_0 \in X$ if and only if it is both, upper and lower semicontinuous in x_0.*
(2) *For a set valued function $\phi : X \to \mathcal{K}(Y)$ and $x_0 \in X$ the following conditions are equivalent.*

[3] The topology induced by d_H does not depend on the special choice of the metric d for the topology on X. For more information cf. for instance [Ku 66] or [Ke 95].

(a) *ϕ is upper semicontinuous in x_0.*
(b) $\overline{\phi}(x_0) \subseteq \phi(x_0)$.
(c) $\overline{\phi}(x_0) = \phi(x_0)$.

(3) *For any set valued function $\phi : X \to \mathcal{K}(Y)$ its upper regularization $\overline{\phi} : X \to \mathcal{K}(Y)$ is upper semicontinuous.*

(4) *Let the set valued function $\phi : X \to \mathcal{K}(Y)$ be upper semicontinuous. Then ϕ has a residual set of continuity points.*

Proof. (1) and (2) are obvious. For (3) pick any $x_0 \in X$, and note that $U_1 \subseteq U_0 \in \mathcal{U}(x_0)$ implies $\bigcup_{x \in U_1} \phi(x) \subseteq \bigcup_{x \in U_0} \phi(x)$. For open U_0 this implies

$$\overline{\phi}(x) = \bigcap_{U \in \mathcal{U}(x)} \overline{\bigcup_{x' \in U} \phi(x')} \subseteq \overline{\bigcup_{x' \in U_0} \phi(x')}$$

for all $x \in U_0$. Thus

$$\bigcup_{x \in U_0} \overline{\phi}(x) \subseteq \overline{\bigcup_{x' \in U_0} \phi(x')}$$

and

$$\overline{\bigcup_{x \in U_0} \overline{\phi}(x)} \subseteq \overline{\bigcup_{x' \in U_0} \phi(x')}.$$

Since this holds for all open $U_0 \in \mathcal{U}(x_0)$ we conclude

$$\overline{\overline{\phi}}(x_0) = \bigcap_{U_0 \in \mathcal{U}(x_0)} \overline{\bigcup_{x \in U_0} \overline{\phi}(x)} \subseteq \bigcap_{U_0 \in \mathcal{U}(x_0)} \overline{\bigcup_{x' \in U_0} \phi(x')} = \overline{\phi}(x_0).$$

By (2) this shows upper semicontinuity of $\overline{\phi}$ at x_0. Since $x_0 \in X$ was arbitrary, (3) is proved.

In order to prove (4) let $D \subseteq X$ denote the set of all points where ϕ is not continuous and, for each $\varepsilon > 0$, D_ε the set of all $x_0 \in X$ such that for each $U \in \mathcal{U}(x_0)$ there is an $x_U \in U$ such that $\phi(x_0)$ is not contained in $\phi(x_U)^\varepsilon$. Note that, by upper semicontinuity of ϕ,

$$D = \bigcup_{\varepsilon > 0} D_\varepsilon = \bigcup_{k=1}^{\infty} D_{\frac{1}{k}}.$$

Assume, by contradiction, that D is not meager. Then, by Baire's theorem, there is an integer $k_0 \geq 1$ and a nonempty open set $O \subseteq X$ such that $O \subseteq \overline{D_\varepsilon}$ with $\varepsilon = \frac{1}{k_0}$. For every $x_0 \in D_\varepsilon$ and $U \in \mathcal{U}(x_0)$ there are

$$x_U \in U$$
$$y_U \in \phi(x_0) \setminus \phi(x_U)^\varepsilon.$$

Fix such an $x_0 \in O \cap D_\varepsilon$. Starting with this x_0 and $U_0 = O$ one can recursively define sequences of points

$$\begin{aligned} &x_0, x_1, \ldots \in D_\varepsilon \subseteq X, \\ &x'_0, x'_1, \ldots \in D_\varepsilon \subseteq X, \\ &y_0, y_1, \ldots \in Y \end{aligned}$$

