The Coherence of Semifilters: a Survey

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1. Some history and motivation

The aim of this survey is to present the principal ideas and results of the theory of semifilters from the book "Coherence of Semifilters" [6], which will be our basic reference. In spite of the fact that the term "semifilter" is new, many results about semifilters are well-known in the framework of filters or ultrafilters.

Our starting point is the principle NCF, the Near Coherence of Filters. It was introduced by Blass [9] and belongs to the most exciting and counter-intuitive set-theoretic principles (with many applications in various fields of mathematics). This principle asserts that any two filters $\mathcal{F}_1, \mathcal{F}_2$ on the set ω of non-negative numbers are near coherent in the sense that for some finite-to-one surjection $\varphi : \omega \to \omega$ the union $\varphi(\mathcal{F}_1) \cup \varphi(\mathcal{F}_2)$ lies in some filter. This is equivalent to saying that any two ultrafilters $\mathcal{U}_1$, $\mathcal{U}_2$ are coherent in the sense that $\varphi(\mathcal{U}_1) = \varphi(\mathcal{U}_2)$ for some finite-to-one surjection $\varphi : \omega \to \omega$ (here $\varphi(\mathcal{U}) = \{\varphi(U) : U \in \mathcal{U}\}$). For those familiar with many different sorts of ultrafilters (selective, Ramsey, P-points, Q-points, etc.) the principle NCF may look suspicious. Indeed, it is false under the Continuum Hypothesis (as well as under Martin's Axiom). Nonetheless, NCF does not contradict ZFC and is true in some models of ZFC constructed by Blass, Shelah [15], [16]. Thus, like many other set-theoretic principles, NCF is independent of ZFC.

The principle NCF arose from the joint work of Blass, Weiss [18] on a problem of Brown, Pearcy and Salinas [20] asking if the ideal of compact operators in the ring of bounded operators on the Hilbert space is the sum of two properly smaller ideals. It turned out that this problem has an affirmative answer if and only if NCF is false and thus is independent of the axiom system ZFC, see Blass [10]. Another important application of NCF concerns the Stone-Čech remainder $\beta\mathbb{H} \setminus \mathbb{H}$ of the half-line $\mathbb{H} = [0, \infty)$. According to a combined result of Rudin [38], Mioduszewski [31] and Blass [10], NCF is equivalent to the statement that the indecomposable continuum $\beta\mathbb{H} \setminus \mathbb{H}$ has only one composant (this contrasts with the classical Mazurkiewicz result that each metrizable indecomposable continuum has uncountably many composants). Quite recently NCF has found applications in Topological Algebra, see Banakh, Nikolas, Sanchis [5] and Selection Principles, see Bartoszyński, Shelah, Tsaban [8], Tsaban, Zdomskyy [42], Eisworth, Just [23]. Some weaker forms of NCF appear in the theory of separately continuous functions. Namely, the existence of a discontinuous separately continuous function $f : X \times Y \to \mathbb{R}$ defined on the product of arbitrary non-discrete separable Tychonov spaces X, Y is equivalent to the near coherence of any two P-filters, see Banakh, Maslyuchenko, Mykhaylyuk [4].

It should be mentioned that all known models of NCF satisfy two formally stronger principles called the Filter Dichotomy and the Semifilter Trichotomy. The Filter Dichotomy says that any filter $\mathcal{F}$ on ω is coherent either to some fixed ultrafilter $\mathcal{U}_0$ or to the Fréchet filter $\mathfrak{Fr}$ consisting of all cofinite subsets of ω. The Semifilter Trichotomy treats families of infinite subsets of ω closed under taking almost supersets and says that any such a family is coherent either to the Fréchet filter $\mathfrak{Fr}$ or to any fixed ultrafilter $\mathcal{U}_0$ or to the family $[\omega]^\omega$ of all infinite subsets of ω.

2. Introducing semifilters

Families of infinite subsets of ω, closed under taking almost supersets, play a crucial role in the whole theory so we have decided to give them a special name: semifilters. In other words, a non-empty family $\mathcal{F}$ of infinite subsets of a set X is called a *semifilter* if

i. $\mathcal{F}$ is closed under taking supersets and

ii. for each element $F \in \mathcal{F}$ and each cofinite set $C \subset \omega$ the intersection $C \cap F \in \mathcal{F}$.

Replacing the condition *ii* by the more familiar

ii'. $F_1 \cap F_2 \in \mathcal{F}$ for any $F_1, F_2 \in \mathcal{F}$

we get the well-known definition of a filter. This explains the choice of the term "semifilter".

It should be mentioned that a filter $\mathcal{F}$ is a semifilter if and only if $\mathcal{F}$ is *free* that is $\cap\mathcal{F} = \emptyset$ which is equivalent to the inclusion $\mathfrak{Fr} \subset \mathcal{F}$. Thus the Fréchet filter $\mathfrak{Fr}$ is the smallest element of the family FF of all free filters on ω. This family has no largest element but has $2^{\mathfrak{c}}$ maximal elements called *ultrafilters*.

3. The lattice SF of semifilters

In contrast to this, the set SF of all semifilters on ω ordered by the ordinary inclusion relation has both a smallest element, the Fréchet filter $\mathfrak{Fr}$, and a largest element, the co-Fréchet semifilter $\mathfrak{Fr}^\perp = [\omega]^\omega$ consisting of all infinite subsets of ω. Semifilters have some advantages compared to filters. In particular, arbitrary unions as well as intersections of semifilters are semifilters. Thus SF is a sublattice of the double power-set $\mathcal{P}(\mathcal{P}(\omega))$ considered as a (complete distributive) lattice with respect to the operations of intersection and union. Being closed under arbitary intersections and unions, SF is a complete distributive lattice. Endowed with the Lawson topology (equivalently, with the Tychonov product topology inherited from $2^{2^\omega} = \mathcal{P}(\mathcal{P}(\omega)) \supset \mathsf{SF}$) this lattice is a supercompact Hausdorff space. The Lawson topology on SF is generated by the binary sub-base $\mathfrak{B}$ consisting of the sets $F^+ = \{\mathcal{F} \in \mathsf{SF} : F \in \mathcal{F}\}$ and $F^- = \{\mathcal{F} \in \mathsf{SF} : F \notin \mathcal{F}\}$ where F runs over the subsets of ω. The binary property of $\mathfrak{B}$ means that each cover of SF by elements of $\mathfrak{B}$ contains a two-element subcover (topological spaces possessing a binary sub-base are called *supercompact*).

Besides the two lattice operations $\cap$ and $\cup$, the lattice SF possesses the important continuous unary operation of *transversal* which assigns to a semifilter $\mathcal{F}$ the semifilter

$$\mathcal{F}^\perp = \big\{E \subset \omega : \forall F \in \mathcal{F} \quad E \cap F \neq \emptyset\big\}$$

called the *dual semifilter* to $\mathcal{F}$. It can be easily shown that $(\mathcal{F}^\perp)^\perp = \mathcal{F}$ and $(\mathcal{F} \cap \mathcal{U})^\perp = \mathcal{F}^\perp \cup \mathcal{U}^\perp$, $(\mathcal{F} \cup \mathcal{U})^\perp = \mathcal{F}^\perp \cap \mathcal{U}^\perp$ and thus the transversal operation $\perp : \mathsf{SF} \to \mathsf{SF}$ is an involutive topological antiisomorphism of SF. Let us observe that the semifilters $\mathfrak{Fr}$ and $[\omega]^\omega$ are dual each to the other. That is why we often denote the semifilter $[\omega]^\omega$ by $\mathfrak{Fr}^\perp$.

Now let us consider the structure of the self-dual semifilters, that is semifilters $\mathcal{F}$ equal to their duals $\mathcal{F}^\perp$. Important examples of such semifilters are ultrafilters. However ultrafilters do not exhaust all possible self-dual semifilters. Let us observe that the inclusion $\mathcal{F} \subset \mathcal{F}^\perp$ is equivalent to the *linkedness* of $\mathcal{F}$ which means that $F_1 \cap F_2 \neq \emptyset$ for all $F_1, F_2 \in \mathcal{F}$ while the inclusion $\mathcal{F}^\perp \subset \mathcal{F}$ is equivalent to the *unsplit* property of $\mathcal{F}$ which means that for each subset $A \subset \omega$

either $A \in \mathcal{F}$ or $\omega \setminus A \in \mathcal{F}$. Consequently, a semifilter $\mathcal{F}$ is self-dual if and only if $\mathcal{F}$ is unsplit and linked if and only if $\mathcal{F}$ is maximal linked. Because of this self-duality property, the maximal linked semifilters will play an important role in studying semifilters. In particular, they have the following approximation property: for any semifilter $\mathcal{F}$ there is a maximal linked semifilter $\mathcal{L}$ with $\mathcal{F} \cap \mathcal{F}^{\perp} \subset \mathcal{L} \subset \mathcal{F} \cup \mathcal{F}^{\perp}$. The set ML of maximal linked semifilters is closed with respect to the Lawson topology on SF and moreover, is supercompact, which makes ML similar to the superextensions considered in van Mill [30]. Other interesting subsets of SF are also closed with respect to the Lawson topology. In particular, so are the sets

- UF of all ultrafilters;
- FF of all free filters;
- CEN of all centered semifilters;
- L_k of k-linked semifilters for $k \geq 2$.

Besides the considered algebraic operations on semifilters, there is an operation of support. By definition, the *support* of a semifilter $\mathcal{F}$ is the filter

$$\operatorname{supp}(\mathcal{F}) = \big\{E \subset \omega : \forall F \in \mathcal{F}\ F \cap E \in \mathcal{F}\big\} \subset \mathcal{F}.$$

It is clear that $\operatorname{supp}(\mathcal{F}) = \mathcal{F}$ if and only if $\mathcal{F}$ is a filter; so the difference $\mathcal{F} \setminus \operatorname{supp}(\mathcal{F})$ shows how far a semifilter $\mathcal{F}$ is from being a filter. It is interesting to note that the support $\operatorname{supp}(\mathcal{L})$ of a maximal linked semifilter is a filter equal to $(\mathcal{L} \wedge \mathcal{L})^{\perp}$ where $\mathcal{L} \wedge \mathcal{L} = \{A \cap B : A, B \in \mathcal{L}\}$. Unlike the transversality operation, the operation of support is discontinuous with respect to the Lawson topology on SF. Its continuity points are semifilters $\mathcal{F}$ with the smallest possible support $\operatorname{supp}(\mathcal{F}) = \mathfrak{Fr}$.

One can show that the lattice SF is topologically isomorphic to the lattice of non-constant monotone Boolean functions on the powerset $(\mathcal{P}(\omega), \subset^*)$ endowed with the almost inclusion preorder.

4. The limit operator on SF

Since SF is a compact Hausdorff space, for any sequence $(\mathcal{U}_n)_{n\in\omega} \subset \mathsf{SF}$ and any ultrafilter $\mathcal{F}$ we can consider the limit $\lim_{\mathcal{F}} \mathcal{U}_n$ of $(\mathcal{U}_n)$ along $\mathcal{F}$. This is a unique point $\mathcal{U}_\infty \in \mathsf{SF}$ such that for any neighborhood $\mathsf{O}(\mathcal{U}_\infty) \subset \mathsf{SF}$ there is an element $F \in \mathcal{F}$ such that $\mathcal{U}_n \in \mathsf{O}(\mathcal{U}_\infty)$ for all $n \in F$. In fact, the limit semifilter $\lim_{\mathcal{F}} \mathcal{U}_n$ admits a direct description: it is generated by the sets $\bigcup_{n\in F} U_n$ where $F \in \mathcal{F}$ and $U_n \in \mathcal{U}_n$ for $n \in F$.

We can take this direct description as a definition of $\lim_{\mathcal{F}} \mathcal{U}_n$ for any sequence $(\mathcal{U}_n)$ of semifilters and any semifilter $\mathcal{F}$ (not necessarily an ultrafilter). In such a way we define the *limit operator* $\lim : \mathsf{SF} \times \mathsf{SF}^\omega \to \mathsf{SF}$ assigning to a pair $(\mathcal{F}, (\mathcal{U}_n))$ the limit semifilter $\lim_{\mathcal{F}} \mathcal{U}_n$ generated by the sets $\bigcup_{n\in F} U_n$ where $F \in \mathcal{F}$ and $U_n \in \mathcal{U}_n$ for $n \in F$. This operator has many nice properties. In particular, it nicely agrees with the duality: $(\lim_{\mathcal{F}} \mathcal{U}_n)^\perp = \lim_{\mathcal{F}^\perp} \mathcal{U}_n^\perp$. Also it preserves some important subsets of SF: $\lim(\mathsf{X} \times \mathsf{X}^\omega) \subset \mathsf{X}$, where $\mathsf{X} \in \{\mathsf{UF}, \mathsf{ML}, \mathsf{FF}, \mathsf{CEN}, \mathsf{L}_k : k \geq 2\}$.

For each fixed sequence $(\mathcal{U}_n)$ of semifilters we can look at the limit operator as a function $\lim^{(\mathcal{U}_n)} : \mathsf{SF} \to \mathsf{SF}$, $\lim^{(\mathcal{U}_n)} : \mathcal{F} \mapsto \lim_{\mathcal{F}} \mathcal{U}_n$ of one variable $\mathcal{F}$. This function turns to be a continuous lattice homomorphism on SF. Moreover, if all the semifilters $\mathcal{U}_n$ are maximal linked, then this homomorphism preserves the transversality operation in the sense that $(\lim^{(\mathcal{U}_n)}(\mathcal{F}))^\perp = \lim^{(\mathcal{U}_n)}(\mathcal{F}^\perp)$.

The homomorphism $\lim^{(\mathcal{U}_n)} : \mathsf{SF} \to \mathsf{SF}$ is injective if the sequence $(\mathcal{U}_n)$ is *separated* in the sense that there is a disjoint sequence of sets $(S_n)_{n\in\omega}$ such that $S_n \in \operatorname{supp}(\mathcal{U}_n)$ for all $n \in \omega$. Separated sequences of ultrafilters $(\mathcal{U}_n)$ can be characterized in topological terms as discrete subspaces of UF. For such sequences of ultrafilters, the operator $\lim^{(\mathcal{U}_n)} : \mathsf{SF} \to \mathsf{SF}$ is an isomorphic embedding of the lattice SF into SF such that $(\lim^{(\mathcal{U}_n)})^{-1}(\mathsf{X}) = \mathsf{X}$ for any $\mathsf{X} \in \{\mathsf{UF}, \mathsf{ML}, \mathsf{FF}, \mathsf{CEN}, \mathsf{L}_k : k \geq 2\}$.

5. Algebraic operations on the lattice SF

As we already know the lattice of semifilters SF contains the set $\mathsf{UF} = \beta\omega\setminus\omega$ of all (free) ultrafilters. The latter set is well studied from various points of view. One of very fruitful approaches to studying UF consists in looking at the Stone-Čech remainder $\mathsf{UF} = \beta\omega \setminus \omega$ as an algebro-topological object, see Protasov [35] or Hindman, Strauss [25]. Such a point of view is based on the fact that each algebraic operation $* : \omega \times \omega \to \omega$ on ω can be extended to an operation $\circ : \beta\omega \times \beta\omega \to \beta\omega$ as follows: given two ultrafilters $\mathcal{F}, \mathcal{U}$ let $\mathcal{F} \circ \mathcal{U}$ be the ultrafilter generated by the sets of the form $\bigcup_{x\in F} x * U_x$ where $F \in \mathcal{F}$ and $U_x \in \mathcal{U}$ for each $x \in F$. Here for subsets $A, B \subset \omega$ we put $A * B = \{a * b : a \in A,\ b \in B\}$.

Exploiting the so extended algebraic operation $\circ : \beta\omega \times \beta\omega \to \beta\omega$ yielded new and transparent proofs of many difficult combinatorial results like van der Waerden or Hindman Theorems, see Protasov [35], Hindman, Strauss [25].

The product $\mathcal{F} \circ \mathcal{U}$ of two free ultrafilters is again a free ultrafilter provided the operation $*$ has finite-to-one left shifts in the sense that for each $a \in \omega$ the left shift $l_a : x \mapsto a * x$ is finite-to-one. It turns out that any such an operation $* : \omega \times \omega \to \omega$ induces a binary operation $\circ$ on SF in the same way as it does on the Stone-Čech compactification of ω.

Namely, given two semifilters $\mathcal{F}, \mathcal{U}$ let $\mathcal{F} \circ \mathcal{U}$ be the semifilter generated by the sets of the form $\bigcup_{x\in F} x * U_x$ where $F \in \mathcal{F}$ and $U_x \in \mathcal{U}$ for each $x \in F$. The operation $\circ : \mathsf{SF} \times \mathsf{SF} \to \mathsf{SF}$ is associative provided so is the operation $*$ on ω (in contrast, the operation $\circ$ need not be commutative even for commutative $*$). Alternatively, $\mathcal{F} \circ \mathcal{U}$ can be defined as the limit semifilter $\lim_{\mathcal{F}} n * \mathcal{U}$, where $n * \mathcal{U}$ is the semifilter generated by the sets $n * U$, $U \in \mathcal{U}$. Applying the known properties of the limit operator, we see that for any fixed semifilter $\mathcal{U}$ the right shift $r_{\mathcal{U}} : \mathcal{F} \mapsto \mathcal{F} \circ \mathcal{U}$ is a continuous homomorphism of the lattice SF. Thus the operation $\circ$ turns SF into right topological semigroup. The operation $\circ$ can have points of joint continuity: for each P-point $\mathcal{U}$ and each semifilter $\mathcal{F}$ the operation $\circ$ is jointly continuous at $(\mathcal{F}, \mathcal{U})$.

It is interesting to notice that the sets $\mathsf{UF}, \mathsf{ML}, \mathsf{FF}, \mathsf{CEN}, \mathsf{L}_k$, $k \geq 2$, are subsemigroups of $(\mathsf{SF}, \circ)$.

6. (Sub)coherence relation

What makes the study of semifilters truly exciting is the coherence relation. Trying to find a true definition of a coherence equivalence on SF one can proceed by the most obvious way and define two semifilters $\mathcal{F}$,$\mathcal{U}$ to be coherent if $\varphi(\mathcal{F}) = \varphi(\mathcal{U})$ for some finite-to-one surjection $\varphi : \omega \to \omega$. Such a definition determines an equivalence relation on the set of ultrafilters however it is not clear why it is transitive of the lattice SF of all semifilters.