and of open sets

$$X \supseteq U_0 \supseteq U'_0 \supseteq U_1 \supseteq U'_1 \supseteq \ldots$$

in the following way. Since $x_n \in D_\varepsilon$, $x'_n \in U_n$ can be taken such that $\phi(x_n)$ is not contained in $\phi(x'_n)^\varepsilon$, i.e. there is a

$$y_n \in \phi(x_n) \setminus \phi(x'_n)^\varepsilon.$$

Upper semicontinuity of ϕ in x'_n guarantees that there is an open $U'_n \in \mathcal{U}(x'_n)$ such that $U'_n \subseteq U_n$ and $\phi(x) \subseteq \phi(x'_n)^{\frac{\varepsilon}{2}}$ for all $x \in U'_n$. $U'_n \subseteq O \subseteq \overline{D_\varepsilon}$, hence there is an $x_{n+1} \in U'_n \cap D_\varepsilon$ and, by upper semicontinuity of ϕ in x_{n+1}, a $U_{n+1} \in \mathcal{U}(x_{n+1})$ with $U_{n+1} \subseteq U'_n$ and $\phi(x) \subseteq \phi(x_{n+1})^{\frac{\varepsilon}{2}}$ for all $x \in U_{n+1}$.

Let now $k \geq 1$. Since $x_{n+k} \in U'_n$ we get the inclusion

$$y_{n+k} \in \phi(x_{n+k}) \subseteq \phi(x'_n)^{\frac{\varepsilon}{2}}$$

which means $d(y_{n+k}, \phi(x'_n)) < \frac{\varepsilon}{2}$. Furthermore, since $y_n \notin \phi(x'_n)^\varepsilon$ we have $d(y_n, \phi(x'_n)) \geq \varepsilon$. Combining these two facts we conclude

$$d(y_n, y_{n+k}) \geq \frac{\varepsilon}{2}.$$

Thus $\{y_n : \ n \in \mathbb{N}\}$ is an infinite discrete and closed subset of the compact space Y which is impossible. This contradiction shows that our assumption is wrong, hence the set D of discontinuities of ϕ is meager. □

Remark. Consider real valued functions φ on X, i.e. $\varphi : X \to Y = [-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$ (two point compactification of $\mathbb{R}$). If $\phi : X \to \mathcal{K}(Y)$ is defined by $\phi(x) = [-\infty, \varphi(x)]$, then ϕ is upper semicontinuous in the sense of set valued functions if and only if φ is upper semicontinuous in the classical, real valued sense. Thus our considerations apply to real valued functions as well.

3. Set limits of continuous functions

Let again X and Y be compact metric spaces. For any sequence $\mathbf{f} = (f_n)_{n\in\mathbb{N}}$ of functions $f_n : X \to Y$ let

$$\phi_{\mathbf{f}}(x) = \bigcap_{N\in\mathbb{N}} \overline{\{f_n(x) : n \geq N\}}$$

be the set valued function which assigns to each $x \in X$ the set $\phi_{\mathbf{f}}(x)$ of all accumulation points of the sequence $\mathbf{f}(x) = (f_n(x))_{n\in\mathbb{N}}$. Let us call a set valued function ϕ a **set limit of continuous functions** if there are continuous f_n such that $\phi = \phi_{\mathbf{f}}$.

For a fixed set valued function $\phi : X \to \mathcal{K}(Y)$, let us use the following notation. For all $A \in \mathcal{K}(Y)$ we consider the sets

$$M(A) = M(A, \phi) = \{x \in X : \phi(x) \cap A \neq \emptyset\},$$

their topological boundaries $\partial M(A) = \overline{M(A)} \setminus M(A)$, the complements $R_A = R_A(\phi) = X \setminus \partial M(A)$ and their intersection $R = R(\phi) = \bigcap_{A \in \mathcal{K}(Y)} R_A$. A point $x \in X$ is called **A-regular** (with respect to ϕ) if $x \in R_A$ and **ϕ-regular** if $x \in R$.