After some attempts we came to the conclusion that the most efficient way to define a coherence relation is to use finite-to-finite multifunctions in place of finite-to-one functions. By a *multifunction* from a set X to a set Y we understand a subset $\Phi \subset X \times Y$ which can be thought as a multi-valued function assigning to a point $x \in X$ the subset $\Phi(x) = \{y \in Y : (x, y) \in \Phi\}$. In such a way multifunctions are identified with their graphs. Often we shall write $\Phi : X \Rightarrow Y$ to stress that Φ is interpreted as a multivalued function. For a subset $A \subset X$ we put $\Phi(A) = \bigcup_{x \in A} \Phi(x)$. The inverse to a multifunction $\Phi \subset X \times Y$ is the multifunction $\Phi^{-1} = \{(y, x) : (x, y) \in \Phi\} : Y \Rightarrow X$ assigning to each $y \in Y$ the set $\Phi^{-1}(y) = \{x \in X : y \in \Phi(x)\}$.

A multivalued function $\Phi : X \Rightarrow Y$ is called *finite-to-finite* if for any finite non-empty sets $A \subset X$, $B \subset Y$ the sets $\Phi(A)$ and $\Phi^{-1}(B)$ are finite and **non-empty**. In the sequel we shall consider exclusively multifunctions from ω to ω.

The class of finite-to-finite multifunctions has some advantages compared to the class of finite-to-one functions because the former class is closed under unions, compositions, and taking the inverse.

Now we are able to define the principal concept of the book – the subcoherence relation on the lattice SF. We shall say that a semifilter $\mathcal{F}$ is *subcoherent* to a semifilter $\mathcal{U}$ and denote this by $\mathcal{F} \Subset \mathcal{U}$ if there is a finite-to-finite multifunction $\Phi : \omega \Rightarrow \omega$ such that $\Phi(\mathcal{F}) \subset \mathcal{U}$ where $\Phi(\mathcal{F}) = \{\Phi(F) : F \in \mathcal{F}\}$. Two semifilters $\mathcal{F}$ and $\mathcal{U}$ are defined to be *coherent* (denoted by $\mathcal{F} \asymp \mathcal{U}$) if $\mathcal{F} \Subset \mathcal{U}$ and $\mathcal{U} \Subset \mathcal{F}$. It is easy to see that the subcoherence relation, being reflexive and transitive, is a preorder on SF and hence the coherence relation is an equivalence relation on SF. By its definition the subcoherence relation resembles the preordering introduced by Rudin [38]: $\mathcal{F} \le_{BR} \mathcal{U}$ if $\varphi^{-1}(\mathcal{F}) \subset \mathcal{U}$ for some finite-to-one function $\varphi : \omega \to \omega$.

There is a more cumbersome but maybe intuitively more acceptable way to define the subcoherence relation using the interval partitions, a standard instrument in studying the coherence of filters. Namely, a semifilter $\mathcal{F}$ is subcoherent to a semifilter $\mathcal{U}$ if and only if there is an increasing number sequence (n_k) such that

$$\bigcup\big\{[n_k, n_{k+3}) : [n_{k+1}, n_{k+2}) \cap F \neq \emptyset\big\} \in \mathcal{U}$$

for each $F \in \mathcal{F}$.

By its spirit, the coherence of filters is a particular case of parallel filters in balleans, the theory created recently by I. Protasov, see Protasov, Banakh [36] and Protasov, Zarichnyi [37]. The coherence of semifilters can also be naturally considered in the framework of multicovered spaces, see Banakh, Zdomskyy [7]. It should be mentioned that our concept of the coherence differs from that of A.Blass [9], [10] who defined two filters $\mathcal{F}$,$\mathcal{U}$ to be coherent if their union $\mathcal{F}\cup\mathcal{U}$ can be enlarged to a filter.

7. Near coherence of semifilters

The machinery of finite-to-finite multifunctions allows to extend the notion of near coherence from filters to semifilters. We define two semifilters $\mathcal{F}$,$\mathcal{U}$ to be *nearly coherent* if there is a finite-to-finite multifunction $\Phi : \omega \Rightarrow \omega$ such that $\Phi(F) \cap \Phi(U) \neq \emptyset$ for any elements $F \in \mathcal{F}$, $U \in \mathcal{U}$. It can be shown that two semifilters $\mathcal{F}$,$\mathcal{U}$ are nearly coherent if and only if $\mathcal{F} \Subset \mathcal{U}^{\perp}$ if and only if $\mathcal{U} \Subset \mathcal{F}^{\perp}$. Hence the near coherence can be expressed via the subcoherence relation.

Note that two maximal linked semifilters $\mathcal{L}$,$\mathcal{U}$ are coherent if and only if they are nearly coherent.

8. A characterization of the semifilters coherent to $\mathfrak{Fr}$ or $\mathfrak{Fr}^{\perp}$

A characterization of the semifilters coherent to the extremal semifilters $\mathfrak{Fr}$ and $\mathfrak{Fr}^{\perp}$ was given by Talagrand [41] in topological terms. Namely, he has shown that for a semifilter $\mathcal{F}$ the following conditions are equivalent:

1. *$\mathcal{F}$ is coherent to the Fréchet filter;*

2. *$\mathcal{F}$ is a meager subspace of the power-set $\mathcal{P}(\omega)$ endowed with the natural compact metrizable topology;*

3. *$\mathcal{F}$ lies in a σ-compact semifilter;*

4. *$\varphi(\mathcal{F}) = \mathfrak{Fr}$ for some monotone surjection $\varphi : \omega \to \omega$;*

5. *for each infinite subset $\Omega \subset \omega$ there is an increasing number sequence $(n_k) \subset \Omega$ such that each set $F \in \mathcal{F}$ meets almost all the half-intervals $[n_k, n_{k+1})$.*

By duality, this characterization implies a characterization of the semifilters coherent to $\mathfrak{Fr}^{\perp}$: *A semifilter $\mathcal{F}$ is coherent to $\mathfrak{Fr}^{\perp}$* if and only if *$\mathcal{F}$ is comeager in $\mathcal{P}(\omega)$* if and only if *$\mathcal{F}^{\perp}$ is meager* if and only if *$\varphi(\mathcal{F}) = \mathfrak{Fr}^{\perp}$ for some monotone surjection $\varphi : \omega \to \omega$.*

A semifilter $\mathcal{F}$ is defined to be *bi-Baire* if both $\mathcal{F}$ and $\mathcal{F}^{\perp}$ are Baire, equivalently, if $\mathfrak{Fr} \not\approx \mathcal{F} \not\approx \mathfrak{Fr}^{\perp}$. Each maximal linked semifilter (in particular, each ultrafilter) is bi-Baire. Bi-Baire semifilters fail to have the Baire property in $\mathcal{P}(\omega)$. Consequently, each semifilter $\mathcal{F}$ which is Borel or analytic as a subspace of $\mathcal{P}(\omega)$ is coherent either to $\mathfrak{Fr}$ or to $\mathfrak{Fr}^{\perp}$.

Looking at Talagrand's characterizations one can see that in some cases the (sub)coherence relation can be expressed via monotone surjections (which are finite-to-one functions). This lead us to the notion of strict subcoherence.

9. The strict subcoherence and regularity of semifilters

We shall say that a semifilter $\mathcal{F}$ is *strictly subcoherent* to a semifilter $\mathcal{U}$ and denote this by $\mathcal{F} \sqsubseteq \mathcal{U}$ if for any monotone surjection $\varphi : \omega \to \omega$ there is a monotone surjection $\psi : \omega \to \omega$ such that $\psi \circ \varphi(\mathcal{F}) \subset \psi \circ \varphi(\mathcal{U})$. The strict subcoherence is the strongest among many possible definitions of subcoherence. Like the subcoherence relation, the strict subcoherence is a preorder on SF.

It is clear that $\mathcal{F} \sqsubseteq \mathcal{U}$ implies $\mathcal{F} \Subset \mathcal{U}$. If the converse happens, then we say that a semifilter is regular. More precisely, we define a semifilter $\mathcal{F}$ to be *regular* if for any semifilter $\mathcal{U}$ the relation $\mathcal{F} \Subset \mathcal{U}$ (resp. $\mathcal{U} \Subset \mathcal{F}$) is equivalent to $\mathcal{F} \sqsubseteq \mathcal{U}$ (resp. $\mathcal{U} \sqsubseteq \mathcal{F}$). The class of regular semifilters is quite wide and includes many important semifilters. Namely, a semifilter $\mathcal{F}$ is regular if $\mathcal{F}$ satisfies one of the conditions: (i) $\mathcal{F}$ is meager or comeager; (ii) $\mathcal{F}^\perp$ is regular; (iii) $\mathcal{F}$ is coherent to a regular semifilter; (iv) $\mathcal{F}$ has non-meager support $\mathrm{supp}(\mathcal{F})$; (v) $\mathcal{F}$ is coherent to a filter.

One may suggest that each semifilter is regular. However this is not so: under the assumption ($\mathfrak{r} = \mathfrak{c}$) there are coherent centered semifilters $\mathcal{F}, \mathcal{U}$ such that $\mathcal{F} \Subset \mathcal{U}$ but $\mathcal{F} \not\subseteq \mathcal{U}$. Here $\mathfrak{r}$, the *reaping number*, is the smallest size $|\mathcal{R}|$ of an unsplit family $\mathcal{R} \subset [\omega]^\omega$ (which means that for any subset $A \subset \omega$ there is $R \in \mathcal{R}$ such that either $R \subset^* A$ or $R \subset^* \omega \setminus A$).

We do not know if any maximal linked semifilter is regular. This would be so if any maximal linked semifilter would have non-meager support. However, under $\mathfrak{r} = \mathfrak{c}$ there are maximal linked semifilters with meager support, see Theorem 5.5.9 of [6].

10. The coherence lattice [SF]

Taking the quotient set [SF] of SF by the coherence equivalence $\asymp$ we arrive to an extremely interesting object called the coherence lattice. Its elements are the coherence classes $[\mathcal{F}] = \{\mathcal{U} \in \mathsf{SF} : \mathcal{U} \asymp \mathcal{F}\}$ of semifilters $\mathcal{F} \in \mathsf{SF}$.

To introduce lattice operations on [SF] we remark that the (sub)coherence relation nicely agrees with the algebraic structure of the lattice SF. Namely, for semifilters $\mathcal{F}_1 \Subset \mathcal{F}_2$ and $\mathcal{U}_1 \Subset \mathcal{U}_2$ we get $\mathcal{F}_1^\perp \Supset \mathcal{F}_2^\perp$, $\mathcal{F}_1 \cup \mathcal{U}_1 \Subset \mathcal{F}_2 \cup \mathcal{U}_2$ and $\mathcal{F}_1 \cap \mathcal{U}_1 \Subset \mathcal{F}_2 \cap \mathcal{U}_2$, see Proposition 5.1.5 of [6]. Thus $\asymp$ is a congruence on [SF] which allows us to introduce the partial order and the lattice operations on [SF] in a standard way:

- $[\mathcal{F}] \vee [\mathcal{U}] = [\mathcal{F} \cup \mathcal{U}]$;
- $[\mathcal{F}] \wedge [\mathcal{U}] = [\mathcal{F} \cap \mathcal{U}]$;
- $[\mathcal{F}]^\perp = [\mathcal{F}^\perp]$;
- $[\mathcal{F}] \le [\mathcal{U}]$ if and only if $\mathcal{F} \Subset \mathcal{U}$.

Being a quotient lattice of the distributive lattice SF, the coherence lattice [SF] is distributive. The smallest element of this lattice is the coherence class $[\mathfrak{Fr}]$

of the Fréchet filter while the largest element is $[\mathfrak{Fr}^\perp]$, the coherence class of the co-Fréchet semifilter. Besides these extreme elements the coherence lattice contains the coherence class $[\mathcal{U}]$ of an ultrafilter. This is all the information on the structure of the coherence lattice that can be proved in ZFC: it is consistent that $|[\mathsf{SF}]| = 3$. On the other hand, it is also consistent that this lattice contains an isomorphic copy of SF and thus has $2^{\mathfrak{c}}$ elements. Thus, like a set-theoretic chameleon the coherence lattice $[\mathsf{SF}]$ changes its properties depending on additional set-theoretic assumptions.

The embedding of the lattice SF into $[\mathsf{SF}]$ is constructed with the help of the limit operator $\lim^{(\mathcal{U}_n)} : \mathsf{SF} \to \mathsf{SF}$ composed with the quotient map $q : \mathsf{SF} \to [\mathsf{SF}]$. Such a composition $q \circ \lim^{(\mathcal{U}_n)} : \mathsf{SF} \to [\mathsf{SF}]$ is injective if the sequence of ultrafilters $(\mathcal{U}_n)$ is *totally separated* in the sense that for any finite-to-one map $\varphi : \omega \to \omega$ the sequence of ultrafilters $(\varphi(\mathcal{U}_n))_{n\in\omega}$ is separated or, equivalently, discrete in UF. Therefore, for a totally separated sequence $(\mathcal{U}_n)$ of ultrafilters the map $h = [\lim^{(\mathcal{U}_n)}] : \mathsf{SF} \to [\mathsf{SF}]$ is an injective lattice homomorphism preserving the transversality operation and such that $h(\mathsf{X}) = [\mathsf{X}]$ and $h(\mathsf{SF} \setminus \mathsf{X}) = [\mathsf{SF}] \setminus [\mathsf{X}]$ for any $\mathsf{X} \in \{\mathsf{UF}, \mathsf{ML}, \mathsf{FF}, \mathsf{CEN}, \mathsf{L}_k : k \geq 2\}$. Constructing totally separated sequences of ultrafilters is a rather non-trivial task and is connected with some topological properties of the lattice $[\mathsf{SF}]$.

11. Topologizing the coherence lattice $[\mathsf{SF}]$

The natural idea to topologize the lattice $[\mathsf{SF}]$ with the quotient topology fails because each coherence class $[\mathcal{F}]$ is dense in SF and thus the quotient topology on $[\mathsf{SF}]$ is antidiscrete. Nonetheless, the coherence classes are closed with respect to a stronger topology on SF, coinciding with the Lawson topology on all subsets of size $< \mathfrak{b}$, where $\mathfrak{b}$, the $\mathfrak{b}$ounding number, is the smallest size of an unbounded subset in $(\mathbb{N}^\omega, \leq^*)$. This topology on SF consists of so-called $\mathfrak{b}_<$-open sets, where a subset U of a topological space X is called $\kappa_<$*-open* if for each subset $C \subset X$ of size $|C| < \kappa$ the intersection $C \cap U$ is relatively open in C. Now for any uncountable regular cardinal $\kappa \leq \mathfrak{b}$ we can consider the topology $[\tau_\kappa]$ on $[\mathsf{SF}]$ consisting of subsets $U \subset [\mathsf{SF}]$ whose preimages $q^{-1}(U)$ under the quotient map are $\kappa_<$-open in SF. It turns out that $[\tau_\kappa]$ is a well-defined T_1-

topology on $[\mathsf{SF}]$ which is $\kappa_<$-*tight* (in the sense that for any subset $A \subset [\mathsf{SF}]$ and any point a in the closure of A there is a subset $C \subset A$ of size $|C| < \kappa$ whose closure contains a) and $\kappa_<$-*bounded* (in the sense that for any subset $C \subset [\mathsf{SF}]$ of size $|C| < \kappa$ and any open cover $\mathcal{U}$ of $[\mathsf{SF}]$ some finite subfamily of $\mathcal{U}$ covers the set C). Among the κ-topologies $[\tau_\kappa]$ on $[\mathsf{SF}]$ the most interesting are extremal ones for $\kappa = \aleph_1$ and $\kappa = \mathfrak{b}$: the topology $[\tau_\mathfrak{b}]$ is the closest to being compact while $[\tau_{\omega_1}]$ is the closest to being Hausdorff. However both topologies coincide on countable subspaces of $[\mathsf{SF}]$.

Returning to totally separated sequences of ultrafilters, one can show that a sequence of ultrafilters $(\mathcal{U}_n)_{n\in\omega}$ is totally separated if and only if the sequence $([\mathcal{U}_n])_{n\in\omega}$ of their coherence classes is discrete in $([\mathsf{SF}], [\tau_{\omega_1}])$, see Proposition 6.4.4 of [6]. Now, if the space $([\mathsf{SF}], [\tau_{\omega_1}])$ were Hausdorff, then the existence of a totally separated sequence of ultrafilters would be equivalent to the infinity of the set $[\mathsf{UF}] = \{[\mathcal{U}] : \mathcal{U} \in \mathsf{UF}\}$ of coherence classes of ultrafilters. Unfortunately, the space $[\mathsf{SF}]$ does not look to be Hausdorff. Nonetheless, using the powerful machinery of cardinal characteristics on the lattice $[\mathsf{SF}]$, one can prove the above equivalence, implying a striking Finite-$2^\mathfrak{c}$ Dichotomy: the size of the set $[\mathsf{UF}]$ either is finite or $2^\mathfrak{c}$. Unfortunately, we do not know if the same dichotomy is true for the lattice $[\mathsf{SF}]$.

12. Algebraic operations on $[\mathsf{SF}]$

As we already know each binary (associative) operation $* : \omega \times \omega \to \omega$ with finite-to-one left shifts induces an (associative) binary operation $\circ : \mathsf{SF} \times \mathsf{SF} \to \mathsf{SF}$ with continuous right shifts. It turns out that the coherence relation $\asymp$ is a congruence on $(\mathsf{SF}, \circ)$ and thus we can define the quotient operation $\bullet$ on the coherence lattice $[\mathsf{SF}]$. However this quotient operation is not interesting: $([\mathsf{SF}], \bullet)$ is a semigroup of right zeros. This follows from the coherence $\mathcal{F} \circ \mathcal{U} \asymp \mathcal{U}$ holding for any semifilters $\mathcal{F}, \mathcal{U}$. This simple fact allows us two make two observations on the structure of the semigroup $(\mathsf{SF}, \circ)$: (1) non-coherent semifilters cannot commute and (2) each left ideal in $(\mathsf{SF}, \circ)$ meets each coherence class and thus is rather large.

13. Cardinal characteristics of semifilters: general theory

An important role in the theory belongs to the cardinal characteristics of semifilters, that is cardinal-valued functions on the lattice SF. Cardinal characteristics carry a valuable information on a semifilter and in some cases allow to identify the semifilter up to the coherence. Before considering some concrete cardinal functions on SF we develop their general theory in order to give an abstract idea of exploiting cardinal characteristics for studying the (sub)coherence relation.

Cardinal functions on the lattice SF can have additional algebraic properties. Namely, we define a cardinal function $\xi(-)$ on SF to be

- $\subset$-*monotone* if $\xi(\mathcal{F}) \le \xi(\mathcal{U})$ for any semifilters $\mathcal{F} \subset \mathcal{U}$;
- a $\cup$-*homomorphism* if $\xi(\mathcal{F} \cup \mathcal{U}) = \max\{\xi(\mathcal{F}), \xi(\mathcal{U})\}$ for any semifilters $\mathcal{F}, \mathcal{U}$;
- a $\cap$-*homomorphism* if $\xi(\mathcal{F} \cap \mathcal{U}) = \min\{\xi(\mathcal{F}), \xi(\mathcal{U})\}$ for any semifilters $\mathcal{F}, \mathcal{U}$;
- a *lattice homomorphism* if ξ is both a $\cup$- and a $\cap$-homomorphism;
- a $\cap_{<\kappa}$-*homomorphism* (resp. $\cup_{<\kappa}$-homomorphism) if $\xi(\cap\mathsf{F}) = \min\{\xi(\mathcal{F}) : \mathcal{F} \in \mathsf{F}\}$ (resp. $\xi(\cup\mathsf{F}) = \sup\{\xi(\mathcal{F}) : \mathcal{F} \in \mathsf{F}\}$) for any family F of semifilters with $|\mathsf{F}| < \kappa$;
- $\Subset$-*monotone* if $\xi(\mathcal{F}) \le \xi(\mathcal{U})$ for any semifilters $\mathcal{F} \Subset \mathcal{U}$;
- $\asymp$-*invariant* if $\xi(\mathcal{F}) = \xi(\mathcal{U})$ for any coherent semifilters $\mathcal{F} \asymp \mathcal{U}$.

It is easy to see that a cardinal function $\xi(-)$ is $\subset$-monotone if it is a $\cap$- or a $\cup$-homomorphism. Also $\xi(-)$ is $\Subset$-monotone if and only if it is $\subset$-monotone and $\asymp$-invariant.

In the sequel we shall be mostly interested in $\asymp$-invariant cardinal characteristics on SF because every $\asymp$-invariant cardinal function $\xi(-)$ induces a cardinal function $\xi[-]$ on the coherence lattice $[\mathsf{SF}]$. So working with $\asymp$-invariant cardinal functions we shall often write $\xi[\mathcal{F}]$ in place of $\xi(\mathcal{F})$ to stress that the value of $\xi(\mathcal{F})$ does not depend on the choice of a particular semifilter from the coherence class $[\mathcal{F}]$. If an $\asymp$-invariant cardinal characteristic $\xi(-)$ is a $\cup_{<\kappa}$-homomorphism (resp. a $\cap_{<\kappa}$-homomorphism) for some uncountable regular cardinal $\kappa \le \mathfrak{b}$, then the induced cardinal characteristic $\xi[-]$ on $[\mathsf{SF}]$ is upper (resp. lower) semicontinuous in the sense that for any cardinal λ the set $\{[\mathcal{F}] \in [\mathsf{SF}] : \xi[\mathcal{F}] > \lambda\}$ (resp. $\{[\mathcal{F}] \in [\mathsf{SF}] : \xi[\mathcal{F}] < \lambda\}$) is open in the κ-topology $[\tau_\kappa]$ on $[\mathsf{SF}]$.

Given a cardinal function $\xi(-)$ on SF there are (at least) three ways to produce an $\asymp$-invariant cardinal function:

- $\xi_{[\mathcal{F}]} = \min\{\xi(\mathcal{U}) : \mathcal{U} \in [\mathcal{F}]\}$, the *minimization*,
- $\xi^{[\mathcal{F}]} = \sup\{\xi(\mathcal{U}) : \mathcal{U} \in [\mathcal{F}]\}$, the *supremization*, and
- $\hat{\xi}[\mathcal{F}] = \min\{\sup_{\mathcal{S}\in\mathsf{SF}} \xi(\mathcal{S}), \xi(\mathcal{U}) : \mathcal{U} \notin \mathcal{F}\}$, the *nonification* of $\xi(-)$.

Both the minimization $\xi_{[-]}$ and supremization $\xi^{[-]}$ of a $\cup$-homomorphism $\xi(-)$ on SF lead to $\asymp$-invariant $\cup$-homomorphisms, while the nonification of any cardinal function $\xi(-)$ on SF yields a $\Subset$-monotone $\cap_{<\mathfrak{b}}$-homomorphism $\hat{\xi}[-]$ on SF.

All applications of cardinal functions to studying the subcoherence relation are based on the following simple fact (actually, a tautology): a semifilter $\mathcal{F}$ is subcoherent to a semifilter $\mathcal{U}$ if $\xi_{[\mathcal{F}]} < \hat{\xi}[\mathcal{U}]$ for some cardinal function $\xi(-)$ on SF.

This simple observation will be used to detect the coherence of all semifilters belonging to some fixed class $\mathsf{F} \subset \mathsf{SF}$ of semifilters. Very often this is equivalent to the strict inequality between suitable small cardinals. As a rule these cardinals are the smallest or largest possible values of suitable cardinal characteristics on the class F. Given a cardinal characteristic $\xi(-)$ on SF and a class of semifilters $\mathsf{F} \subset \mathsf{SF}$ let

$$\xi_{\mathsf{F}} = \min\big\{\xi(\mathcal{F}) : \mathcal{F} \in \mathsf{F}\big\},$$
$$\xi^{\mathsf{F}} = \sup\big\{\xi(\mathcal{F}) : \mathcal{F} \in \mathsf{F}\big\}$$

be the critical values of $\xi(-)$ on F. Observe that the minimization $\xi_{[\mathcal{F}]}$ and the supremization $\xi^{[\mathcal{F}]}$ are just critical values of $\xi(-)$ on the coherence class $[\mathcal{F}]$ of a semifilter $\mathcal{F}$.

In case of strict inequality $\xi_{\mathsf{F}} < \hat{\xi}^{\mathsf{F}}$ the coherence of all semifilters of a class $\mathsf{F} \subset \mathsf{SF}$ can be characterized very easily: the latter happens if and only if $\xi^{\mathsf{F}} < \hat{\xi}_{\mathsf{F}}$.

This characterization will be applied to establish the coherence of all semifilters belonging to some concrete classes. Among such concrete classes the most important classes are:

– the class BS of bi-Baire semifilters (that is semifilters $\mathcal{F} \notin [\mathfrak{F}r] \cup [\mathfrak{F}r^{\perp}]$);

– the class BF of Baire filters (that is, filters $\mathcal{F} \notin [\mathfrak{F}r]$);

– the class ML of maximal linked semifilters;

– the class UF of ultrafilters.

It is clear that $\mathsf{UF} \subset \mathsf{ML} \cap \mathsf{BF} \subset \mathsf{ML} \cup \mathsf{BF} \subset \mathsf{BS}$.

Given a cardinal characteristic ξ on SF we shall write ξ_b, ξ_f, ξ_u, ξ_l in place of ξ_{BS}, ξ_{BF}, ξ_{UF}, ξ_{ML}, respectively. Also we write ξ^b, ξ^f, ξ^u, ξ^l in place of ξ^{BS}, ξ^{BF}, ξ^{UF}, ξ^{ML}, respectively.

The following diagram describes the interplay between these cardinals.

$$\begin{tikzcd}
& \xi^b \arrow[dl, dash] \arrow[dr, dash] & \\
\xi^f \arrow[dr, dash] & & \xi^l \arrow[dl, dash] \\
& \xi^u \arrow[d, dash] & \\
& \xi_u \arrow[dl, dash] \arrow[dr, dash] & \\
\xi_f \arrow[dr, dash] & & \xi_l \arrow[dl, dash] \\
& \xi_b &
\end{tikzcd}$$

Among these cardinals the critical values ξ_l and ξ^l occupy a special place because of the following two

Polarization Formulas. Let $\xi(-)$ be a $\subset$-monotone cardinal function on SF. Then for any semifilter $\mathcal{F}$

$$\min\left\{\xi_{[\mathcal{F}]}, \hat{\xi}[\mathcal{F}^{\perp}]\right\} \le \xi_l \qquad \text{and} \qquad \max\left\{\xi_{[\mathcal{F}]}, \hat{\xi}[\mathcal{F}^{\perp}]\right\} \ge \hat{\xi}^l.$$

The critical values ξ_l and $\hat{\xi}^l$ have the following extremal property:

– $\xi_l = \min\big\{\max\{\xi(\mathcal{F}),\xi(\mathcal{F}^\perp)\} : \mathcal{F}\in\mathsf{SF}\big\}$ if $\xi(-)$ is a $\cup$-homomorphism;

– $\xi^l = \sup\big\{\min\{\xi(\mathcal{F}),\xi(\mathcal{F}^\perp)\} : \mathcal{F}\in\mathsf{SF}\big\}$ if $\xi(-)$ is a $\cap$-homomorphism.

This extremal property of ξ_l and ξ^l can be easily derived from the approximation property of maximal linked semifilters (asserting that for any semifilter $\mathcal{F}$ there is a maximal linked semifilter $\mathcal{L}$ with $\mathcal{F}\cap\mathcal{F}^\perp\subset\mathcal{L}\subset\mathcal{F}\cup\mathcal{F}^\perp$).

In light of the extremal properties of the cardinals ξ_l and ξ^l it is natural to call a semifilter $\mathcal{F}$

– ξ-*minimal* if $\max\{\xi(\mathcal{F}),\xi(\mathcal{F}^\perp)\}\le\xi_l$;

– ξ-*maximal* if $\min\{\xi(\mathcal{F}),\xi(\mathcal{F}^\perp)\}\ge\xi^l$.

It turns out that under the assumption ($\xi_l<\hat{\xi}^l$) ξ-minimal and $\hat{\xi}$-maximal semifilters are unique up to coherence and can characterized as follows, see Theorem 7.4.5 in [6].

Theorem 13.1 (The First Fundamental Theorem). *If $\xi_l<\hat{\xi}^l$ for some $\asymp$-invariant cardinal function $\xi[-]$ on SF, then for any semifilter $\mathcal{F}$ the following conditions are equivalent:*

1. *$\mathcal{F}$ is ξ-minimal;*
2. *$\mathcal{F}$ is $\hat{\xi}$-maximal;*
3. $\max\{\xi[\mathcal{F}],\xi[\mathcal{F}^\perp]\}<\hat{\xi}^l$;
4. $\min\{\hat{\xi}[\mathcal{F}],\hat{\xi}[\mathcal{F}^\perp]\}>\xi_l$.

Moreover any two semifilters satisfying the conditions 1 – 4 are coherent.

In fact, for an $\asymp$-invariant $\cup$-homomorphism $\xi[-]$ on SF the strict inequality $\xi_l<\hat{\xi}^l$ is equivalent to the coherence of all ξ-minimal semifilters, see Theorem 7.4.6 of [6].

The First Fundamental Theorem implies that under $\xi_l<\hat{\xi}^l$ all the semifilters from a class $\mathsf{F}\subset\mathsf{SF}$ containing a ξ-minimal semifilter are coherent if and only if $\max\{\xi[\mathcal{F}],\xi[\mathcal{F}^\perp]\}<\hat{\xi}^l$ for any semifilter $\mathcal{F}\in\mathsf{F}$ if and only if $\min\{\hat{\xi}[\mathcal{F}],\hat{\xi}[\mathcal{F}^\perp]\}>\xi_l$ for any $\mathcal{F}\in\mathsf{F}$.

Even more interesting situation appears under the inequality $\xi_l<\widehat{\hat{\xi}}^{\,l}$.

Theorem 13.2 (The Second Fundamental Theorem). *If* $\xi_l < \overset{\frown l}{\hat{\xi}}$ *for some cardinal function* $\xi(-)$ *on* SF*, then the lattice* SF *contains at most two noncoherent maximal linked semifilters. More precisely, a maximal linked semifilter* $\mathcal{L}$ *is coherent to*

- *a unique* ξ*-minimal semifilter if* $\hat{\xi}[\mathcal{L}] > \xi_l$;
- *a unique* $\hat{\xi}$*-minimal semifilter if* $\hat{\xi}[\mathcal{L}] < \overset{\frown l}{\hat{\xi}}$.

This theorem explains the nature of the striking result of Blass, Mildenberger [14] who proved that under $\mathfrak{r} < \mathfrak{s}$ there are at most two noncoherent ultrafilters. To derive the Blass-Mildenberger dichotomy from the Second Fundamental Theorem it suffices to find an $\asymp$-invariant cardinal characteristic $\xi[-]$ on SF with $\xi_l = \mathfrak{r}$ and $\overset{\frown l}{\hat{\xi}} \geq \mathfrak{s}$. It turns out that for such a cardinal characteristic it suffices to take the minimization $\pi\chi_{[-]}$ of the π-character $\pi\chi(-)$ of a semifilter. At this point we leave the general theory of cardinal characteristics and turn to their concrete representatives.

14. Cardinal characteristics of semifilters: the four levels of complexity

The cardinal characteristics of semifilters appearing in practice fall into four complexity categories:

1. Cardinal characteristics of semifilters determined by their inner structure (as a rule they are not $\asymp$-invariant);

2. $\asymp$-Invariant cardinal characteristics obtained after minimizations or supremizations of some cardinal characteristics of the first complexity level;

3. Cardinal characteristics obtained by nonifications of the cardinal characteristics of the second complexity level;

3'. Cardinal characteristics of some external objects determined by a semifilter, close by their properties to the cardinal characteristics of the third level;

4. Cardinal characteristics obtained after nonifications of the cardinal characteristics of the third complexity level.

15. Cardinal characteristics of the first complexity level

On the first complexity level we shall encounter 7 cardinal characteristics of a semifilter $\mathcal{F}$:

- $\chi(\mathcal{F}) = \min\{|\mathcal{B}| : \mathcal{B} \subset \mathcal{F}\ \forall F \in \mathcal{F}\ \exists B \in \mathcal{B}\ [B \subset^* F]\}$, the *character*, equal to the smallest size of a base for $\mathcal{F}$;
- $\pi\chi(\mathcal{F}) = \min\{|\mathcal{B}| : \mathcal{B} \subset [\omega]^\omega\ \forall F \in \mathcal{F}\ \exists B \in \mathcal{B}\ [B \subset^* F]\}$, the *$\pi$-character*, equal to the smallest size of a π-base for $\mathcal{F}$;
- $\mathfrak{p}(\mathcal{F}) = \min\{\mathfrak{c}^+, |\mathcal{B}| : \mathcal{B} \subset \mathcal{F}\ \forall F \in \mathcal{F}\ \exists B \in \mathcal{B}\ [F \not\subset^* B]\}$, the *filter number*, equal to the smallest size of a subfamily $\mathcal{B} \subset \mathcal{F}$ having no infinite pseudointersection in $\mathcal{F}$ or $\mathfrak{c}^+$ if $\mathcal{F}$ has an infinite pseudointersection in $\mathcal{F}$;
- $\pi\mathfrak{p}(\mathcal{F}) = \min\{\mathfrak{c}^+, |\mathcal{B}| : \mathcal{B} \subset \mathcal{F}\ \forall F \in [\omega]^\omega\ \exists B \in \mathcal{B}\ [F \not\subset^* B]\}$, the *linkedness number*, equal to the smallest size of a subfamily $\mathcal{B} \subset \mathcal{F}$ having no infinite pseudointersection, or $\mathfrak{c}^+$ if $\mathcal{F}$ has an infinite pseudointersection;
- the *tower number* $\mathfrak{t}(\mathcal{F})$ is the smallest length λ of a $\subset^*$-decreasing sequence $(T_\alpha)_{\alpha<\lambda} \subset \mathcal{F}$ having no pseudointersection in $\mathcal{F}$; if no such a sequence exists, we put $\mathfrak{t}(\mathcal{F}) = \mathfrak{c}^+$;
- the *unilink number* $\mathrm{ul}(\mathcal{F})$ equal to the minimal size $|\mathsf{L}|$ of a family L of linked semifilters with $\mathcal{F} \subset \cup\mathsf{L}$;
- the *almost disjointness number* $\mathrm{ad}(\mathcal{F}) = \sup\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{F}$ is almost disjoint$\}$, where a family $\mathcal{A}$ is *almost disjoint* if any two distinct members $A, B \in \mathcal{A}$ have finite intersection.

Let us note that $\mathfrak{p}(\mathcal{F})$ and $\chi(\mathcal{F})$ as well as $\pi\mathfrak{p}(\mathcal{F})$ and $\pi\chi(\mathcal{F})$ form two dual pairs. Observe also that the filter and linkedness numbers $\mathfrak{p}(\mathcal{F})$ and $\pi\mathfrak{p}(\mathcal{F})$ allow to estimate the linkedness level of linked semifilters while the unilink and almost disjointness numbers $\mathrm{ul}(\mathcal{F})$ and $\mathrm{ad}(\mathcal{F})$ help to classify non-linked semifilters.

For each semifilter $\mathcal{F}$ these cardinal characteristics are related as follows:

$$\mathrm{ad}(\mathcal{F}) \le \mathrm{ul}(\mathcal{F}) \le \pi\chi(\mathcal{F}) \le \chi(\mathcal{F}) \quad \text{and} \quad \mathfrak{p}(\mathcal{F}) \le \min\{\mathfrak{t}(\mathcal{F}), \pi\mathfrak{p}(\mathcal{F})\}.$$

For any filter $\mathfrak{p}(\mathcal{F}) = \mathfrak{t}(\mathcal{F})$. The equality $\chi(\mathcal{F}) = 1$, equivalent to $\mathfrak{p}(\mathcal{F}) = \mathfrak{c}^+$, means that $\mathcal{F}$ is generated by some infinite set $B \subset \omega$ in the sense that $\mathcal{F} = \{F \in [\omega]^\omega : B \subset^* F\}$. On the other hand, the equality $\pi\chi(\mathcal{F}) = 1$, equivalent to $\pi\mathfrak{p}(\mathcal{F}) = \mathfrak{c}^+$, means that $\mathcal{F}$ has an infinite pseudointersection $B \subset \omega$ in the sense that $B \subset^* F$ for each $F \in \mathcal{F}$.