Lemma 2. *Given a set limit $\phi = \phi_{\mathbf{f}}$ ($\mathbf{f} = (f_n)_{n \in \mathbb{N}}$) of continuous functions $f_n : X \to Y$, then for each $A \in \mathcal{K}(Y)$ the set R_A of A-regular points is residual.*

Proof. Note that $M(A) = \bigcap_{N \in \mathbb{N}, k \geq 1} O_{N,k}$ with $O_{N,k} = \bigcup_{n \geq N} f_n^{-1}(A^{\frac{1}{k}})$. By definition, each $O_{N,k}$ is dense in $\overline{M(A)}$. Since the $A^{\frac{1}{k}}$ are open and the f_n are continuous, the $O_{N,k}$ are open as well. Thus $M(A)$ is residual in $\overline{M(A)}$ which means that the set $\partial M(A) = \overline{M(A)} \setminus M(A)$ of A-singular points is meager in the compact set $\overline{M(A)} \subseteq X$ and thus in X. This means that R_A is residual. □

Although $\mathcal{K}(Y)$ is uncountable in general, for a continuous limit ϕ the intersection $R = \bigcap_{A \in \mathcal{K}(Y)} R_A$ nevertheless turns out to be residual as well.

Proposition 3. *For a set limit ϕ of continuous functions, the set R of ϕ-regular points is residual.*

Proof. Since Y is a compact metric space there is a countable open basis $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ for the topology on Y. Since $\mathcal{B}$ is countable, Lemma 2 implies that the set $R_{\mathcal{B}} = \bigcap_{n \in \mathbb{N}} R_{\overline{B_n}}$ is residual. We have to show $R_{\mathcal{B}} = R$, i.e. that every $x \in R_{\mathcal{B}}$ is ϕ-regular. Pick therefore any $A \in \mathcal{K}(Y)$ and $x_0 \in R_{\mathcal{B}}$. In order to see that $x_0 \in R_A = (X \setminus \overline{M(A)}) \cup M(A)$ we assume $x_0 \in \overline{M(A)}$ and derive from that $x_0 \in M(A)$. Consider the system $\mathcal{K}_0$ of all $K \in \mathcal{K}(Y)$ such that $x_0 \in R_K$. $\mathcal{K}_0$ contains all $\overline{B_n} \in \mathcal{B}$ and is closed under finite unions. By compactness, for each $\varepsilon > 0$ there is a finite covering $A \subseteq B_\varepsilon = \bigcup_{i=1}^k B_{n_i}$ with $\overline{B_\varepsilon} = \bigcup_{i=1}^k \overline{B_{n_i}} \subseteq A^\varepsilon$ and $\overline{B_\varepsilon} \in \mathcal{K}_0$. Since $x_0 \in \overline{M(A)} \subseteq \overline{M(\overline{B_\varepsilon})}$ is $\overline{B_\varepsilon}$-regular we conclude that $x_0 \in M(\overline{B_\varepsilon})$, i.e. there is a $y_\varepsilon \in \phi(x_0) \cap \overline{B_\varepsilon} \subseteq A^\varepsilon$. The sequence $(y_{2^{-m}})_{m \in \mathbb{N}}$ has an accumulation point $y_0 \in \phi(x_0) \cap \bigcap_{m \in \mathbb{N}} A^{2^{-m}} = \phi(x_0) \cap A$. Thus $x_0 \in M(A)$ and we are done. □

Proposition 4. *Given the set valued function $\phi : X \to \mathcal{K}(Y)$, let x_0 be ϕ-regular. Then ϕ is upper semicontinuous in x_0.*

Proof. For arbitrary $\varepsilon > 0$ we have to find a $U \in \mathcal{U}(x_0)$ such that $\phi(x) \subseteq \phi(x_0)^\varepsilon$ for all $x \in U$. Assume, by contradiction, that for each $U \in \mathcal{U}(x_0)$ there are $x_U \in U$ and $y_U \in \phi(x_U) \setminus \phi(x_0)^\varepsilon$. Consider $A = Y \setminus \phi(x_0)^\varepsilon \in \mathcal{K}(Y)$.