For the extremal semifilters $\mathfrak{Fr}, \mathfrak{Fr}^\perp$ these cardinal characteristics take extremal values:

$$\mathrm{ad}(\mathfrak{Fr}) = \mathrm{ul}(\mathfrak{Fr}) = \pi\chi(\mathfrak{Fr}) = \chi(\mathfrak{Fr}) = 1, \quad \mathfrak{t}(\mathfrak{Fr}) = \mathfrak{p}(\mathfrak{Fr}) = \pi\mathfrak{p}(\mathfrak{Fr}) = \mathfrak{c}^+$$

and

$$\mathrm{ad}(\mathfrak{Fr}^\perp) = \mathrm{ul}(\mathfrak{Fr}^\perp) = \pi\chi(\mathfrak{Fr}^\perp) = \chi(\mathfrak{Fr}^\perp) = \mathfrak{c},$$

$$\mathfrak{p}(\mathfrak{Fr}^\perp) = \pi\mathfrak{p}(\mathfrak{Fr}^\perp) = 2, \quad \mathfrak{t}(\mathfrak{Fr}^\perp) = \mathfrak{t}.$$

Among these seven cardinal characteristics the π-character $\pi\chi(-)$ and the linkedness number $\pi\mathfrak{p}(-)$ are the most important. The first of them is $\subset$-increasing while the second is a $\subset$-decreasing cardinal function on SF. More precisely, we shall be interested in the minimization $\pi\chi_{[-]}$ of the π-character and the supremization $\pi\mathfrak{p}^{[-]}$ of the linkedness number $\pi\mathfrak{p}(-)$. Thus we obtain two $\asymp$-invariant cardinal functions $\pi\chi_{[-]}$ and $\pi\mathfrak{p}^{[-]}$ having parallel properties. We start with the cardinal characteristic $\pi\chi_{[-]}$.

16. The minimization $\pi\chi[-]$ of the π-character

For typographical reasons, given a semifilter $\mathcal{F}$ we shall write $\pi\chi[\mathcal{F}]$ instead of $\pi\chi_{[\mathcal{F}]}$, where $\pi\chi_{[\mathcal{F}]} = \min\{\pi\chi(\mathcal{U}) : \mathcal{U} \in [\mathcal{F}]\}$. After the minimization, the π-character becomes a $\Subset$-monotone $\cup$-homomorphism $\pi\chi[-]$ on SF. On the extremal semifilters $\mathfrak{Fr}$ and $\mathfrak{Fr}^\perp$ the cardinal function $\pi\chi[-]$ takes its extremal values: $\pi\chi[\mathfrak{Fr}] = 1$ and $\pi\chi[\mathfrak{Fr}^\perp] = \mathrm{ul}_{[\mathfrak{Fr}^\perp]} = \mathrm{ad}_{[\mathfrak{Fr}^\perp]} = \mathfrak{c}$.

As an application of the equality $\mathrm{ul}_{[\mathfrak{Fr}^\perp]} = \mathfrak{c}$ we obtain a simple proof of Plewik's Theorem [33] stating that the intersection $\mathcal{F} = \bigcap_{\alpha<\kappa} \mathcal{U}_\alpha$ of $\kappa < \mathfrak{c}$ ultrafilters is non-meager. Indeed, assuming that $\mathcal{F}$ is meager, we will get that its dual $\mathcal{F}^\perp = \bigcup_{\alpha<\kappa} \mathcal{U}_\alpha^\perp$ is comeager and has $\mathrm{ul}(\mathcal{F}^\perp) \le \kappa < \mathfrak{c}$, which contradicts $\mathrm{ul}_{[\mathfrak{Fr}^\perp]} = \mathfrak{c}$.

The cardinal characteristic $\pi\chi[-]$ is a $\cup_{<\mathfrak{b}}$-homomorphism while $\min\{\mathfrak{d}, \pi\chi[-]\}$ is a $\cap_{\le\omega}$-homomorphism on SF. Thus $\min\{\mathfrak{d}, \pi\chi[-]\}$ is continuous with respect to the ω_1-topology $[\tau_{\omega_1}]$ on $[\mathsf{SF}]$.

The critical values of the cardinal function $\pi\chi[-]$ coincide with the two classical small cardinals $\mathfrak{b}$ and $\mathfrak{r}$. More precisely, $\pi\chi_b = \pi\chi_f = \mathfrak{b}$ and $\pi\chi_l = \pi\chi_u = \mathfrak{r}$. The equality $\pi\chi_b = \pi\chi_f = \mathfrak{b}$ is the combined result of Solomon [40] (who proved that each semifilter $\mathcal{F}$ with $\pi\chi(\mathcal{F}) < \mathfrak{b}$ is meager) and P. Simon (who constructed a non-meager filter $\mathcal{F}$ with $\pi\chi(\mathcal{F}) = \mathfrak{b}$). The quality $\pi\chi_l = \pi\chi_u = \mathfrak{r}$ follows from the inequality $\pi\chi[\mathcal{U}] \ge \mathfrak{r}$ held for any unsplit (in particular, any maximal linked) semifilter and the existence of an ultrafilter $\mathcal{U}$ with $\pi\chi(\mathcal{U}) = \mathfrak{r}$ (the latter was proved by Balcar, Simon [2]). Actually a more general result is true: *each filter $\mathcal{F}$ can be enlarged to a ultrafilter $\mathcal{U}$ with* $\pi\chi(\mathcal{U}) \le \max\{\chi(\mathcal{F}), \mathfrak{r}\}$, see Theorem 8.4.4 of [6].

In contrast to the lower critical values $\pi\chi_b = \pi\chi_f$, $\pi\chi_l = \pi\chi_u$ the critical values $\pi\chi^b, \pi\chi^f, \pi\chi^l, \pi\chi^u$ of the cardinal function $\pi\chi[-]$ are not so definite and depend on additional set-theoretic assumptions. For example, the inequality $\pi\chi^u < \mathfrak{d}$ is equivalent to NCF, see Theorem 12.3.3 of [6].

There is an interesting interplay between $\pi\chi[\mathcal{F}]$ from one side and $\pi\chi[\mathcal{F}^\perp]$ and $\mathrm{ad}[\mathcal{F}^\perp]$ from the other. Namely, we have two Polarization Formulas:

$$\max\left\{\pi\chi[\mathcal{F}], \pi\chi[\mathcal{F}^\perp]\right\} \ge \mathfrak{r} \quad \text{and} \quad \max\left\{\pi\chi[\mathcal{F}], \mathrm{ad}[\mathcal{F}^\perp]\right\} \ge \min\{\mathfrak{r}, \mathfrak{d}\}$$

holding for any semifilter $\mathcal{F}$. The latter polarization formula is a motivation for introducing versions $\mathfrak{r}_\kappa$ and $\mathfrak{u}_\kappa$ of the classical small cardinals $\mathfrak{r}$ and $\mathfrak{u}$, parametrized by a cardinal κ:

$$\mathfrak{r}_\kappa = \min\left\{\pi\chi(\mathcal{F}) : \mathcal{F} \text{ is a semifilter with } \mathrm{ad}(\mathcal{F}^\perp) < \kappa\right\} \quad \text{and}$$

$$\mathfrak{u}_\kappa = \min\left\{\chi(\mathcal{F}) : \mathcal{F} \text{ is a filter with } \mathrm{ad}(\mathcal{F}^\perp) < \kappa\right\}.$$

The values of the cardinals $\mathfrak{r}_\kappa$ and $\mathfrak{u}_\kappa$ will not change if we replace the cardinal functions $\chi(-)$ and $\mathrm{ad}(-)$ by their minimizations $\chi[-] = \chi_{[-]}$ and $\mathrm{ad}[-] =$

$\mathrm{ad}_{[-]}$, so that

$$\mathfrak{r}_\kappa = \min\big\{\, \pi\chi[\mathcal{F}] : \mathcal{F} \text{ is a semifilter with } \mathrm{ad}[\mathcal{F}^\perp] < \kappa\big\};$$

$$\mathfrak{u}_\kappa = \min\big\{\chi[\mathcal{F}] : \mathcal{F} \text{ is a filter with } \mathrm{ad}[\mathcal{F}^\perp] < \kappa\big\},$$

see Proposition 8.8.2 of [6]. By Theorem 8.8.2 [6], $\min\{\mathfrak{r},\mathfrak{d}\} \le \mathfrak{r}_k \le \mathfrak{u}$ for each cardinal $\kappa \le \mathfrak{c}$. Studying the interplay between the cardinals $\mathfrak{r}_\kappa$ and $\mathfrak{u}_\kappa$ we prove the inequality $\min\{\mathfrak{u}_\kappa,\mathfrak{d}\} \le \mathfrak{r}_\kappa$, see Theorem 8.8.3 of [6]. For $\kappa = 2$ this inequality turns into the inequality $\min\{\mathfrak{u},\mathfrak{d}\} \le \mathfrak{r}$ discovered by Aubrey [1].

17. The function representation $\nearrow\mathcal{F}$ of a semifilter

It turns out that the value of the $\pi\chi$-character $\pi\chi[\mathcal{F}]$ of the coherence class $[\mathcal{F}]$ is coded in covering properties of the function representation $\nearrow\mathcal{F}$. By definition, the *function representation* of a semifilter $\mathcal{F}$ is the subset $\nearrow\mathcal{F} = \{\mathrm{next}_F : F \in \mathcal{F}\} \subset \omega^\omega$, where $\mathrm{next}_F : \omega \to \omega$, $\mathrm{next}_F : n \mapsto \min F \cap [n+1,\infty)$. In Theorem 8.5.3 of [6] it is proved that $\mathrm{bc}(\nearrow\mathcal{F}) = \min\{\pi\chi[\mathcal{F}],\mathfrak{d}\}$, where $\mathrm{bc}(X)$, the *bounded covering number* of a subset $X \subset \omega^\omega$ equals the smallest size $|D|$ of a set $D \subset \omega^\omega$ such that for each $x \in X$ there is $y \in D$ with $x \le^* y$. Let us also note that $\mathfrak{d}$ equals the bounded covering number of ω^ω. The equality $\mathrm{bc}(\nearrow\mathcal{F}) = \min\{\mathfrak{d},\pi\chi[\mathcal{F}]\}$ has a philosophical value since the definition of the cardinal characteristic $\pi\chi[\mathcal{F}]$ uses the subcoherence relation while $\mathrm{bc}(\nearrow\mathcal{F})$ does not.

The function representation $\nearrow\mathcal{F}$ is a powerful tool for proving coherence properties of semifilters with small $\pi\chi$-characters. In particular, using the function representation one can show that each semifilter $\mathcal{F}$ with $\pi\chi[\mathcal{F}] < \mathfrak{d}$ is subcoherent to a filter $\tilde{\mathcal{F}}$ with $\chi(\tilde{\mathcal{F}}) = \pi\chi[\mathcal{F}]$ and satisfies $\pi\chi[\mathcal{F}] \le \mathfrak{r}$. This implies that any maximal linked semifilter $\mathcal{L}$ with $\pi\chi[\mathcal{L}] < \mathfrak{d}$ is coherent to an ultrafilter and thus is regular.

The near coherence of semifilters also can be easily translated into dominating properties of their function representations. According to Theorem 8.5.6 of [6], two semifilters $\mathcal{F}$,$\mathcal{U}$ are nearly coherent if and only if the set

$$\max(\nearrow\mathcal{F},\nearrow\mathcal{U}) = \big\{\max(f,g) : f \in\nearrow\mathcal{F},\ g \in\nearrow\mathcal{U}\big\}$$

is not dominating in the poset $(\omega^\omega, \leq)$, where a subset D of the poset $(\omega^\omega, \leq)$ is *dominating* if for each $x \in \omega^\omega$ there is $y \in D$ with $x \leq y$.

This characterization implies that two semifilters $\mathcal{F}, \mathcal{U}$ are near coherent provided $\max\{\pi\chi[\mathcal{F}], \pi\chi[\mathcal{U}]\} < \mathfrak{d}$. For ultrafilters this result was proved by Blass [9].

18. The nonification $\widehat{\pi\chi}[-]$ of $\pi\chi[-]$

The latter near coherence condition can be used to estimate the nonification $\widehat{\pi\chi}[\mathcal{F}]$ of a semifilter $\mathcal{F}$ with $\pi\chi[\mathcal{F}^\perp] < \mathfrak{d}$. For such a semifilter $\mathcal{F}$ we will get $\widehat{\pi\chi}[\mathcal{F}] \geq \mathfrak{d}$. Indeed, any semifilter $\mathcal{S}$ with $\pi\chi[\mathcal{S}] < \mathfrak{d}$ is near coherent to $\mathcal{F}^\perp$ and hence subcoherent to $\mathcal{F}$. Thus we arrive to the polarization formula $\max\{\pi\chi[\mathcal{F}^\perp], \widehat{\pi\chi}[\mathcal{F}]\} \geq \mathfrak{d}$ holding for each semifilter $\mathcal{F}$. On the other hand, $\min\{\pi\chi[\mathcal{F}^\perp], \widehat{\pi\chi}[\mathcal{F}]\} \leq \mathfrak{r}$. Indeed, assuming that $\widehat{\pi\chi}[\mathcal{F}] > \mathfrak{r}$ and taking an ultrafilter $\mathcal{U}$ with $\pi\chi(\mathcal{U}) = \mathfrak{r} < \widehat{\pi\chi}[\mathcal{F}]$ we will get $\mathcal{U} \Subset \mathcal{F}$ and thus $\mathcal{F}^\perp \Subset \mathcal{U}^\perp = \mathcal{U}$ which yields $\pi\chi[\mathcal{F}^\perp] \leq \pi\chi[\mathcal{U}] = \mathfrak{r}$.

Under the assumption $\mathfrak{r} < \mathfrak{d}$ the inequality $\max\{\pi\chi[\mathcal{F}^\perp], \widehat{\pi\chi}[\mathcal{F}]\} \geq \mathfrak{d}$ implies that $\widehat{\pi\chi}[\mathcal{U}] \geq \mathfrak{d}$ for each ultrafilter $\mathcal{U}$ with $\pi\chi[\mathcal{U}] = \mathfrak{r}$, which allows us to prove that $\widehat{\pi\chi}^l = \mathfrak{d}$. Therefore the assumption $\mathfrak{r} < \mathfrak{d}$ implies $\pi\chi_l = \mathfrak{r} < \mathfrak{d} = \widehat{\pi\chi}^l$ and we can apply the First Fundamental Theorem to establish the uniqueness of $\pi\chi$-minimal semifilters. Following the general rule we define a semifilter $\mathcal{F}$ to be *$\pi\chi$-minimal* if $\max\{\pi\chi[\mathcal{F}], \pi\chi[\mathcal{F}^\perp]\} \leq \pi\chi_l = \mathfrak{r}$. It turns out that under $\mathfrak{r} < \mathfrak{d}$ all $\pi\chi$-minimal semifilters are coherent (in fact, the inequality $\mathfrak{r} < \mathfrak{d}$ is equivalent to the coherence of all $\pi\chi$-minimal ultrafilters). Moreover, a semifilter $\mathcal{F}$ is coherent to a $\pi\chi$-minimal ultrafilter if and only if $\max\{\pi\chi[\mathcal{F}], \pi\chi[\mathcal{F}^\perp]\} < \mathfrak{d}$ if and only if $\min\{\widehat{\pi\chi}[\mathcal{F}], \widehat{\pi\chi}[\mathcal{F}^\perp]\} > \mathfrak{r}$.

This result implies a characterization of NCF due to Blass [9]: NCF holds if and only if all ultrafilter are coherent if and only if each ultrafilter is $\pi\chi$-minimal if and only if $\pi\chi[\mathcal{U}] < \mathfrak{d}$ for any ultrafilter if and only if $\widehat{\pi\chi}[\mathcal{U}] > \mathfrak{r}$ for any ultrafilter $\mathcal{U}$. By analogy we can characterize the coherence of all semifilters from a given class $\mathsf{F} \subset \mathsf{SF}$ containing a $\pi\chi$-minimal semifilter, see Theorem 8.6.5 of [6].

19. The "Ideal" cardinal characteristics of semifilters

It turns out that the nonification $\widehat{\pi\chi}[-]$ (which belongs to the third complexity level in our hierarchy) can be thought as one of four cardinal characteristics $\mathrm{add}[-]$, $\mathrm{cov}[-]$, $\mathrm{non}[-]$, $\mathrm{cof}[-]$ having the "ideal" origin. The cardinal characteristics $\mathrm{add}(\mathcal{I})$, $\mathrm{cov}(\mathcal{I})$, $\mathrm{non}(\mathcal{I})$, $\mathrm{cof}(\mathcal{I})$ are defined for any family $\mathcal{I}$ of subsets of a set X with $\cup\mathcal{I} = X \notin \mathcal{I}$ as follows:

$$\mathrm{add}(\mathcal{I}) = \min\big\{|\mathcal{C}| : \mathcal{C} \subset \mathcal{I} \quad \cup\mathcal{C} \notin \mathcal{I}\big\},$$

$$\mathrm{non}(\mathcal{I}) = \min\big\{|A| : A \subset X \ A \notin \mathcal{I}\big\},$$

$$\mathrm{cov}(\mathcal{I}) = \min\big\{|\mathcal{C}| : \mathcal{C} \subset \mathcal{I} \quad \cup\mathcal{C} = X\big\},$$

$$\mathrm{cof}(\mathcal{I}) = \min\big\{|\mathcal{C}| : \mathcal{C} \subset \mathcal{I}\ \forall A \in \mathcal{I}\ \exists C \in \mathcal{C} \text{ with } A \subset C\big\}.$$

These cardinal characteristics are classical tools for studying various ideals $\mathcal{I}$, i.e., families of sets closed under unions and taking subsets. In fact, the small cardinals $\mathfrak{b}$ and $\mathfrak{d}$ are nothing else but the cardinal characteristics $\mathrm{add}(\mathcal{B}) = \mathrm{non}(\mathcal{B}) = \mathfrak{b}$ and $\mathrm{cov}(\mathcal{B}) = \mathrm{cof}(\mathcal{B}) = \mathfrak{d}$ of the ideal $\mathcal{B}$ of bounded subsets of the poset $(\omega^\omega, \le^*)$.