Then $x_U \in M(A) \cap U$ for all $U \in \mathcal{U}(x_0)$, showing that $x_0 \in \overline{M(A)}$. By regularity of x_0 this is possible only if $x_0 \in M(A)$, yielding the contradiction

$$\emptyset = \phi(x_0) \cap Y \setminus \phi(x_0)^\varepsilon = \phi(x_0) \cap A \neq \emptyset.$$

□

Example. Note that Proposition 4 does not hold for lower instead of upper semicontinuity. As a simple example take $X = [0,1]$, Y containing at least two points $y_0 \neq y_1$ and $\phi : X \to \mathcal{K}(Y)$ defined by $\phi(x_0) = \phi(0) = \{y_0, y_1\}$ and $\phi(x) = \{y_0\}$ for $x \neq x_0 = 0$. For $A \in \mathcal{K}(Y)$ we have $M(A) = X$ if $y_0 \in A$, $M(A) = \{0\}$ if $y_0 \notin A$ and $y_1 \in A$, and $M(A) = \emptyset$ if $y_0, y_1 \notin A$. In all cases $M(A)$ is closed, hence $\overline{M(A)} = M(A)$ and all $x \in X$ are ϕ-regular. But ϕ is not lower semicontinuous in $x_0 = 0$.

Theorem 1. *Let X and Y be compact metric spaces and the set valued function $\phi : X \to \mathcal{K}(Y)$ be a set limit of continuous functions. Then ϕ has a residual set of points of upper semicontinuity.*

Proof. Follows immediately by combination of Propositions 3 and 4. □

Example. Theorem 1 does not hold if upper is replaced by lower semicontinuity. As an illustration consider $X = Y = [0,1]$ (with the natural metric) and two sequences $b_0, b_1, b_2, \ldots$ and $c_0, c_1, c_2, \ldots$ in X such that $B = \{b_n : n \in \mathbb{N}\}$ and $C = \{c_n : n \in \mathbb{N}\}$ are disjoint and dense subsets of $X = [0,1]$. Let $f_n : X \to Y$, $n \in \mathbb{N}$, be continuous functions with $f_n(b_i) = 0$ and $f_n(c_i) = 1$ for $i = 0, 1, 2, \ldots, n$ and $\phi = \phi_{\mathbf{f}}$ the corresponding set limit of continuous functions. We are going to show that ϕ is not lower semicontinuous at any point $x_0 \in X$. First note that $\phi(b_n) = \{0\}$ and $\phi(c_n) = \{1\}$ for all $n \in \mathbb{N}$. Fix an ε with $0 < \varepsilon < \frac{1}{2}$ and pick any $x_0 \in X$. Since $\emptyset \neq \phi(x_0) \subseteq [0,1]$ we have $\phi(x_0) \cap [\varepsilon, 1] \neq \emptyset$ (first case) or $\phi(x_0) \cap [0, 1-\varepsilon] \neq \emptyset$ (second case). Take any $U \in \mathcal{U}(x_0)$. Then there are $b_U \in B \cap U$ and $c_U \in C \cap U$. The choice $x = b_U$ (first case) resp. $x = c_U$ (second case) shows that $\phi(x_0)$ is not contained in $\phi(x)^\varepsilon$. Thus ϕ is not lower semicontinuous in x_0.[4]

4. Applications

Let us now focus on topological dynamics, i.e. we consider a continuous transformation $T : X \to X$ of the compact metric space X. For each $x \in X$, we write

$$O(x) = \overline{\{T^n x : n \in \mathbb{N}\}}$$

[4]Recall that functions of Baire class 1, i.e. functions which are a pointwise limit of continuous functions, have a residual set of continuity points, cf. Theorem 1 on p. 394 in [Ku 66]. As a consequence we see that set limits of continuous functions, in general, cannot be represented as pointwise limits in $\mathcal{K}(Y)$.