Each non-comeager semifilter $\mathcal{F}$ generates an ideal $\downarrow[\mathcal{F}] = \{\mathcal{U} \in \mathsf{SF} : \mathcal{U} \Subset \mathcal{F}\}$ in the set SF of all semifilters. So we can consider the cardinal characteristics of this ideal and think of them as cardinal characteristics of the semifilter $\mathcal{F}$. In fact, we go a bit further and introduce cardinal characteristics $\mathrm{add}(\mathsf{F})$, $\mathrm{non}(\mathsf{F})$, $\mathrm{cov}(\mathsf{F})$, and $\mathrm{cof}(\mathsf{F})$ of any family $\mathsf{F} \in \mathsf{SF}$ with $\cup\mathsf{F} = \mathfrak{Fr}^\perp \notin \mathsf{F}$. Given such a family F let $\downarrow\mathsf{F} = \{\mathcal{S} \subset \mathcal{P}(\omega) : \mathcal{S} \subset \mathcal{F} \text{ for some } \mathcal{F} \in \mathsf{F}\}$ and consider the cardinals

$$\mathrm{add}(\mathsf{F}) = \min\big\{|\mathsf{C}| : \mathsf{C} \subset \mathsf{F} \text{ and } \cup\mathsf{C} \notin \downarrow\mathsf{F}\big\};$$

$$\mathrm{cov}(\mathsf{F}) = \min\big\{|\mathsf{C}| : \mathsf{C} \subset \mathsf{F} \text{ and } \cup\mathsf{C} = \mathfrak{Fr}^\perp\big\};$$

$$\mathrm{cof}(\mathsf{F}) = \min\big\{|\mathsf{C}| : \mathsf{C} \subset \mathsf{F} \text{ and } \downarrow\mathsf{C} = \downarrow\mathsf{F}\big\}.$$

To define the cardinal $\mathrm{non}(\mathsf{F})$ note that infinite subsets of ω play the role of "points" in $\mathcal{P}([\omega]^\omega) \supset \mathsf{SF}$. So it is natural to put

$$\mathrm{non}(\mathsf{F}) = \min\big\{|\mathcal{B}| : \mathcal{B} \subset [\omega]^\omega \text{ is such that } \mathcal{B} \notin \downarrow\mathsf{F}\big\}.$$

The cardinal characteristics $\mathrm{add}(\mathsf{F})$, $\mathrm{cov}(\mathsf{F})$, $\mathrm{non}(\mathsf{F})$, $\mathrm{cof}(\mathsf{F})$ have duals defined for any family $\mathsf{F} \subset \mathsf{SF}$ with $\cap\mathsf{F} = \mathfrak{Fr} \notin \mathsf{F}$ as follows:

$$\begin{aligned}
\mathrm{add}^{\perp}(\mathsf{F}) &= \mathrm{add}(\mathsf{F}^{\perp}) = \min\big\{|\mathsf{C}| : \mathsf{C} \subset \mathsf{F},\ \cap\mathsf{C} \notin \uparrow \mathsf{F}\big\};\\
\mathrm{cov}^{\perp}(\mathsf{F}) &= \mathrm{cov}(\mathsf{F}^{\perp}) = \min\big\{|\mathsf{C}| : \mathsf{C} \subset \mathsf{F},\ \cap\mathsf{C} = \mathfrak{Fr}\big\};\\
\mathrm{cof}^{\perp}(\mathsf{F}) &= \mathrm{cof}(\mathsf{F}^{\perp}) = \min\big\{|\mathsf{C}| : \mathsf{C} \subset \mathsf{F}, \uparrow\mathsf{C} = \uparrow\mathsf{F}\big\};\\
\mathrm{non}^{\perp}(\mathsf{F}) &= \mathrm{non}(\mathsf{F}^{\perp}),
\end{aligned}$$

where $\mathsf{F}^{\perp} = \big\{\mathcal{F}^{\perp} : \mathcal{F} \in \mathsf{F}\big\}$ and $\uparrow\mathsf{F} = \big\{\mathcal{B} \subset \mathcal{P}(\omega) : \mathcal{B} \supset \mathcal{F}$ for some $\mathcal{F} \in \mathsf{F}\big\}$.

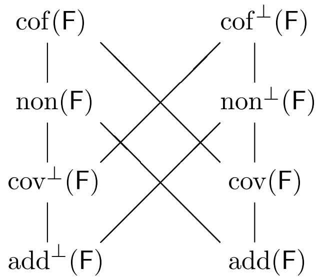
The cardinal characteristics of any family $\mathsf{F} \subset \mathsf{SF}$ with $\cup\mathsf{F} = \mathfrak{Fr}^{\perp} \notin \mathsf{F}$ are related as expected:

$$\mathrm{add}(\mathsf{F}) \le \min\big\{\mathrm{cov}(\mathsf{F}), \mathrm{cf}(\mathrm{non}(\mathsf{F})), \mathrm{cf}(\mathrm{cof}(\mathsf{F}))\big\}$$

and

$$\mathrm{cof}(\mathsf{F}) \ge \max\big\{\mathrm{non}(\mathsf{F}), \mathrm{cov}(\mathsf{F})\big\}.$$

There is also a non-expected inequality $\mathrm{cov}^{\perp}(\mathsf{F}) \le \mathrm{non}(\mathsf{F})$ holding for any $\asymp$-invariant family F of bi-Baire semifilter. The $\asymp$-invariantness of F means that $[\mathcal{F}] \subset \mathsf{F}$ for any semifilter $\mathcal{F} \in \mathsf{F}$. The interplay between the cardinal characteristics of such a family F is described by the following symmetric diagram:



The inequality $\mathrm{cov}^{\perp}(\mathsf{F}) \le \mathrm{non}(\mathsf{F})$ follows from the deep Theorem 9.2.5 of [6] asserting that for any $\asymp$-invariant family of semifilters $\mathsf{F} \subset \mathsf{SF} \setminus [\mathfrak{Fr}]$ a semifilter $\mathcal{F}$ with $\pi\chi[\mathcal{F}] < \mathrm{cov}^{\perp}(\mathsf{F})$ is strictly subcoherent to the intersection $\cap\mathsf{C}$ of any family $\mathsf{C} \subset \mathsf{F}$ with size $|\mathsf{C}| < \mathrm{cov}^{\perp}(\mathsf{F})$. This theorem implies also that for any $\asymp$-invariant family $\mathsf{F} \subset \mathsf{SF} \setminus [\mathfrak{F}r]$ containing a semifilter $\mathcal{F} \in \mathsf{F}$ with $\pi\chi[\mathcal{F}] < \mathrm{cov}^{\perp}(\mathsf{F})$ the cardinal $\mathrm{cov}^{\perp}(\mathsf{F})$ equals $\mathrm{add}^{\perp}(\mathsf{F})$ and thus is regular.

These results show the importance of the cardinal characteristics $\mathrm{cov}(\mathsf{F})$. It turns out that $\mathrm{cov}(\mathsf{F})$ is large if all semifilters $\mathcal{F} \in \mathsf{F}$ have small cardinal characteristics of the first level. In Theorem 9.2.7 of [6] we show that for a family $\mathsf{F} \subset \mathsf{SF}$ with $\cup\mathsf{F} = \mathfrak{F}r^{\perp} \notin \mathsf{F}$ its covering number

- $\mathrm{cov}(\mathsf{F}) \le \mathfrak{c}$;
- $\mathrm{cov}(\mathsf{F}) \le \mathfrak{d}$ if F is $\asymp$-invariant;
- $\mathrm{cov}(\mathsf{F}) = \mathfrak{c}$ provided $\sup\limits_{\mathcal{F}\in\mathsf{F}} \mathrm{ad}(\mathcal{F}) < \mathfrak{c}$;
- $\mathrm{cov}(\mathsf{F}) \ge \mathrm{cf}(\mathfrak{c})$ provided $\mathrm{ad}(\mathcal{F}) < \mathfrak{c}$ for any $\mathcal{F} \in \mathsf{F}$;
- $\mathrm{cov}(\mathsf{F}) \ge \mathfrak{d}$ provided $\sup\limits_{\mathcal{F}\in\mathsf{F}} \pi\chi[\mathcal{F}] < \mathfrak{d}$; **Please, check if the item is correct.**
- $\mathrm{cov}(\mathsf{F}) \ge \mathrm{cf}(\mathfrak{d})$ provided $\pi\chi[\mathcal{F}] < \mathfrak{d}$ for all $\mathcal{F} \in \mathsf{F}$;
- $\mathrm{cov}(\mathsf{F}) \ge \mathfrak{s}$ provided $\sup\limits_{\mathcal{F}\in\mathsf{F}} \pi\mathfrak{p}(\mathcal{F}^{\perp}) < \mathfrak{s}$;
- $\mathrm{cov}(\mathsf{F}) \ge \mathrm{cf}(\mathfrak{s})$ provided $\pi\mathfrak{p}(\mathcal{F}^{\perp}) < \mathfrak{s}$ for all $\mathcal{F} \in \mathsf{F}$.

The values of the cardinal characteristic $\mathrm{cov}^{\perp}(\mathsf{F})$ for various families of semifilters $\mathsf{F} \subset \mathsf{SF}$ lead to interesting and important small cardinals:

$$\begin{aligned}
\mathfrak{h} &= \mathrm{cov}^{\perp}(\mathsf{F}), \text{ where } \mathsf{F} \text{ is the family of all semifilters } \mathcal{F} \text{ with } \pi\chi(\mathcal{F}) > 1;\\
\mathfrak{g} &= \mathrm{cov}^{\perp}(\mathsf{F}), \text{ where } \mathsf{F} \text{ is the family of all non-meager semifilters;}\\
\mathfrak{g}_f &= \mathrm{cov}^{\perp}(\mathsf{F}), \text{ where } \mathsf{F} \text{ is the family of all non-meager filters;}\\
\mathfrak{g}_u &= \mathrm{cov}^{\perp}(\mathsf{F}), \text{ where } \mathsf{F} \text{ is the family of all semifilters coherent to ultrafilters;}\\
\mathfrak{g}_l &= \mathrm{cov}^{\perp}(\mathsf{F}), \text{ where } \mathsf{F} \text{ is the family of all semifilters } \mathcal{F} \asymp \mathcal{F}^{\perp}.
\end{aligned}$$

One may wonder why we do not add to this list the cardinal $\mathrm{cov}^{\perp}(\mathsf{UF})$ where UF is the class of ultrafilters. But according to a result of Plewik [34] the cardinal $\mathrm{cov}^{\perp}(\mathsf{UF})$ equals $\mathfrak{c}$ and thus is not interesting.

The first two cardinals $\mathfrak{h}$ and $\mathfrak{g}$ are well-known to set theorists and naturally appear in many different contexts. For example $\mathfrak{h}$ can be equivalently defined an the smallest size of a family $\mathcal{N}$ of nowhere dense subsets of the Stone-Čech remainder $\omega^* = \beta\omega \setminus \omega$ with dense union $\cup\mathcal{N}$. The cardinal $\mathfrak{h}$ was introduced by B. Balcar, J. Pelant and P. Simon as the smallest height of a tree π-base for the Stone-Čech remainder ω^* (that is why the letter $\mathfrak{h}$ appears).

Another well-known cardinal in the above list is $\mathfrak{g}$, the groupwise density number, introduced by Blass [11] as the smallest size of a collection of groupwise dense families with empty intersection. The cardinal $\mathfrak{g}_f$ was recently considered by Mildenberger [27] and the cardinal $\mathfrak{g}_u$ (which is equal to $\mathrm{cov}(\mathfrak{D}_{\mathrm{fin}})$) was studied in [29] and [39].

Investigating the properties of the small cardinals $\mathfrak{g}, \mathfrak{g}_f, \mathfrak{g}_l, \mathfrak{g}_u$, we show in [6, §9.3] that $\mathfrak{g}_u \geq \mathfrak{g}_l \geq \max\{\mathfrak{g}, \mathfrak{b}\}$ and $\mathfrak{g}_f \leq \mathrm{cf}(\mathfrak{c})$. Since the inequality $\mathfrak{g} < \mathfrak{g}_f$ is consistent [19], the latter result strengthen the lower bound $\mathfrak{g} \leq \mathrm{cf}(\mathfrak{c})$ proved earlier by Blass [13, 8.7].

Applying the mentioned general results about the cardinal characteristics of families of semifilters to the coherence classes of semifilters we arrive to the following "ideal" cardinal characteristics $\mathrm{add}[\mathcal{F}], \mathrm{cov}[\mathcal{F}], \mathrm{non}[\mathcal{F}], \mathrm{cof}[\mathcal{F}]$ defined for any non-comeager semifilter $\mathcal{F}$:

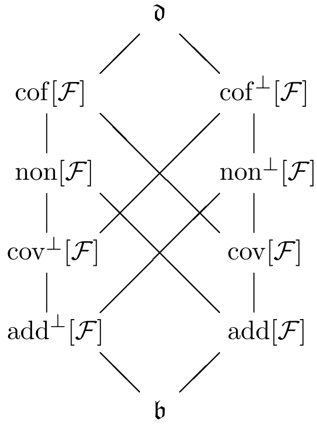
- $\mathrm{add}[\mathcal{F}] = \mathrm{add}([\mathcal{F}]) = \min\big\{|\mathsf{C}| : \mathsf{C} \subset [\mathcal{F}] \ \ \cup\mathsf{C} \not\asymp \mathcal{F}\big\}$,
- $\mathrm{cov}[\mathcal{F}] = \mathrm{cov}([\mathcal{F}]) = \min\big\{|\mathsf{C}| : \mathsf{C} \subset [\mathcal{F}] \ \ \cup\mathsf{C} = \mathfrak{Fr}^\perp\big\}$,
- $\mathrm{cof}[\mathcal{F}] = \mathrm{cof}([\mathcal{F}]) = \min\big\{|\mathsf{C}| : \mathsf{C} \subset [\mathcal{F}]\, \forall \mathcal{E} \in [\mathcal{F}]\, \exists \mathcal{C} \in \mathsf{C} \text{ with } \mathcal{E} \subset \mathcal{C}\big\}$,
- $\mathrm{non}[\mathcal{F}] = \mathrm{non}([\mathcal{F}]) = \min\big\{\chi(\mathcal{B}) : \mathcal{B} \in \mathsf{SF} \text{ and } \mathcal{B} \not\leq \mathcal{F}\big\}$.

These cardinal characteristics have duals defined for any non-meager semifilter $\mathcal{F}$:

- $\mathrm{add}^\perp[\mathcal{F}] = \mathrm{add}(\mathcal{F}^\perp) = \min\big\{|\mathsf{C}| : \mathsf{C} \subset [\mathcal{F}] \ \ \cap\mathsf{C} \not\asymp \mathcal{F}\big\}$,
- $\mathrm{cov}^\perp[\mathcal{F}] = \mathrm{cov}[\mathcal{F}^\perp] = \min\big\{|\mathsf{C}| : \mathsf{C} \subset [\mathcal{F}] \ \ \cap\mathsf{C} = \mathfrak{Fr}\big\}$,
- $\mathrm{cof}^\perp[\mathcal{F}] = \mathrm{cof}[\mathcal{F}^\perp] = \min\big\{|\mathsf{C}| : \mathsf{C} \subset [\mathcal{F}]\, \forall \mathcal{E} \in [\mathcal{F}]\, \exists \mathcal{C} \in \mathsf{C} \text{ with } \mathcal{C} \subset \mathcal{E}\big\}$,
- $\mathrm{non}^\perp[\mathcal{F}] = \mathrm{non}[\mathcal{F}^\perp] = \min\big\{\chi(\mathcal{B}) : \mathcal{B} \in \mathsf{SF} \text{ and } \mathcal{F} \not\leq \mathcal{B}^\perp\big\}$.

It is important to notice that for any non-comeager semifilter $\mathcal{F}$ the cardinal $\mathrm{non}[\mathcal{F}]$ coincides with the nonification $\widehat{\pi\chi}[\mathcal{F}]$ of the π-character (this partly justifies the choice of the term "nonification" for $\widehat{\pi\chi}[-]$).

For any bi-Baire semifilter $\mathcal{F}$ its "ideal" cardinal characteristics are related as follows:



Let us observe that the cardinal characteristics $\mathrm{add}[\mathcal{F}]$, $\mathrm{cov}[\mathcal{F}]$, $\mathrm{non}[\mathcal{F}]$, $\mathrm{cof}[\mathcal{F}]$ are defined only for non-comeager semifilters $\mathcal{F}$. If we want to extend them onto comeager semifilters, the unique reasonable way to do that is to put $\mathrm{add}[\mathfrak{F}r^{\perp}] = \mathrm{cov}[\mathfrak{F}r^{\perp}] = \mathfrak{b}$ and $\mathrm{non}[\mathfrak{F}r^{\perp}] = \mathrm{cof}[\mathfrak{F}r^{\perp}] = \mathfrak{d}$.

Among the eight "ideal" cardinal characteristics of semifilters, $\mathrm{cov}^{\perp}[-]$ and $\mathrm{non}[-]$ seem to be the most important. The first of them, $\mathrm{cov}^{\perp}[-]$, is a $\Subset$-monotone $\cup$-homomorphism while the second, $\mathrm{non}[-]$, is a $\Subset$-monotone $\cap$-homomorphism on the lattice SF.

Applying the above-mentioned theorem (estimating $\mathrm{cov}(\mathsf{F})$ for various families of semifilters), we arrive to the following important Polarization Formula holding for any semifilter $\mathcal{F}$:

$$\max\left\{\mathrm{cov}^{\perp}[\mathcal{F}], \pi\chi[\mathcal{F}^{\perp}]\right\} \geq \mathfrak{d}.$$

Since $\mathrm{cov}^{\perp}[\mathcal{F}] \leq \mathrm{non}[\mathcal{F}] \leq \widehat{\pi\chi}[\mathcal{F}]$ this formula strengthen the polarization formula $\max\left\{\widehat{\pi\chi}[\mathcal{F}], \pi\chi[\mathcal{F}^{\perp}]\right\} \geq \mathfrak{d}$ proved earlier with help of the function representation.

If $\mathcal{F}$ is a semifilter with $\pi\chi[\mathcal{F}] < \mathrm{cov}^\perp[\mathcal{F}]$, then the cardinal $\mathrm{cov}^\perp[\mathcal{F}] = \mathrm{add}^\perp[\mathcal{F}]$ is regular, see Theorem 9.4.2 of [6]. This theorem implies that the cardinal $\mathfrak{d}$ is regular under $(\mathfrak{r} < \mathfrak{d})$. Indeed, take any ultrafilter $\mathcal{U}$ with $\pi\chi[\mathcal{U}] = \mathfrak{r} < \mathfrak{d}$ and derive from $\max\left\{\mathrm{cov}^\perp[\mathcal{U}], \pi\chi[\mathcal{U}^\perp]\right\} \geq \mathfrak{d}$ that $\mathfrak{d} = \mathrm{cov}^\perp[\mathcal{U}] > \mathfrak{r} = \pi\chi[\mathcal{U}]$ and thus the cardinal $\mathfrak{d} = \mathrm{cov}^\perp[\mathcal{U}] = \mathrm{add}^\perp[\mathcal{U}]$ is regular.

20. The relation $\leq_{\mathcal{F}}$ and its cardinal characteristics

It turns out that the cardinal characteristics $\mathrm{cov}^\perp[-]$ and $\mathrm{non}[-]$ give lower and upper bounds for another two cardinal characteristics $\mathfrak{b}(-)$ and $\mathfrak{q}(-)$ having their origins in Non-standard Arithmetics. To define the cardinal characteristics $\mathfrak{b}(-)$ and $\mathfrak{q}(-)$ observe that each semifilter $\mathcal{F}$ generates two relations $\leq_{\mathcal{F}}$ and $=_{\mathcal{F}}$ on the countable product ω^ω: $f \leq_{\mathcal{F}} g$ (resp. $f =_{\mathcal{F}} g$) if the set $\{n \in \omega : f(n) \leq g(n)\} \in \mathcal{F}$ (resp. $\{n \in \omega : f(n) = g(n)\} \in \mathcal{F}$). In case of the Fréchet filter $\mathcal{F} = \mathfrak{Fr}$ the relation $\leq_{\mathcal{F}}$ coincides with the usual preorder $\leq^*$ of eventual dominance.

If $\mathcal{F}$ is a filter, then $\leq_{\mathcal{F}}$ is a preorder and $=_{\mathcal{F}}$ is an equivalence relation generated by this preorder on ω^ω. Moreover, the preorder $\leq_{\mathcal{U}}$ is total (that is any two elements of ω^ω are $\leq_{\mathcal{U}}$-comparable) provided $\mathcal{U}$ is an ultrafilter. In the latter case the total preorder $\leq_{\mathcal{U}}$ is well-studied in Set and Model Theories: the quotient space $\omega^\omega/\mathcal{U}$ of ω^ω by the equivalence relation $=_{\mathcal{U}}$, endowed with natural arithmetic and order structures is a non-standard model of arithmetics.

Two cardinal characteristics play an important role in studying the ultrapower $\omega^\omega/\mathcal{U}$: the cofinality $\mathrm{cof}(\omega^\omega/\mathcal{U})$ of $\omega^\omega/\mathcal{U}$ and the coinitiality $\mathrm{coin}(\omega^{\uparrow\omega}/\mathcal{U})$ of its top sky. By the *top sky* of an ultrapower $\omega^\omega/\mathcal{U}$ we understand the quotient-image $\omega^\omega/\mathcal{U}$ of the set $\omega^{\uparrow\omega}$ of all functions $f : \omega \to \omega$ with $\lim_{n\to\infty} f(n) = +\infty$.

In the case of (semi)filters the cardinals $\mathrm{cof}(\omega^\omega/\mathcal{F})$, $\mathrm{coin}(\omega^\omega/\mathcal{F})$ are split up into four important cardinal characteristics $\mathfrak{b}(\mathcal{F}), \mathfrak{b}^\perp(\mathcal{F})$ and $\mathfrak{q}(\mathcal{F}), \mathfrak{q}^\perp(\mathcal{F})$ which are tightly connected with the "ideal" cardinal characteristics of $\mathcal{F}$. To define them take a semifilter $\mathcal{F}$ and call a subset $A \subset \omega^{\uparrow\omega}$

– $\leq_{\mathcal{F}}$-*bounded* if there is a function $b \in \omega^{\uparrow\omega}$ such that $a \leq_{\mathcal{F}} b$ for all $a \in A$;

- $\le_{\mathcal{F}}$-*dominating* if for each $x \in \omega^{\uparrow\omega}$ there is $a \in A$ with $x \le_{\mathcal{F}} a$;
- $\ge_{\mathcal{F}}$-*bounded* if there is a function $b \in \omega^{\uparrow\omega}$ such that $b \le_{\mathcal{F}} a$ for all $a \in A$;
- $\ge_{\mathcal{F}}$-*dominating* if for each $x \in \omega^{\uparrow\omega}$ there is $a \in A$ with $a \le_{\mathcal{F}} x$.

Given a semifilter $\mathcal{F}$ let

- $\mathfrak{b}(\mathcal{F})$ be the smallest size of a $\le_{\mathcal{F}}$-unbounded subset $B \subset \omega^{\uparrow\omega}$;
- $\mathfrak{q}(\mathcal{F})$ be the smallest size of a $\ge_{\mathcal{F}}$-unbounded subset $B \subset \omega^{\uparrow\omega}$;
- $\mathfrak{b}^{\perp}(\mathcal{F})$ be the smallest size of a $\le_{\mathcal{F}}$-dominating subset $D \subset \omega^{\uparrow\omega}$;
- $\mathfrak{q}^{\perp}(\mathcal{F})$ be the smallest size of a $\ge_{\mathcal{F}}$-dominating subset $D \subset \omega^{\uparrow\omega}$.

The cardinal functions $\mathfrak{b}(-), \mathfrak{q}(-)$ are $\Subset$-monotone lattice homomorphisms on SF. Moreover, $\mathfrak{b}^{\perp}(\mathcal{F}) = \mathfrak{b}(\mathcal{F}^{\perp})$ and $\mathfrak{q}^{\perp}(\mathcal{F}) = \mathfrak{q}(\mathcal{F}^{\perp})$ for each semifilter $\mathcal{F}$. In particular, $\mathfrak{b}(\mathcal{L}) = \mathfrak{b}^{\perp}(\mathcal{L})$ and $\mathfrak{q}(\mathcal{L}) = \mathfrak{q}^{\perp}(\mathcal{L})$ for any maximal linked semifilter $\mathcal{L}$.

For the extremal semifilters $\mathfrak{Fr}$ and $\mathfrak{Fr}^{\perp}$ the cardinal characteristics $\mathfrak{b}(-)$ and $\mathfrak{q}(-)$ take their extremal values: $\mathfrak{b}(\mathfrak{Fr}) = \mathfrak{q}(\mathfrak{Fr}) = \mathfrak{b}$ and $\mathfrak{b}(\mathfrak{Fr}^{\perp}) = \mathfrak{q}(\mathfrak{Fr}^{\perp}) = \mathfrak{d}$. The $\subset$-monotonicity of $\mathfrak{b}(-)$ and $\mathfrak{q}(-)$ implies that the cardinal characteristics $\mathfrak{b}(-), \mathfrak{q}(-)$ take their values in the interval $[\mathfrak{b}, \mathfrak{d}]$. The cardinals $\mathfrak{b}(\mathcal{F}), \mathfrak{q}(\mathcal{F})$ are regular for any filter $\mathcal{F}$. These are the only restrictions on the values of $\mathfrak{b}(-)$ and $\mathfrak{q}(-)$: according to a forcing result of Canjar [21] for any regular cardinals $\kappa, \lambda \in [\mathfrak{b}, \mathfrak{d}]$ it is consistent that there is an ultrafilter $\mathcal{U}$ with $\mathfrak{b}(\mathcal{U}) = \kappa$ and $\mathfrak{q}(\mathcal{U}) = \lambda$. Moreover, in ZFC there always exists a ultrafilter $\mathcal{U}$ with $\mathfrak{b}(\mathcal{U}) = \mathfrak{q}(\mathcal{U}) = \mathrm{cf}(\mathfrak{d})$, see Canjar [22].

The cardinal characteristics of semifilters discussed above have counterparts defined for a family of semifilters $\mathsf{F} \subset \mathsf{SF}$ as follows:

- $\mathfrak{b}^{\perp}(\mathsf{F}) = \min\big\{|D| : D \subset \omega^{\omega} \quad \forall (\mathcal{F}, f) \in \mathsf{F} \times \omega^{\omega}\ \exists g \in D \text{ with } f \le_{\mathcal{F}} g\big\}$;
- $\mathfrak{q}^{\perp}(\mathsf{F}) = \min\big\{|D| : D \subset \omega^{\uparrow\omega}\ \forall (\mathcal{F}, f) \in \mathsf{F} \times \omega^{\uparrow\omega}\ \exists g \in D \text{ with } g \le_{\mathcal{F}} f\big\}$;
- $\mathfrak{b}(\mathsf{F}) = \min\big\{|\mathsf{P}| : \mathsf{P} \subset \mathsf{F} \times \omega^{\omega} \quad \forall g \in \omega^{\omega}\ \exists (\mathcal{F}, f) \in \mathsf{P} \text{ with } f \not\le_{\mathcal{F}} g\big\}$;
- $\mathfrak{q}(\mathsf{F}) = \min\big\{|\mathsf{P}| : \mathsf{P} \subset \mathsf{F} \times \omega^{\uparrow\omega} \quad \forall g \in \omega^{\uparrow\omega}\ \exists (\mathcal{F}, f) \in \mathsf{P} \text{ with } g \not\le_{\mathcal{F}} f\big\}$.

It follows from the definitions that $\mathfrak{b}(\mathsf{F}) \le \mathfrak{b}(\mathcal{F}) = \mathfrak{b}^\perp(\mathcal{F}^\perp) \le \mathfrak{b}^\perp(\mathsf{F}^\perp)$ for any semifilter $\mathcal{F} \in \mathsf{F}$. In particular, $\mathfrak{b}([\mathcal{F}]) \le \mathfrak{b}(\mathcal{F}) \le \mathfrak{b}^\perp([\mathcal{F}^\perp])$. In contrast to the cardinal characteristics $\mathfrak{b}(\mathcal{F}), \mathfrak{q}(\mathcal{F})$ which can be distinct, their "family" counterparts $\mathfrak{b}(\mathsf{F})$ and $\mathfrak{q}(\mathsf{F})$ are equal for any $\asymp$-invariant family F of semifilters (in particluar, for each coherence class). The same concerns their duals $\mathfrak{b}^\perp(\mathsf{F}) = \mathfrak{q}^\perp(\mathsf{F})$.

As we said, $\mathrm{cov}^\perp[-]$ and $\mathrm{non}[-]$ give lower and upper bounds for $\mathfrak{b}(-)$ and $\mathfrak{q}(-)$. The following diagram describes the interplay between the cardinal characteristics (of the third complexity level) of a semifilter $\mathcal{F}$ and these of its support $\mathrm{supp}(\mathcal{F})$:

$$\begin{tikzcd}[column sep=small]
 & & \max\{\mathfrak{b}^\perp(\mathrm{supp}(\mathcal{F})), \mathfrak{q}^\perp(\mathrm{supp}(\mathcal{F}))\} \arrow[d, dash] \arrow[dr, dash] & & & \\
 & & \mathrm{cof}[\mathcal{F}] \arrow[d, dash] \arrow[dddddr, dash] & \mathrm{cof}^\perp[\mathcal{F}] \arrow[d, dash] \arrow[dddddl, dash] & & \\
 & & \mathrm{non}[\mathcal{F}] \arrow[d, dash] \arrow[dddddr, dash] & \mathrm{non}^\perp[\mathcal{F}] \arrow[d, dash] \arrow[dddddl, dash] & & \\
\mathfrak{q}^\perp([\mathcal{F}^\perp]) \arrow[d, dash] \arrow[rr, phantom, "="] & & \mathfrak{b}^\perp([\mathcal{F}^\perp]) \arrow[d, dash] & \mathfrak{b}^\perp([\mathcal{F}]) \arrow[d, dash] \arrow[rr, phantom, "="] & & \mathfrak{q}^\perp([\mathcal{F}]) \arrow[d, dash] \\
\mathfrak{q}(\mathcal{F}) \arrow[d, dash] & & \mathfrak{b}(\mathcal{F}) \arrow[d, dash] & \mathfrak{b}^\perp(\mathcal{F}) \arrow[d, dash] & & \mathfrak{q}^\perp(\mathcal{F}) \arrow[d, dash] \\
\mathfrak{q}([\mathcal{F}]) \arrow[rr, phantom, "="] & & \mathfrak{b}([\mathcal{F}]) \arrow[d, dash] & \mathfrak{b}([\mathcal{F}^\perp]) \arrow[d, dash] \arrow[rr, phantom, "="] & & \mathfrak{q}([\mathcal{F}^\perp]) \\
 & & \mathrm{cov}^\perp[\mathcal{F}] \arrow[d, dash] & \mathrm{cov}[\mathcal{F}] \arrow[d, dash] & & \\
 & & \mathrm{add}^\perp[\mathcal{F}] \arrow[d, dash] & \mathrm{add}[\mathcal{F}] \arrow[dl, dash] & & \\
 & & \min\{\mathfrak{b}(\mathrm{supp}(\mathcal{F})), \mathfrak{q}(\mathrm{supp}(\mathcal{F}))\} & & &
\end{tikzcd}$$

Since the support $\mathrm{supp}(\mathcal{F})$ of any filter $\mathcal{F}$ coincides with $\mathcal{F}$, the above diagram implies the equalities

$$\min\big\{\mathfrak{b}(\mathcal{F}), \mathfrak{q}(\mathcal{F})\big\} = \mathfrak{b}([\mathcal{F}]) = \mathfrak{q}([\mathcal{F}]) = \mathrm{cov}^\perp[\mathcal{F}] = \mathrm{add}^\perp[\mathcal{F}] \le \mathrm{add}[\mathcal{F}]$$

holding for any filter $\mathcal{F}$. For an ultrafilter $\mathcal{U}$ we have, in addition, $\mathrm{cof}[\mathcal{U}] = \mathrm{non}[\mathcal{U}] = \mathfrak{b}^\perp([\mathcal{U}]) = \mathfrak{q}^\perp([\mathcal{U}]) = \max\big\{\mathfrak{b}(\mathcal{U}), \mathfrak{q}(\mathcal{U})\big\}$. Thus the "ideal" cardinal characteristics of any ultrafilter $\mathcal{U}$ are completely determined by the values of

$\mathfrak{b}(\mathcal{U})$ and $\mathfrak{q}(\mathcal{U})$. This fact has a philosophical value since the "ideal" cardinal characteristics are defined with help of the subcoherence relation while $\mathfrak{b}(-)$, $\mathfrak{q}(-)$ are not.

Thus the values of cardinal characteristics $\mathfrak{b}(-)$ and $\mathfrak{q}(-)$ determine the values of other cardinal characteristics of ultrafilters. As we said even for an ultrafilter $\mathcal{U}$ the cardinal characteristics $\mathfrak{b}(\mathcal{U})$ and $\mathfrak{q}(\mathcal{U})$ can be different. However, any such an ultrafilter $\mathcal{U}$ is coherent to a Q-point according to a result of Laflamme, Zhu [26]. To understand the nature of this result, let us introduce two oriented modifications of the subcoherence relation. We shall say that a semifilter $\mathcal{F}$ is *right* (resp. *left*) *subcoherent* to a semifilter $\mathcal{U}$ and denote this by $\mathcal{F} \underset{\rightarrow}{\subset} \mathcal{U}$ (resp. $\mathcal{F} \underset{\leftarrow}{\subset} \mathcal{U}$) if for any finite-to-one function $f : \omega \to \omega$ there is a finite-to-finite multifunction Φ on ω such that $\Phi(\mathcal{F}) \subset \mathcal{U}$ and $\Phi(n) \subset [f(n), \infty)$ (resp. $\Phi(n) \subset [0, f(n)]$) for almost all $n \in \omega$. The right (left) subcoherence agrees well with duality and the usual subcoherence relation, see Proposition 10.2.2 of [6]:

- $\mathcal{F} \underset{\rightarrow}{\subset} \mathcal{U}$ if and only if $\mathcal{U}^\perp \underset{\leftarrow}{\subset} \mathcal{F}^\perp$;
- $\mathcal{F} \Subset \mathcal{U}$ provided $\mathcal{F} \underset{\rightarrow}{\subset} \mathcal{U}$ or $\mathcal{F} \underset{\leftarrow}{\subset} \mathcal{U}$;
- $\mathcal{F}' \Subset \mathcal{F} \underset{\rightarrow}{\subset} \mathcal{U} \Subset \mathcal{U}'$ implies $\mathcal{F}' \underset{\rightarrow}{\subset} \mathcal{U}'$;
- $\mathcal{F}' \Subset \mathcal{F} \underset{\leftarrow}{\subset} \mathcal{U} \Subset \mathcal{U}'$ implies $\mathcal{F}' \underset{\leftarrow}{\subset} \mathcal{U}'$.