for the orbit closure and

$$\omega(x) = \bigcap_{N \in \mathbb{N}} \overline{\{T^n x : n \geq N\}}$$

for the ω-limit set of x. Recall that x is called transitive if $O(x) = X$ and recurrent if $x \in \omega(x)$. Note that $O(x) = \omega(x)$ whenever x is recurrent.

The set valued function

$$\phi : X \to Y = \mathcal{K}(X),$$
$$x \mapsto \omega(x),$$

is a set limit $\phi = \phi_{\mathbf{f}}$, $\mathbf{f} = (T^n)_{n \in \mathbb{N}}$, of the continuous functions T^n and thus, by Theorem 1, has a residual set of points of upper semicontinuity. Suppose now that there is at least one transitive and recurrent point $x_0 \in X$, i.e. $O(x_0) = \omega(x_0) = X$.[5] Obviously $\phi = \phi \circ T$ is T-invariant, hence $\omega(T^n x_0) = \omega(x_0) = X$ for all $n \in \mathbb{N}$, yielding that the upper regularization $\overline{\phi}$ of ϕ takes the constant value X. Since ϕ has a residual set of points of upper semicontinuity we have

$$\omega(x) = \phi(x) = \overline{\phi}(x) = X$$

for all x from a residual subset of X. (Note that in our case all transitive points are recurrent as well.) Thus we have derived from our previous results the following well known fact.[6]

Corollary 5. *Let X be a compact metric space, $T : X \to X$ continuous, and assume $\omega(x_0) = X$ for some $x_0 \in X$. Then $\omega(x) = X$ for all x from a residual subset of X.*

If we omit the assumption on x_0, we still can apply Theorem 1. This means that the equation $\phi(x) = \overline{\phi}(x) = \omega(x)$ holds residually. Furthermore, by Proposition 1, $\overline{\phi}$ is upper semicontinuous (assertion (3)) and thus has a residual set of continuity points (assertion (4)). Combination of these facts yields:

Corollary 6. *Let X be a compact metric space and $T : X \to X$ continuous. Then the mapping*

$$\phi : X \to \mathcal{K}(X),$$
$$x \mapsto \omega(x),$$

is T-invariant and has a residual set of points of upper semicontinuity, i.e. residually fulfills $\phi(x) = \overline{\phi}(x)$ ($\overline{\phi}$, by Proposition 1, having a residual set of continuity points).

Example. Let us, for illustration, consider the example of the two dimensional torus $X = \mathbb{T}^2$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and the transformation $T : X \to X$, $(x, y) \mapsto (x, y + x)$. Then $\omega(x, y) = \phi(x, y) = \{x\} \times \mathbb{T}$ for irrational x and

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[5] Note that, given transitivity of x_0, the recurrence condition is relevant only if x_0 is an isolated point of X in which case X is finite.

[6] Cf., for a homeomorphism T, for instance [Wa 79], Theorem 5.8.

$\phi(x,y) = \{x\} \times (y + C_n)$ with $C_n = \{0, \frac{1}{n}, \ldots, 1 - \frac{1}{n}\}$ if $x = \frac{k}{n} \in \mathbb{Q}$ with coprime integers $n \neq 0, k$. In the space $\mathcal{K}(\mathbb{T})$ we have $\lim_{n\to\infty} C_n = \mathbb{T}$. This easily implies that ϕ is continuous at (x,y) if $x \notin \mathbb{Q}$. But ϕ is not upper semicontinuous on the meager set of all (x,y) where x is rational. Thus the upper regularization $\overline{\phi} : (x,y) \mapsto \{x\} \times \mathbb{T}$ coincides with ϕ if and only if x is irrational.