The main result concerning oriented subcoherences is Theorem 10.2.4 of [6] asserting that $\mathcal{F} \underset{\rightarrow}{\subset} \mathcal{U}$ (resp. $\mathcal{F} \underset{\leftarrow}{\subset} \mathcal{U}$) implies $\mathfrak{b}(\mathcal{F}) \le \mathfrak{q}(\mathcal{U})$ (resp. $\mathfrak{q}(\mathcal{F}) \le \mathfrak{b}(\mathcal{U})$). For an ultrafilter $\mathcal{U}$ the relations $\mathcal{U} \underset{\rightarrow}{\subset} \mathcal{U}$ and $\mathcal{U} \underset{\leftarrow}{\subset} \mathcal{U}$ are equivalent to the absence of a Q-point in the coherence class $[\mathcal{U}]$ of $\mathcal{U}$, see Proposition 10.2.3 [6]. For such ultrafilters $\mathcal{U}$ the characteristics $\mathfrak{b}(\mathcal{U})$, $\mathfrak{q}(\mathcal{U})$, $\mathrm{add}[\mathcal{U}]$, $\mathrm{cov}[\mathcal{U}]$, $\mathrm{non}[\mathcal{U}]$, $\mathrm{cof}[\mathcal{U}]$ all are equal.

It is interesting to note that each of the cardinal characteristics $\mathrm{non}[-]$, $\mathrm{cov}^\perp[-]$, $\mathfrak{b}(-)$, $\mathfrak{q}(-)$ is responsible for a special sort of subcoherence: for semifilters $\mathcal{F}, \mathcal{U}$

- $\pi\chi[\mathcal{F}] < \mathrm{non}[\mathcal{U}]$ implies $\mathcal{F} \Subset \mathcal{U}$;
- $\pi\chi[\mathcal{F}] < \mathrm{cov}^\perp[\mathcal{U}]$ implies $\mathcal{F} \sqsubseteq \mathcal{U}$;

– $\pi\chi[\mathcal{F}] < \mathfrak{b}(\mathcal{U})$ implies $\mathcal{F} \underset{\leftarrow}{\subset} \mathcal{U}$;

– $\pi\chi[\mathcal{F}] < \mathfrak{q}(\mathcal{U})$ implies $\mathcal{F} \underset{\rightarrow}{\subset} \mathcal{U}$.

We derive from this that the cardinal functions non$[-]$, $\mathrm{cov}^{\perp}[-]$, $\mathfrak{b}(-)$ and $\mathfrak{q}(-)$ cannot exceed $\pi\chi[-]$ too much. More precisely, for any semifilter $\mathcal{F}$ with $\pi\chi[\mathcal{F}] < \mathfrak{d}$ we get

$\mathrm{non}[\mathcal{F}] \le \mathrm{next}_{\mathrm{non}[\mathsf{SF}]}(\pi\chi[\mathcal{F}])$, $\mathrm{cov}^{\perp}[\mathcal{F}] \le \mathrm{next}_{\mathrm{cov}[\mathsf{SF}]}(\pi\chi[\mathcal{F}])$,

$\mathfrak{b}(\mathcal{F}) \le \mathrm{next}_{\mathfrak{b}[\mathsf{SF}]}(\pi\chi[\mathcal{F}])$, $\mathfrak{q}(\mathcal{F}) \le \mathrm{next}_{\mathfrak{q}[\mathsf{SF}]}(\pi\chi[\mathcal{F}])$,

where $\mathrm{next}_X(\kappa)$ is a cardinal function assigning to a set X of cardinals and a cardinal $\kappa < \sup X$ the cardinal $\mathrm{next}_X(\kappa) = \min X \setminus [0, \kappa]$.

21. Constructing non-coherent semifilters

It turns out that under some additional set-theoretic assumptions it is possible to construct large families of pairwise non-coherent or even incomparable semifilters (possessing some additional properties). Two semifilters $\mathcal{F}$,$\mathcal{U}$ are *incomparable* if neither $\mathcal{F} \Subset \mathcal{U}$ nor $\mathcal{U} \Subset \mathcal{F}$. Maximal linked semifilters are incomparable if and only if they are not coherent. In particular, we show in [6, Ch.11] that

– under $\mathfrak{d} < \mathfrak{c}$ the lattice SF contains at least $\mathfrak{c}$ pairwise non-coherent semifilters;

– under $\max\{\mathfrak{r}, \mathfrak{d}\} < \mathfrak{c}$ the lattice SF contains at least $\mathfrak{c}$ pairwise non-coherent filters;

– under $\mathfrak{r} \ge \mathfrak{d}$ the lattice SF contains strictly more than $\mathfrak{r}$ pairwise non-coherent $\pi\chi$-minimal ultrafilters;

– under $\mathfrak{r} \ge \mathfrak{d}$ the lattice SF contains strictly more than $\mathfrak{d}$ pairwise non-coherent $\pi\chi$-minimal ultrafilters $\mathcal{U}$ with $\mathfrak{b}(\mathcal{U}) = \mathfrak{q}(\mathcal{U}) = \mathrm{cf}(\mathfrak{d})$;

– under $\mathfrak{t} = \mathfrak{d}$ the lattice SF contains strictly more than $\mathfrak{d}$ pairwise incomparable non-meager filters $\mathcal{F}$ with $\chi(\mathcal{F}) = \mathfrak{p}(\mathcal{F}) = \mathfrak{t}$;

– under $\mathfrak{r} \geq \mathfrak{d}$ (resp. $\mathfrak{b} = \mathfrak{d}$) for every $n \geq 2$ the lattice SF contains strictly more than $\mathfrak{r}$ pairwise non-coherent (resp. pairwise incomparable) maximal n-linked semifilters which fails to be coherent to $(n+1)$-linked semifilters.

– under $\mathfrak{r} \geq \mathfrak{d}$ the lattice contains a totally separated sequence of ultrafilters $(\mathcal{U}_n)$. For this sequence the operator $\lim^{[\mathcal{U}_n]} : \mathsf{SF} \to [\mathsf{SF}]$ is an isomorphic embedding of SF into $[\mathsf{SF}]$ preserving many classes of semifilters. In particular, $|[\mathsf{SF}]| = |[\mathsf{UF}]| = 2^{\mathfrak{c}}$ under $\mathfrak{r} \geq \mathfrak{d}$.

The principal result here is Theorem 11.6.7 of [6] producing a totally separated sequence of ultrafilters (and consequently an embedding of SF into $[\mathsf{SF}]$) from any infinite sequence of non-coherent ultrafilters. This theorem implies the Banakh-Blass Dichotomy that the number of coherence classes of ultrafilters is either finite or $2^{\mathfrak{c}}$.

22. Total coherence

The results on the existence of non-coherent semifilters combined with coherence criteria obtained with the help of cardinal characteristics help us to characterize the principle NCF and its variations in the terms of inequalities between suitable small cardinals. Namely, in [6, Ch.12] we show that

– All Simon semifilters (i.e., semifilters $\mathcal{F}$ with $\pi\chi[\mathcal{F}] = \mathfrak{b}$) are coherent if and only if all centered Simon semifilters are coherent if and only if $\mathfrak{b} < \mathrm{non}_b$.

– All Simon filters are coherent if and only if $\mathfrak{b} < \mathrm{non}_f$.

– All $\pi\chi$-minimal ultrafilters are coherent if and only if all $\pi\chi$-minimal semifilters are coherent if and only if $\mathfrak{r} < \mathfrak{d}$.

– The principle NCF holds if and only if all ultrafilters are coherent if and only if $\mathfrak{r} < \mathrm{non}_u$ if and only if $\mathfrak{r} < \mathrm{add}_u = \mathfrak{g}_u = \mathfrak{b}_u = \mathfrak{q}_u = \mathrm{non}_u = \mathfrak{d}$.

– All maximal linked semifilters are coherent if and only if $\mathfrak{r} < \mathrm{non}_l$ if and only if $\mathfrak{r} < \mathrm{add}_l = \mathfrak{g}_l = \mathfrak{b}_l = \mathfrak{q}_l = \mathrm{non}_l = \mathfrak{d}$.

- the Filter Dichotomy holds if and only if all non-meager filters are coherent if and only if $\mathfrak{r} < \mathfrak{g}_f$ if and only if $\mathfrak{b} = \mathfrak{r} < \mathrm{add}_f = \mathfrak{g}_f = \mathfrak{d} = \mathfrak{c}$.

- the Semifilter Trichotomy holds if and only if all bi-Baire semifilters are coherent if and only if all non-meager centered semifilters are coherent if and only if all centered Simon semifilters are coherent and all non-meager filters are coherent if and only if $\mathfrak{r} < \mathfrak{g}$ if and only if $\mathfrak{b} = \mathfrak{r} < \mathfrak{g} = \mathrm{non}_b = \mathfrak{d} = \mathfrak{c}$.

23. The supremization of the linkedness number $\pi\mathfrak{p}(-)$

Now we return to some cardinal characteristics of the second complexity level and study the supremization $\pi\mathfrak{p}^{[-]}$ of the linkedness number $\pi\mathfrak{p}(-)$. For typographical reasons we shall write $\pi\mathfrak{p}[\mathcal{F}]$ in place of $\pi\mathfrak{p}^{[\mathcal{F}]}$. By its properties the supremization $\pi\mathfrak{p}[-]$ resembles the minimization $\pi\chi[-]$ of the π-character. The role of the small cardinals $\mathfrak{r}$ and $\mathfrak{d}$ (which are of crucial importance in applications of $\pi\chi[-]$) play another two classical small cardinals: $\mathfrak{b}$ and $\mathfrak{s}$. We recall that $\mathfrak{s}$ is the smallest size of a splitting family $\mathcal{S} \subset \mathcal{P}(\omega)$ (the splitting property of $\mathcal{S}$ means that for each infinite subset $I \subset \omega$ there is a set $S \in \mathcal{S}$ such that both the sets $I \cap S$ and $I \setminus S$ are infinite).

For the cardinal function $\pi\mathfrak{p}[-]$ in [6, 13.2.1] we can prove the polarization formulas

$$\min\left\{\mathrm{non}[\mathcal{F}], \pi\mathfrak{p}[\mathcal{F}]\right\} \le \mathfrak{b} \quad \text{and} \quad \max\left\{\mathrm{cov}^{\perp}[\mathcal{F}], \pi\mathfrak{p}[\mathcal{F}]\right\} \ge \mathfrak{s}$$

which resemble the polarization formulas

$$\min\left\{\mathrm{non}^{\perp}[\mathcal{F}], \pi\chi[\mathcal{F}]\right\} \le \mathfrak{r} \quad \text{and} \quad \max\left\{\mathrm{cov}[\mathcal{F}], \pi\chi[\mathcal{F}]\right\} \ge \mathfrak{d}$$

holding for the cardinal function $\pi\chi[-]$.

The former polarization formula allows to prove the near coherence of semifilters $\mathcal{F}, \mathcal{U}$ satisfying the inequality $\max\{\mathrm{cov}^{\perp}[\mathcal{F}], \mathrm{cov}^{\perp}[\mathcal{U}]\} < \mathfrak{s}$ (this is a counterpart of the near coherence of any two semifilters $\mathcal{F}$, $\mathcal{U}$ with $\max\{\pi\chi[\mathcal{F}], \pi\chi[\mathcal{U}]\} < \mathfrak{d}$). These two near coherence conditions allow us to establish the Blass-Mildenberger Dichotomy asserting that under $\mathfrak{r} < \mathfrak{s}$ there are at most two non-coherent maximal linked semifilters ($\pi\chi$-minimal and $\pi\mathfrak{p}$-maximal). Indeed, if $\mathcal{L}$ is a maximal linked semifilter with $\mathrm{cov}[\mathcal{L}] > \mathfrak{r}$, then $\pi\chi[\mathcal{L}] \leq \mathfrak{r}$ (by the polarization formula $\min\{\mathrm{cov}[\mathcal{L}], \pi\chi[\mathcal{L}]\} \leq \mathfrak{r}$) and $\mathcal{L}$ is (nearly) coherent to each $\pi\chi$-minimal ultrafilter. If $\mathrm{cov}[\mathcal{L}] \leq \mathfrak{r}$, then $\mathrm{cov}[\mathcal{L}] \leq \mathfrak{r} < \mathfrak{s}$ and $\mathcal{L}$ is (nearly) coherent to each maximal linked semifilter $\mathcal{U}$ with $\mathrm{cov}[\mathcal{U}] < \mathfrak{s}$ (by the polarization formula $\max\{\mathrm{cov}^{\perp}[\mathcal{U}], \pi\mathfrak{p}[\mathcal{U}]\} \geq \mathfrak{s}$, any such a maximal linked semifilter $\mathcal{U}$ is $\pi\mathfrak{p}$-maximal). These results imply also an easy proof of the Mildenberger inequality $\mathfrak{s} \leq \mathrm{cf}(\mathfrak{d})$: Assuming that $\mathfrak{s} > \mathrm{cf}(\mathfrak{d})$ we would get that the cardinal $\mathfrak{d}$ is singular and thus $\mathfrak{r} \geq \mathfrak{d}$. In this case by Corollary 11.2.2 of [7], there are two non-coherent ultrafilters $\mathcal{U}_1, \mathcal{U}_2$ with $\mathrm{cov}[\mathcal{U}_1] = \mathrm{cov}[\mathcal{U}_2] = \mathrm{cf}(\mathfrak{d})$. On the other hand, they should be (nearly) coherent by the near coherence condition $\max\{\mathrm{cov}[\mathcal{U}_1], \mathrm{cov}[\mathcal{U}_2]\} = \mathrm{cf}(\mathfrak{d}) < \mathfrak{s}$, which is a contradiction.

It is interesting to mention that under the assumption $\mathfrak{r} < \mathfrak{g}$ (equivalent to the Semifilter Trichotomy) the cardinal characteristics $\pi\chi[-]$ and $\pi\mathfrak{p}[-]$ completely determine the subcoherence relation: a semifilter $\mathcal{F}$ is subcoherent to a semifilter $\mathcal{U}$ if and only if $\pi\chi[\mathcal{F}] \leq \pi\chi[\mathcal{U}]$ if and only if $\pi\mathfrak{p}[\mathcal{F}] \geq \pi\mathfrak{p}[\mathcal{U}]$, see Proposition 13.9.2 of [6].

Under the assumption $\mathfrak{r} < \mathfrak{s}$ we have a weaker result: a semifilter $\mathcal{F}$ is subcoherent to a maximal linked semifilter $\mathcal{L}$ if and only if $\pi\chi[\mathcal{F}] \leq \pi\chi[\mathcal{L}]$ if and only if $\pi\mathfrak{p}[\mathcal{F}] \geq \pi\mathfrak{p}[\mathcal{L}]$, see Proposition 13.9.3 of [6].

24. Some cardinal characteristics of rapid semifilters

In this section we calculate cardinal characteristics of rapid semifilters. A semifilter $\mathcal{F}$ is defined to be *rapid* if for any function $f : \omega \to \omega$ there is $F \in \mathcal{F}$ whose enumerating function $e_F : \omega \to F$ exceeds f. It is easy to see that Q-points are rapid ultrafilters. In Theorem 13.8.1 of [6], we show that $\pi\chi[\mathcal{F}] \geq \mathfrak{d}$, $\mathrm{non}^{\perp}[\mathcal{F}] \leq \mathfrak{r}$, $\pi\mathfrak{p}[\mathcal{F}] \leq \mathfrak{b}$, and $\mathrm{cov}^{\perp}[\mathcal{F}] \geq \mathfrak{s}$ for any rapid semifilter $\mathcal{F}$. We derive from this that under NCF or $\mathfrak{r} < \mathfrak{s}$ each ultrafilter is coherent to a P-point and no ultrafilter is rapid (this was first noticed by A. Blass). Moreover, under $\mathfrak{r} < \mathfrak{g}$ all rapid semifilters are comeager, see Corollary 13.8.3 [6].

25. Inequalities between some critical values

The following inequalities between critical values of cardinal characteristics are proved in Theorem 12.1.9 and Proposition 13.3.1 of [6].

- $\mathfrak{b} \le \mathrm{add}_b \le \mathrm{cov}_b = \max\{\mathfrak{b}, \mathfrak{g}\} = \min\{\mathfrak{b}_b, \mathfrak{q}_b\} \le \max\{\mathfrak{b}_b, \mathfrak{q}_b\} \le \mathrm{non}_b \le \mathrm{cof}_b$.
- $\max\{\mathfrak{g}_l, \mathfrak{b}_b, \mathfrak{q}_b\} \le \mathfrak{b}(\mathsf{ML}) = \mathfrak{q}(\mathsf{ML}) \le \mathfrak{g}_u$.
- $\mathrm{add}_f \ge \mathrm{add}_f^{\perp} = \mathrm{cov}_f^{\perp} = \max\{\mathfrak{b}, \mathfrak{g}_f\} = \min\{\mathfrak{b}_f, \mathfrak{q}_f\} \le \max\{\mathfrak{b}_f, \mathfrak{q}_f\} \le \mathrm{non}_f \le \mathrm{cof}_f \le \mathrm{cof}_u$.
- $\max\{\mathfrak{b}_f, \mathfrak{q}_f\} \le \mathfrak{g}_u = \mathfrak{b}(\mathsf{UF}) = \mathfrak{q}(\mathsf{UF})$.
- $\mathrm{add}_l \le \mathrm{cov}_l \le \min\{\mathfrak{b}_l, \mathfrak{q}_l\} \le \max\{\mathfrak{b}_l, \mathfrak{q}_l\} \le \mathrm{non}_l \le \mathrm{cof}_l \le \mathrm{cof}_u$.
- $\mathrm{add}_u = \mathrm{cov}_u = \min\{\mathfrak{b}_u, \mathfrak{q}_u\} \le \max\{\mathfrak{b}_u, \mathfrak{q}_u\} \le \mathrm{non}_u = \mathrm{cof}_u \le \mathrm{cf}(\mathfrak{d})$.
- $\mathrm{cov}_b = \max\{\mathfrak{b}, \mathfrak{g}\} \le \mathfrak{g}_l \le \mathrm{cov}_l$ and $\mathrm{cov}_f^{\perp} = \max\{\mathfrak{b}, \mathfrak{g}_f\} \le \mathfrak{g}_u \le \mathrm{cov}_u$.
- $\pi\mathfrak{p}^l = \pi\mathfrak{p}^u \le \pi\mathfrak{p}^f = \pi\mathfrak{p}^b$ and $\mathfrak{p} \le \pi\mathfrak{p}^u \le \mathfrak{s}$.