If we are interested in limit measures instead of ω-limit sets we have to take $Y = \mathcal{M}(X)$ (instead of $Y = X$). Apart from this all arguments are very similar. The announced topological counterpart of Birkhoff's ergodic theorem can be formulated as follows.

Theorem 2. *Let X be a compact metric space and $T : X \to X$ continuous. Then the mapping*

$$\phi : X \to \mathcal{K}(\mathcal{M}(X,T)),$$
$$x \mapsto \mathcal{M}(T,x),$$

is T-invariant and has a residual set of points of upper semicontinuity, i.e. residually fulfills $\phi(x) = \overline{\phi}(x)$ ($\overline{\phi}$, by Proposition 1, having a residual set of continuity points).

If $\omega(x_0) = X$ for some $x_0 \in X$, then there is a set $\mathcal{M}_0(T) \subseteq \mathcal{M}(X,T)$ (only depending on T) such that $\overline{\phi}(x) = \mathcal{M}_0(T)$ holds residually. $\mathcal{M}_0(T)$ is the union over all $\mathcal{M}(T,x)$ with transitive x.

Proof. The T-invariance of ϕ is obvious. For the convenience of the reader let us first recall here the argument from [GSW 07], Lemma 2.17., in order to justify that $\phi : X \to \mathcal{K}(\mathcal{M}(X,T))$, i.e. that all $\mu \in \phi(x) = \mathcal{M}(T,x)$ are T-invariant: $\mu \in \mathcal{M}(T,x)$ implies that, for the sequence of points $x_j = T^j x \in X$, $j \in \mathbb{N}$, and a suitable sequence $N_0 < N_1 < N_2 < \ldots \in \mathbb{N}$, we have

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{j=0}^{N_k-1} \delta_{T^j x} = \mu.$$

It suffices to show that, for arbitrary continuous $f : X \to \mathbb{R}$, one has $\int f\,d\mu = \int f \circ T\,d\mu$. Indeed this follows since the two limits

$$\int f\,d\mu = \lim_{k\to\infty} \frac{1}{N_k} \sum_{j=0}^{N_k-1} f(T^j x)$$

and

$$\int f \circ T\,d\mu = \lim_{k\to\infty} \frac{1}{N_k} \sum_{j=0}^{N_k-1} f \circ T(T^j x) = \lim_{k\to\infty} \frac{1}{N_k} \sum_{j=1}^{N_k} f(T^j x)$$

coincide (use that the first limit exists and that f is bounded).

After the above considerations it remains to prove the second assertion in the theorem which concerns the case $\omega(x_0) = X$. Let, at this point by definition,

$$\mathcal{M}_0(T) = \bigcup_{x\colon \text{transitive}} \mathcal{M}(T,x).$$

Note that, by T-invariance of ϕ, $\mathcal{M}(T,T^n x) = \mathcal{M}(T,x)$ for all $x \in X$ and $n \in \mathbb{N}$. For each transitive x this implies that the set of all $x' \in X$ with $\phi(x') \supseteq \mathcal{M}(T,x)$ is dense in X, yielding $\overline{\phi}(x) \supseteq \mathcal{M}_0(T)$ for all $x \in X$. By Corollary 5 the set of all transitive x is residual. Since $\phi(x) = \mathcal{M}(T,x) \subseteq \mathcal{M}_0(T)$ for all transitive x we also have the inclusion $\phi(x) \subseteq \mathcal{M}_0(T)$ residually. Hence, by a combination of Proposition 1 and Theorem 1, we residually have $\phi(x) \subseteq \mathcal{M}_0(T) \subseteq \overline{\phi}(x) = \phi(x)$, proving the theorem. □

Example. In the case $\omega(x_0) = X$ not every transitive and recurrent $x \in X$ has to satisfy $\mathcal{M}(T,x) = \mathcal{M}_0(T)$. Consider the one sided full shift on the two letters 0 and 1, i.e. $X = \{0,1\}^{\mathbb{N}}$ and

$$\begin{aligned} T = \sigma : \{0,1\}^{\mathbb{N}} &\to \{0,1\}^{\mathbb{N}}, \\ x = (a_n)_{n\in\mathbb{N}} &\mapsto \sigma(x) = (a_{n+1})_{n\in\mathbb{N}}. \end{aligned}$$

If x is a sequence containing all finite binary words but seperated by very long blocks of 0's, then x is transitive and recurrent. For sufficiently long blocks of 0's $\mathcal{M}(T,x)$ is a singleton, only containing the point measure δ_{x_0} concentrated in the constant sequence $x_0 = 000\ldots$. But $\mathcal{M}_0(T) = \mathcal{M}(X,T)$ is much bigger (cf. also [Wi 10] or [DGS 76].)

Example. To see that Theorem 2 does not hold with lower instead of upper continuity consider again the shift space $X = \{0,1\}^{\mathbb{N}}$ with $T = \sigma : X \to X$, $(a_n)_{n\in\mathbb{N}} \mapsto (a_{n+1})_{n\in\mathbb{N}}$, and let ϕ be as in Theorem 2. Consider the point measures $\delta_{x_0} = y_0, \delta_{x_1} = y_1 \in Y = \mathcal{M}(X,T)$ concentrated in the constant sequences $x_0 = 000\ldots$ resp. $x_1 = 111\ldots \in X$. In order to imitate the situation in the example after Theorem 1, let the sets $B, C \subseteq X$ consist of all eventually constant sequences finally taking the value 0 resp. 1. The sets B and C are dense in X with $\phi(b) = \{y_0\}$ for all $b \in B$ and $\phi(c) = \{y_1\}$ for all $c \in C$. One picks an $\varepsilon < \frac{d(y_0,y_1)}{2}$ where d is a compatible metric for the topology on Y and argues as in the example after Theorem 1 in order to conclude that ϕ is not lower semicontinuous at any point $x \in X$.

Example. For illustration of the nontransitive case consider once more the example given after Corollary 6 with the transformation $T : (x,y) \mapsto (x, y+x)$ on $X = \mathbb{T}^2$. The values $\phi(x,y) = \mathcal{M}(T,(x,y))$ are singletons in $\mathcal{M}(X)$, namely the uniform distribution measures on $\{x\} \times \mathbb{T}$ for irrational x and on $\{\frac{k}{n}\} \times (y + C_n)$ for $x = \frac{k}{n}$ with coprime integers $n \neq 0, k$. The upper regularization $\overline{\phi}$ coincides with ϕ whenever x is irrational. For rational $x = \frac{k}{n}$ with coprime k, n the set $\overline{\phi}(x,y)$ consists of two members, namely the uniform distribution measures on $\{x\} \times (y + C_n)$ and on $\{x\} \times \mathbb{T}$.

Example. The set $\mathcal{M}_0(T)$ in Theorem 2 is a subset of $\mathcal{M}(X,T)$ which (in contrast to several other simple examples where equality holds) does not necessarily contain all T-invariant measures $\mu \in \mathcal{M}(X,T)$. An example is the shift $T = \sigma$ on the shift orbit closure $X = O(x) = \overline{\{T^n x : \ n \in \mathbb{N}\}}$ of the one-sided infinite sequence $x = (a_n)_{n \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ which is uniquely defined by the implicit equation

$$x = a_0 a_1 a_2 \ldots = 0^1 1^1 a_0 0^2 1^2 a_0 a_1 0^3 1^3 a_0 a_1 a_2 \ldots .$$

(The notation $0^n, 1^n$ is for a block of n times the symbol 0 resp. 1.) If $x_1 = 111\ldots$ denotes the constant sequence of infinitely many 1's then the point measure δ_{x_1} concentrated in x_1 turns out to be T-invariant but not a member of $\mathcal{M}_0(T)$ (cf. [Wi 10]).