26. The consistent structures of the coherence lattice [SF]

With all the knowledge at hand, we return to studying the structure of the coherence lattice [SF] and some its subsets. We start with estimating the size of the set [UF] of all coherence classes of ultrafilters:

- $|[\mathsf{UF}]| = 1$ if and only if $\mathfrak{r} < \mathrm{non}_u$ if and only if $r < \mathrm{cov}_u = \mathfrak{d}$.
- $|[\mathsf{UF}]| \le 2$ if $\mathfrak{r} < \mathfrak{s}$.
- $|[\mathsf{UF}]| = 2^{\mathfrak{c}}$ if $\mathfrak{r} \ge \mathfrak{d}$.
- $|[\mathsf{UF}]| \ge 2^{\mathfrak{c}}$ if and only if $|[\mathsf{UF}]| \ge \aleph_0$.

The size of the coherence lattice [SF] is evaluated in Theorem 14.4.1 of [6] as follows:

1. $|[\mathsf{SF}]| \geq 3$;
2. $|[\mathsf{SF}]| = 3$ if and only if $|[\mathsf{SF}]| \leq 4$ if and only if $(\mathfrak{r} < \mathfrak{g})$.
3. $|[\mathsf{SF}]| \geq 5$ if $\mathfrak{b} < \mathfrak{r}$.
4. $|[\mathsf{SF}]| = 2^{\mathfrak{c}}$ if $\mathfrak{r} \geq \mathfrak{d}$ or $|[\mathsf{UF}]| \geq \aleph_0$.
5. $|[\mathsf{SF}]| \geq M(n)$ if $|[\mathsf{UF}]| = n < \omega$.
6. $|[\mathsf{SF}]| \geq \mathfrak{g}$ if $\mathfrak{g} < \mathfrak{b}$.
7. $|[\mathsf{SF}]| \geq \mathfrak{g}_f$ if $\mathfrak{g}_f < \mathfrak{b}$.
8. $|[\mathsf{SF}]| \geq \mathfrak{c}$ if $\mathfrak{d} < \mathfrak{c}$.

Here $M(n)$, the Dedekind function, equals the number of all monotone Boolean functions of n variables.

Observe that under $\mathfrak{r} < \mathfrak{g}$ the coherence lattice $[\mathsf{SF}]$, being finite, is complete. On the other hand, it fails to be complete under $\mathfrak{b} = \mathfrak{d}$, see Theorem 14.5.3 of [6].

27. Some applications

In this section we collect some applications of the coherence in various fields of mathematics. We tried to select applications in which the presence of the coherence and semifilters would not be immediately evident.

We start with studying the composant structure of the Stone-Čech remainder $\mathbb{H}^* = \beta\mathbb{H} \setminus \mathbb{H}$ of the half-line $\mathbb{H} = [0, \infty)$. The space $\mathbb{H}^*$ is known to be an example of a non-metrizable indecomposable continuum (a continuum is *indecomposable* if it is not the union of two smaller continua). The composant $C(x)$ of a point x of an indecomposable continuum X is the set of all points $y \in X$ which can be connected with x by a proper subcontinuum of X. It is easy to see that two composants of an indecomposable continuum either coincide or are disjoint. According a classical result of Mazurkiewicz each metrizable indecomposable continuum X has uncountably many composants; moreover, X contains an uncountable closed subset intersecting each composant in at most one point. In contrast, the number of composants of $\mathbb{H}^*$ equals the size of the set $[\mathsf{UF}] = \{[\mathcal{U}] : \mathcal{U}$ is an ultrafilters$\}$ and can vary from 1 under NCF till $2^{\mathfrak{c}}$ under $\mathfrak{r} \geq \mathfrak{d}$. More precisely, $\mathbb{H}^*$ has either finitely many or else $2^{\mathfrak{c}}$ composants. In the latter case $\mathbb{H}^*$ contains a closed subset C homeomorphic to $\beta\omega$ and intersecting each composant in at most one point. This is a combined result of Mioduszewski [31], Rudin [38], Blass [10] and Banakh, Blass [3].

Our next application of semifilters appear in calulating the additivity of the Menger property. We recall that a topological space X is *Menger* if for each sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \omega$, whose union $\bigcup_{n\in\omega} \mathcal{V}_n$ covers the space X. It is easy to see that the countable union of Menger spaces is Menger. In Bartoszyński, Shelah, Tsaban [8] asked about the smallest number $\mathrm{add}(\mathsf{Menger})$ of Menger subspaces of the real line whose union is not Menger. They proved that $\mathfrak{b} \leq \mathrm{add}(\mathsf{Menger}) \leq \mathrm{cf}(\mathfrak{d})$ and asked if the equality $\mathfrak{b} = \mathrm{add}(\mathsf{Menger})$ can be proved in ZFC. We show that this is not so because of the lower bound $\mathrm{add}(\mathsf{Menger}) \geq \mathrm{cov}_b = \max\{\mathfrak{b}, \mathfrak{g}\}$. This lower bound proved with help of semifilters is due to Zdomskyy [43].

Our third application of semifilters concerns the problem of the existence of a discontinuous separately continuous function $f : X \times Y \to \mathbb{R}$ defined on the product of any non-discrete separable Tychonov spaces X, Y. We show that this problem has an affirmative solution if and only if any two P-filters $\mathcal{F}, \mathcal{U}$ are nearly coherent (formally this is a weaker than NCF). This is an unpublished result of Banakh, Maslyuchenko, Mykhaylyuk [4].

28. Some open problems

One of the most intriguing open problems related to semifilters is Protasov's reformulation of the Owings problem. We recall that Owings [32] asked if for any partition $\omega = A_1 \cup A_2$ one of the cells of the partition contains the sum set $A + A$ for some infinite $A \subset \omega$. It should be mentioned that Hindman [24] constructed a partition $\omega = A_1 \cup A_2 \cup A_3$ such that no cell of this partition contains a sum set $A + A$ with $A \in \mathfrak{Fr}^\perp$.

Problem 28.1 (Protasov). *Show that the semifilter $\mathcal{S}$ generated by the family $\{A + A : A \in \mathfrak{Fr}^\perp\}$ is comeager. Is $\mathcal{S}$ unsplit?*

Let us note that the unsplit property of the semifilter $\mathcal{S}$ is equivalent to the affirmative solution of the Owings problem.

Many questions can be posed on the consistency of strict inequalities between small cardinals appearing as the extremal values of cardinal characteristics considered in this book. There are 8×2 such cardinal characteristics ($\pi\chi[-]$, $\pi\mathfrak{p}[-]$, $\mathrm{add}[-]$, $\mathrm{cov}[-]$, $\mathfrak{b}(-)$, $\mathfrak{q}(-)$, $\mathrm{non}[-]$, $\mathrm{cof}[-]$ plus their duals) and four basic classes of semifilters (BS, BF, ML, UF). In such a way there appear $8 \times 2 \times 4 \times 2 = 128$ small cardinals and $C^2_{128} = 8128$ possible questions concerning the relation between these cardinals. Of course there is no physical possibility to state all of them.

In this context it would be helpful for the reader to know the existing consistency results on the strict inequalities between some of the classical small cardinals. Except for the four last columns the following table is taken from Blass [13, §11] and describes the values of the small cardinals in some models of ZFC obtained by iterated forcing (which adds Cohen, Random, Sacks, Hechler, Laver, Mathias or Miller reals to ground models). In the table "Blass" stands for the model with $\mathfrak{b} < \mathfrak{g} = \mathfrak{u}$ produced by Blass [12] (he starts with a model of GCH and does the countable-support iterated forcing construction for ω_2 steps using the Miller's superperfect forcing at limit stages and the Cohen forcing at successor stages); "BS" stands for the models of $\mathfrak{g} \le \mathfrak{u} < \mathfrak{d}$ constructed by Blass and Shelah [17] and studied recently by Mildenberger [28]. "Brendle" stands for the model of $\mathfrak{g} < \mathfrak{g}_f$ constructed by Brendle [19] and "MTS" for the model constructed by Mildenberger, Shelah, Tsaban [29]. For the value of the cardinal $\mathfrak{g}_f$ in this model, see Mildenberger [27]. For filling the cells of the table we used the (in)equalities between critical values of cardinal characteristics from Section 25.

Problem 28.2. *Try to determine the new cardinals in the forcing extensions, where there are question marks in the table.*

Problem 28.3. *Explore the structure of the coherence lattice* $[\mathsf{SF}]$ *in known models of ZFC with* $\mathfrak{r} < \mathfrak{d}$.

	Cohen	Random	Sacks	Hechler	Laver	Mathias	Miller	Blass	BS	Brendle	MST
$\mathfrak{b}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$
$\mathfrak{d}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{g}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{g}_f$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$
$\mathfrak{g}_u$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{h}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{p}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\pi\mathfrak{p}^b$	?	?	?	?	?	?	$\aleph_1$	$\aleph_1$	?	?	?
$\pi\mathfrak{p}^u$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	?	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{r}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{r}_\kappa$	$\mathfrak{c}$	?	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{s}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	?	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{t}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$
$\mathfrak{u}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$
add_b	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	?	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$
cov_b	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\aleph_1$
$\mathbf{x}$	?	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	?
$\mathbf{y}$	?	$\aleph_1$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\aleph_1$	$\mathfrak{c}$	$\mathfrak{c}$

Here $\mathbf{x}$, $\mathbf{y}$, are any cardinals from the sets

$$X = \{\mathfrak{g}_l, \mathrm{cov}_f, \mathrm{cov}_l, \mathfrak{b}_i, \mathfrak{q}_i, \mathrm{non}_i, \mathrm{cof}_i : i \in \{b, f, l\}\},$$
$$Y = \{\mathfrak{g}_u, \mathrm{add}_u, \mathrm{cov}_u, \mathfrak{b}_u, \mathfrak{q}_u, \mathrm{non}_u, \mathrm{cof}_u\},$$

respectively.

Problem 28.4. *Is $\mathfrak{b}_i \neq \mathfrak{q}_i$ consistent for some $i \in \{b, f, l, u\}$? Is $\mathrm{non}_b < \mathrm{non}_u$ consistent?*[1]

Problem 28.5. *Which (finite) lattices can occur as isomorphic copies of the coherence lattice* $[\mathsf{SF}]$ *under various set-theoretic assumptions? Is any such a finite lattice isomorphic to the lattice* $\mathsf{M}(\mathcal{P}(n))$ *of monotone Boolean functions*

[1]Some results related to Problems 28.2–28.4 can be found in recent papers of Brendle [19], Mildenberger [27], [28], and Mildenberger, Shelah, Tsaban [29].

over $n = |[\mathsf{UF}]|$ *variables?*

According to the Banakh-Blass Dichotomy, $|[\mathsf{UF}]| < \aleph_0$ if and only if $|[\mathsf{UF}]| < 2^{\mathfrak{c}}$. Is the same thue for the coherence lattice $[\mathsf{SF}]$ or its subset $[\mathsf{ML}]$?

Problem 28.6. *Can* $[\mathsf{SF}]$ *or* $[\mathsf{ML}]$ *have infinite size* $< 2^{\mathfrak{c}}$*? Is the equality* $|[\mathsf{SF}]| = \aleph_0$ *consistent?* (If the latter happens, then $|[\mathsf{UF}]| < \aleph_0$ and $\mathfrak{b} \le \mathfrak{g} \le \mathfrak{r} < \mathfrak{d} = \mathfrak{c}$.)

References

[1] Aubrey, J.: Combinatorics for the dominating and unsplitting numbers, *The Journal of Symbolic Logic* **69** (2004), 482–498.

[2] Balcar, B. and Simon, P.: On minimal π-character of points in extremally disconnected spaces, *Topology and its Applications* **41** (1991), 133–145.

[3] Banakh, T. and Blass, A.: The number of near-coherence classes of ultrafilter is either finite or $2^{\mathfrak{c}}$, *Proceedings of the Barcelona Set Theory Year (2003/2004)*, to appear.

[4] Banakh, T., Mykhaylyuk, V. and Maslyuchenko, O.: Discontinuous separately continuous functions and near coherence of P-filters, *Real Analysis Exchange*, to appear.

[5] Banakh, T., Nikolas, P. and Sanchis, M.: Filter games and pathologic subgroups of $\mathbb{R}^\omega$, *Journal of the Australian Mathematical Society*, to appear.

[6] Banakh, T. and Zdomskyy, L.: Coherence of Semifilters, (`www.franko.lviv.ua/faculty/mechmat/Departments/Topology/booksite.html`)

[7] Banakh, T. and Zdomskyy, L.: Selection Principles and Games on Multicovered Spaces, *Quad. Mat.* **18**, Dept. Math., Seconda Univ. Napoli, Caserta (2006), 1–52.

[8] Bartoszyński, T., Shelah, S. and Tsaban, B.: Additivity properties of topological diagonalizations, *The Journal of Symbolic Logic* **68** (2003), 1254–1260.

[9] Blass, A.: Near coherence of filters, I. Cofinal equivalence of models of arithmetic, *Notre Dame Journal of Formal Logic* **27** (4) (1986), 579–591.

[10] Blass, A.: Near coherence of filters, II. Applications to operator ideals, the Stone-Čech remainder of a half-line, order ideals of sequences, and slenderness of groups, *Transactions of the American Mathematical Society* **300** (1987), 557–581.

[11] Blass, A.: Applications of superperfect forcing and its relatives, in *Set Theory and its Applications* (J. Steprāns and S. Watson, eds.), Lecture Notes in Mathematics **1401**, Springer-Verlag, Berlin (1989), 18–40.

[12] Blass, A.: Personal communication.

[13] Blass, A.: Combinatorial cardinal characteristics of the continuum, in *Handbook of Set Theory* (M. Foreman, A. Kanamori and M. Magidor, eds.), Kluwer.

[14] Blass, A. and Mildenberger, H.: On the cofinality of ultrapowers, *The Journal of Symbolic Logic* **64** (1999), 727–736.

[15] Blass, A. and Shelah, S.: There may be simple $P_{\aleph_1}$ and $P_{\aleph_2}$-point and the Rudin-Keisler ordering may be downward directed, *Annals of Pure and Applied Logic* **33** (1987), 213–243.

[16] Blass, A. and Shelah, S.: Near coherence of filters, III. A simplified consistency proof, *Notre Dame Journal of Formal Logic* **30** (4) (1989), 530–538.

[17] Blass, A. and Shelah, S.: Ultrafilters with small generating sets, *Israel Journal of Mathematics* **65** (1989), 259–270.

[18] Blass, A. and Weiss, G.: A characterization and sum decomposition of operator ideals, *Transactions of the American Mathematical Society* **246** (1978), 407–417.

[19] Brendle, J.: Distinguishing groupwise density numbers, preprint.

[20] Brown, A., Pearcy, C. and Salinas, N.: Ideals of compact operators on Hilbert space, *The Michigan Mathematical Journal* **19** (1971), 373–384.

[21] Canjar, R.M.: Countable ultraproducts without CH, *Annals of Pure and Applied Logic* **37** (1988), 1–79.

[22] Canjar, R.M.: Cofinalities of countable ultraproducts: the existence theorem, *Notre Dame Journal of Formal Logic* **30** (1989), 539–542.

[23] Eisworth, T. and Just, W.: NCF and combinatorics of open covers, unpublished note.

[24] Hindman, N.: Partitions and sums of integers with repetitions, *Journal of Combinatorial Theory Series A* **27** (1979), 19–32.

[25] Hindman, N. and Strauss, D.: *Algebra in the Stone-Cech Compactification: Theory and Applications*, De Gruyter Expositions in Mathematics **27**, Walter de Gruyter, Berlin (1998).

[26] Laflamme, C. and Zhu, J.P.: The Rudin-Blass ordering of ultrafilters, *The Journal of Symbolic Logic* **63** (1998), 584–592.

[27] Mildenberger, H.: The groupwise density number for filters, preprint.

[28] Mildenberger, H.: There may be infinitely many near-coherence classes under $\mathfrak{u} < \mathfrak{d}$, preprint.

[29] Mildenberger, H., Shelah, S. and Tsaban, B.: Covering the Baire space by families which are not finitely dominating, *Annals of Pure and Applied Logic* **140** (2006), 60–71.

[30] van Mill, J.: *Supercompactness and Wallman Spaces*, Mathematical Centre Tracts **85**, Amsterdam (1977).

[31] Mioduszewski, J.: On composants of $\beta\mathbb{R} - \mathbb{R}$, *Proceedings of the Conference on Topology and Measure I* (Zinnovitz, 1974) (J. Flashsmeyer, Z. Frolik and F. Terpe, eds.), Ernst-Moritz-Arndt-Universität zu Greifwald (1978), 257–283.

[32] Owings, J.: Problem E2494, *American Mathematical Monthly* **81** (1974), 902.

[33] Plewik, S.: Intersections and unions of ultrafilters without the Baire property, *Bulletin de l'Académie Polonaise des Sciences* **35** (1987), 805–808.

[34] Plewik, S.: Ideals of the second category, *Fundamenta Mathematicae* **138** (1991), 23–26.

[35] Protasov, I.: *Combinatorics of Numbers*, Monograph Series vol. **2**, VNTL Publ., Lviv (1997).

[36] Protasov, I. and Banakh, T.: *Ball Structures and Colorings of Graphs and Groups*, Monograph Series vol. **11**, VNTL Publ., Lviv (2003).

[37] Protasov, I. and Zarichnyi, M.: *A Quest for Asymptology*, in preparation.

[38] Rudin, M.E.: Composants of $\beta\mathbb{N}$, *Proceedings of the Washington State University Conference on General Topology* (1970), 117–119.

[39] Shelah, S. and Tsaban, B.: Critical cardinalities and additivity properties of combinatorial notions of smalleness, *Journal of Applied Analysis* **9** (2003), 149–162.

[40] Solomon, R.C.: Families of sets and functions, *Czechoslovak Mathematical Journal* **27** (1977), 556–559.

[41] Talagrand, M.: Compacts de fonctiones measurables et filtres non mesurables, *Studia Mathematica* **67** (1980), 13–43.

[42] Tsaban, B. and Zdomskyy, L.: Menger's covering property and groupwise density, *The Journal of Symbolic Logic* **71** (2006), 1053–1056.

[43] Zdomskyy, L.: A semifilter approach to selection principles, *Commentationes Mathematicae Universitatis Carolinae* **46** (2005), 525–539.