The step from Corollary 6 to Theorem 2 can be iterated by investigating not just the accumulation points but the distribution of the sequence of the measures $\mu_n \in Y = \mathcal{M}(X)$ and so on. Then the spaces $\mathcal{M}(\mathcal{M}(X))$, $\mathcal{M}(\mathcal{M}(\mathcal{M}(X)))$ etc. come into play and quite similar results in terms of Baire categories and upper semicontinuity follow. All of them would reflect the principle mentioned in the introduction: toplogically the typical behaviour is as irregular as possible.

As a further application of Theorem 1 let us look at the typical behaviour of arbitrary sequences, not necessarily induced by iteration of a transformation T. Instead of the space X itself we accordingly consider the compact metric space $X^{\mathbb{N}}$ of all sequences $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ on X. An open set O is determined by restrictions to finitely many coordinates, the tail of the sequence remaining arbitrary. Thus, for each nonempty open set $O \subseteq X^{\mathbb{N}}$ and $x \in X$ there is a sequence contained in O which finally takes the constant value x. Consider the continuous mappings (projections)

$$f_n : X^{\mathbb{N}} \to X,$$
$$\mathbf{x} = (x_k)_{k \in \mathbb{N}} \mapsto x_n,$$

and their set limit ϕ in the sense of Section 3. Then for each $x \in X$ the set of all $\mathbf{x} \in X^{\mathbb{N}}$ with $x \in \phi(\mathbf{x})$ is dense. Hence $x \in \overline{\phi}(\mathbf{x})$ for all $\mathbf{x} \in X^{\mathbb{N}}$. This holds for all $x \in X$, hence $\overline{\phi}$ takes the constant value X, and Theorem 1 implies that sequences in X are residually dense in X. Similar arguments applied to

$$f_n : X^{\mathbb{N}} \to \mathcal{M}(X)$$
$$(x_k)_{k \in \mathbb{N}} \mapsto \mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{x_k}$$

for $n \geq 1$ show that a topologically typical sequence has all $\mu \in \mathcal{M}(X)$ as limit measures. This has been proved in [Wi 97] where such sequences have been called **maldistributed**.

Corollary 7. *For a compact metric space X, the set of all dense sequences and (even stronger) the set of all maldistributed sequences on X are residual subsets of $X^{\mathbb{N}}$.*

Slight modifications of the above arguments can also be used to derive Baire results on certain spaces S of subsequences as treated in [GSW 07]. We conclude with a result of this type (observed by G. Barat, oral communication) which shows that there is a typical set of limit measures, provided that S is, in a rather weak sense, stable under changes of finitely many members of its sequences.

Corollary 8. *Let X be a compact metric space and S a closed or, more generally, a G_δ (hence Baire) subspace of the space $X^{\mathbb{N}}$ of all sequences on X. Assume that S has the following property: for all $\mathbf{x} \in S$ and nonempty open sets $O_0, \dots, O_{n-1} \subseteq X$ there is an $\mathbf{x}' = (x'_n)_{n\in\mathbb{N}}$ with $(x'_0, \dots, x'_{n-1}) \in O_0 \times \dots \times O_{n-1}$ and $\mathcal{M}(\mathbf{x}') \supseteq \mathcal{M}(\mathbf{x})$.*

Then there is a set $\mathcal{M}_0 \subseteq \mathcal{M}(X)$ such that $\mathcal{M}(\mathbf{x}) = \mathcal{M}_0$ holds for all $\mathbf{x}$ from a residual subset of S.

Proof. For $\phi = \phi_{\mathbf{f}}$ with $\mathbf{f} = (f_n)_{n\in\mathbb{N}}$, $f_n : \mathbf{x} = (x_n)_{n\in\mathbb{N}} \mapsto \mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{x_k}$, the assumption on S guarantees that $\overline{\phi}$ takes a constant value $\mathcal{M}_0$ with the desired properties. □

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